Partitioning Permutations into Increasing and Decreasing Subsequences

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Introduction

We will write a permutation on n elements as a list (e.g. $\mathcal{P} = (5, 1, 2, 3, 4)$); by '$\mathcal{P}(4) = 3$' we mean that the 4th element of the list is 3.

Given permutations $\mathcal{P}$ on n elements and $\mathcal{S}$ on $m \leq n$ elements, we say that $\mathcal{P}$ contains the sub-permutation $\mathcal{S}$ if there exists $\phi : \{1, \ldots, m\} \rightarrow \{1, \ldots, n\}$ such that $\mathcal{P}(\phi(i)) < \mathcal{P}(\phi(j))$ if and only if $\mathcal{S}(i) < \mathcal{S}(j)$.

For example, $(5, 1, 2, 3, 4)$ contains $(2, 1)$: $(5, 1, 2, 3, 4)$.

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A question we can ask, given a permutation $\mathcal{P}$, is ‘can it be partitioned into r increasing and s decreasing subsequences?’

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$(5, 1, 2, 3, 4)$, or $(5, 1, 2, 3, 4)$, or $(5, 1, 2, 3, 4)$, et cetera.

We will say ‘$\mathcal{P}$ has an (r, s)-partition’.

If a permutation contains a decreasing subsequence of length $r+1$, then it clearly does not have an $(r,0)$-partition.

In fact a permutation has an $(r,0)$-partition if and only if it does not contain the sub-permutation $(r+1, r, \ldots, 2, 1)$.

A permutation has a $(1,1)$-partition if and only if it contains neither $(2,1,4,3)$ nor $(3,4,1,2)$.

For any r and s there exists a minimal set $\mathcal{F}_{r,s}$ of forbidden permutations: a permutation has an (r, s)-partition if and only if it contains no member of $\mathcal{F}_{r,s}$.

Is this set finite in general? If so, how big are the forbidden permutations?

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A permutation on n elements must contain a monotonic subsequence of length $\sqrt{n}$: consider a long increasing subsequence within a member of $\mathcal{F}_{r,s}$:

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A permutation on n elements must contain a monotonic subsequence of length $\sqrt{n}$: consider a long increasing subsequence within a member of $\mathcal{F}_{r,s}$:

$$(\color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{blue}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \quad, \color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{blue}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{blue}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet}, \color{red}{\bullet})$$

If we remove one element we can find an (r, s)-partition.

Theorem. *Kezdy, Snevily and Wang (1996): The set $\mathcal{F}_{r,s}$ is always finite.*

Proof. Let $\mathcal{P} \in \mathcal{F}_{r,s}$ be a permutation on n elements.

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Proof. Let $\mathcal{P} \in \mathcal{F}_{r,s}$ be a permutation on n elements.

$$\mathcal{P} = ({\color{red}\bullet}, {\color{blue}\bullet}, {\color{red}\bullet}, {\color{red}\bullet}, {\color{blue}\bullet}, {\color{red}\bullet}, {\color{red}\bullet}, {\color{blue}\bullet}, {\color{red}\bullet}, {\color{red}\bullet}, {\color{red}\bullet}, {\color{blue}\bullet}, {\color{red}\bullet}, {\color{red}\bullet}, \quad , {\color{blue}\bullet}, {\color{red}\bullet}, {\color{red}\bullet}, {\color{red}\bullet}, {\color{red}\bullet}, {\color{blue}\bullet}, {\color{blue}\bullet}, {\color{blue}\bullet}, {\color{red}\bullet})$$

Since $\mathcal{P}$ is minimal, if we remove the xth element of $\mathcal{P}$ we can find an (r, s)-partition of the result. Let R_x be the union of the increasing subsequences in the partition.

R_x contains no decreasing sequence of length $r + 1$; $\mathcal{P} - R_x - \{x\}$ contains no increasing sequence of length $s + 1$.

The $R_x, x \in \{1, \ldots, n\}$ are not too different:

$\mathcal{P} - R_y - \{y\}$ can be partitioned into s decreasing sequences, so certainly $R_x - R_y - \{y\}$ can be.

But R_x can be partitioned into r increasing sequences, so it cannot contain a decreasing sequence of length $r + 1$.

Thus $|R_x - R_y - \{y\}| \leq rs$, so $|R_x - R_y| \leq rs + 1$.

Finally, $|R_x \Delta R_y| = |(