Wednesday 21st May Queen Mary

Partial latin squares, partial gerechte designs, list colouring and Hall's condition

A.J.W. Hilton (QMUL)

A list assignment to a graph G is a map $L : V(G) \to 2^{\mathscr{C}}$, where $\mathscr{C}$ is a set of colours. The list assignment satisfies Hall's condition if

$$\sum_{\sigma \in \mathscr{C}} \alpha(L, H, \sigma) \geq |V(H)|$$

for all subgraphs H of G, where $\alpha(L, H, \sigma)$ is the independence number of the subgraph of H induced by those vertices of H which have σ in their lists.

Given a partial $n \times n$ latin square P with n symbols $\sigma_1, \sigma_2, \ldots, \sigma_n$, it is natural to define the list $L(v)$ for each cell v to be:

(i) if a cell v is empty in P then $L(v)$ is the subset of $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ consisting of all symbols which do not already occur in the row and column containing v;

(ii) if v is not empty in P and σ is the symbol in v, then $L(v) = \{\sigma\}$.

It is easy to interpret this in terms of graph theory.

Cropper made the following conjecture:
Cropper's conjecture. A partial latin square can be completed if and only if it satisfies Hall's condition.

We shall discuss this conjecture and various analogues relating to sudokus, gerechte designs and pairs of mutually orthogonal latin squares.

Completing partial gerechte designs

EMIL VAUGHAN

A *gerechte skeleton* of order n is an $n \times n$ array whose cells are partitioned into n regions containing n cells each. A *gerechte design* of order n consists of a gerechte skeleton of order n, together with an assignment of a symbol from the set $\{1, \dots, n\}$ to each cell, such that each symbol occurs once in each row, once in each column, and once in each region.
Gerechte designs were introduced by W. U. Behrens in 1956, but interest in them was revived in 2005 with the arrival of sudoku puzzles in the newspapers. There has been much research about completing partial latin squares. We will look at some problems involving the completion of partial gerechte designs. The first of these problems is that of characterising which gerechte skeletons can be completed to gerechte designs.

Extremal t-intersecting sub-families of hereditary families

Peter Borg

University of Malta

21st May 2008

A family $\mathcal{A}$ of sets is said to be *t-intersecting* if any two sets in $\mathcal{A}$ intersect in at least t-elements; a 1-intersecting family is also simply called *intersecting*. For a family $\mathcal{F}$ and $T \subseteq \bigcup_{F \in \mathcal{F}} F$, let $\mathcal{F}\langle T \rangle := \{F \in \mathcal{F} \colon T \subseteq F\}$. If T has size say t, then we call the trivially t-intersecting family $\mathcal{F}\langle T \rangle$ a *t-star of $\mathcal{F}$*; a 1-star is also simply called a *star*. The classical Erdős-Ko-Rado (EKR) Theorem says that, if $n \geq 2r$, then the size of an intersecting sub-family of $\binom{[n]}{r}$ is at most $\binom{n-1}{r-1}$, i.e. the size of a star of $\binom{[n]}{r}$. Erdős, Ko and Rado also showed that the largest t-intersecting sub-families of $\binom{[n]}{r}$ are the t-stars if n is sufficiently large (later Ahlswede and Khachatrian remarkably obtained a characterisation of the largest t-intersecting sub-families of $\binom{[n]}{r}$ for any n, r and t).

A family $\mathcal{H}$ is said to be *hereditary* if any subset of any set in $\mathcal{H}$ is also in $\mathcal{H}$. The power set 2^X of a set X is the most simple example of a hereditary family, but there are various other interesting examples, such as the family of independent sets of a graph or matroid. We say that a set M is *$\mathcal{H}$-maximal* if M is not a subset of any set of $\mathcal{H} \backslash \{M\}$. If $\mathcal{H}$ is a hereditary family and $X_1, ..., X_k$ are the $\mathcal{H}$-maximal sets in $\mathcal{H}$, then clearly $\mathcal{H} = 2^{X_1} \cup ... \cup 2^{X_k}$; in other words, a hereditary family is a union of power sets. We denote by $\mu(\mathcal{H})$ the size of a smallest $\mathcal{H}$-maximal set in $\mathcal{H}$.

The famous Chvátal conjecture says that any hereditary family has a star that is a largest intersecting sub-family. A simple EKR result says that this is true if $\mathcal{H}$ is $2^{[n]}$ (or rather, a power set); however, for $n > t \geq 2$, the t-stars of $2^{[n]}$ are *not* the largest intersecting sub-families (a characterisation of the largest ones was obtained by Katona), and hence the conjecture does not generalise to the t-intersection case. A generalised form of another nice conjecture, made by Holroyd and Talbot, is the following uniform version of Chvátal's conjecture: if $\mathcal{H}$ is hereditary and $\mu(\mathcal{H}) \geq 2r$, then at least one of the largest intersecting sub-families of $\mathcal{H}^{(r)} := \{H \in \mathcal{H} \colon |H| = r\}$ is a star. The EKR Theorem confirms the case $\mathcal{H} = 2^{[n]}$. The speaker has recently proved the natural t-intersection generalisation of this conjecture for $\mu(\mathcal{H})$ sufficiently large, hence generalising the EKR Theorem for t-intersecting families. The talk will revolve around this result. In particular, there will be a description of difficulties encountered, ideas involved, and possible improvements.

Completion of partial latin squares

Roland Häggkvist

I shall discuss a number of results and open questions on the problem type: Given a family of partial $n \times n$ latin squares on n symbols, determine if each member of the family can be completed to a latin square. (A partial $n \times n$ square is an $n \times n$ array where every cell is filled with at most one symbol from $1, 2, \ldots, n$ in such a way that every symbol occurs at most once in every row and column. A latin square of order n is a partial latin square without empty cells.)

Typical families of completable partial latin squares include the following: (i) partial latin squares where each of the n rows, columns and symbols is used is used at most cn times with $c < 0.00001$ (Theorem (Chetwynd and H)) and probably $c < 0.25$ (Conjecture)); (ii) partial latin squares with at most $n-1$ cells filled (proved for $n > 1111$ (H 1976), and for all n (Smetaniuk 1982)

A conjectured extension of this (circulating for more than 25 years): A partial $n \times n$ latin square all of whose occupied cells occur in a $r \times s$ rectangle with at most $n - r$ empty cells in each row and at most $n - s$empty cells in each column is completable.

Another conjectured extension: Any partial $nr \times nr$ latin square whose occupied cells lie in $r - 1$ disjoint $r \times r$ squares c an be completed. This has been proved for $n = 3$ (Denley and H)

Some easy completely untouched problem types appear to be: There exists a bounded function $f(k)$ (possibly growing linearly in k) such that every partial $n \times n$ latin square whose occupied cells are all cells in k rows, k columns and underlying k fixed symbols is completable for $n > f(k)$. Here even the case $k = 1$ is open.

And marginally touched: There exist a bounded function $g(k)$ such that every partial $n \times n$ latin square whose occupied cells occur as all cells in k rows and k columns is completable if n is greater than or equal to $g(k)$. Here $g(2) = 6$ by a recent result by Adams, Bryant and Buchanan. Note incidentally that the case where the intersection between the rows and columns forms a latin $k \times k$ square always is completable.

Pursuit and evasion

Imre Leader (Cambridge)

In a typical continuous pursuit and evasion problem, like the famous `man and lion' puzzle, the problem is considered solved once one has given a winning strategy for the pursuer or the evader. But why should it be the case that one or other player has a winning strategy? And why cannot they both have winning strategies?

Of course, if we consider instead a discrete problem then these issues do not arise, so a key object of study is the relationship between a continuous problem and its discrete approximations.
After reviewing some background, we will present some positive results, as well as some conjectures and counterexamples.

(Joint work with Béla Bollobás)

Something new (all Buchsteiner quasigroups are loops) and something old (a still-unsolved problem of integer sequences).

Donald Keedwell (Surrey)

The new concept of nuclear amalgamation has highlighted particular kinds of loops (and quasigroups): namely, the loops of Bol-Moufang type and the Buchsteiner and conjugacy-closed loops. We shall show that Buchsteiner quasigroups which are not loops do not exist but that there are many proper conjugacy-closed quasigroups. (The corresponding question for quasigroups of Bol-Moufang type has already been resolved.)

It was shown in the 1960s that, for every order that a particular type of integer sequence S exists, there is a pair of orthogonal latin squares. It was also shown at that time that sequences S exist for all orders n up to 20 except 2 and 6 and that the number of different sequences S which exist increases rapidly with n. It seems almost certain therefore, but is still unproved, that sequences S exist for all n>6.

Thursday 22nd May LSE

Turán problems in the hypercube

John Talbot (UCL)

How many edges can a graph on n vertices contain if it has no subgraph isomorphic to a forbidden graph F? Such forbidden subgraph problems have a long history dating back to results of Mantel and Turán.

We consider a related problem for set systems. Given a forbidden family of sets F we say that $H \subseteq \mathcal{P}([n])$ is *F-free* if for every $A \subseteq [n]$, H does not contain a subfamily isomorphic to $\{A \Delta f : f \in F\}$. Our question is: how large can $H \subseteq \mathcal{P}([n])$ be if it is F-free? This can be reformulated as a forbidden configuration problem for subsets of the vertices of the n-dimensional hypercube $\mathcal{Q}_n = \{0,1\}^n$.

Given $F \subseteq \mathcal{Q}_d$ we say that $S \subseteq \mathcal{Q}_n$ is *F-free* if every embedding $i : \mathcal{Q}_d \to \mathcal{Q}_n$ satisfies $i(F) \nsubseteq S$. We determine the asymptotic density of the largest F-free subsets of $\mathcal{Q}_n$ for a variety of F. In particular we generalise the sole non-trivial prior result in this area, for $F = \mathcal{Q}_2$ due to E.A. Kostochka. Many natural questions remain open.

This is joint work with Robert Johnson.

Convergence issues of congestion games

Petra Berenbrink
(Simon Fraser, Canada)

In congestion games, a set of users and a set of resources are given. Each user wants to get access to one of several possible subsets of the resources. For example, in a network congestion game the users would like to access one out of several paths connecting their resource-destination pair. The cost of such an assignment depends on the maximum number of users choosing any resource.

A Nash Equilibrium for such a game is a situation where no player can improve its cost by switching to another strategy (here: another subset of the resources). I this talk I will present some simple processes converging to Nash Equilibria and will analyse the convergence time.

Clique Percolation

Oliver Riordan

Derényi, Palla and Vicsek introduced the following dependent percolation model, in the context of finding communities in networks. Starting with a random graph G generated by some rule, form an auxiliary graph G' whose vertices are the k-cliques of G, in which two vertices are joined if the corresponding cliques share $k-1$ vertices. They considered in particular the case where $G = G(n,p)$, and found heuristically the threshold for a giant component to appear in G'. Béla Bollobás and I have given a rigorous proof of this result, as well as many extensions. The model turns out to be very interesting due to the essential global dependence present in G'. In this talk I will attempt to describe one of the strategies we use for handling this dependence, splitting it into 'positive' and 'negative' parts, and using Bayes' Theorem to bound the effect of the negative dependence.

"The Norman Biggs Lecture"

$\pi_1(|G|)$, earrings, and limits of free groups

Reinhard Diestel

University of Hamburg

We characterize the fundamental group $\pi_1(|G|)$ of the space $|G|$ formed by a locally finite graph G and its ends, by embedding it canonically as a subgroup in the inverse limit of the free groups $\pi_1(G')$ with $G' \subset G$ finite.

As an intermediate step, we characterize $\pi_1(|G|)$ combinatorially as a group of infinite words.

(This is joint work with Philipp Sprüssel. The paper containing these results can be found at `http://www.math.uni-hamburg.de/home/diestel/papers/Homotopy.pdf`)

A combinatorial proof of an identity for spanning trees

David Wagner

If one increases the conductance of a wire in a (linear, resistive) electrical network then the effective conductance of the whole network cannot decrease. This physically intuitive property is implied by a combinatorial identity for spanning trees in a graph that can be traced back to Maxwell. The previously-known proofs all involve a significant amount of linear algebra and symbolic manipulations. We give a short, self-contained, and purely graph-theoretic proof of the identity. The motivation for this is to provide insight into a conjectured analogue for spanning forests, for which the algebraic techniques are not available.
This is a joint work with Josef Cibulka, Jan Hladky, and Mike LaCroix.

Cycles in directed graphs

Deryk Osthus

A fundamental result of Dirac states that a minimum degree of $|G|/2$ guarantees a Hamilton cycle in an undirected graph G. There is an analogue of this for digraphs due to Ghouila-Houri which states that every digraph D whose minimum out- and minimum indegrees are at least $|D|/2$ contains a Hamilton cycle. I will discuss the following analogue of the latter result for oriented graphs: every sufficiently large oriented graph G with minimum out- and indegrees at least $(3|G|-4)/8$ contains a Hamilton cycle. This bound is best possible and answers a question of Thomassen from 1979 for large oriented graphs. I will also discuss further results, e.g. (i) an approximate solution to a conjecture of Nash-Williams concerning a digraph analogue of Chvátal's theorem as well as (ii) results on cycles of given length in oriented graphs of large minimum degree. (Joint work with Peter Keevash, Luke Kelly, Daniela Kühn and Andrew Treglown.)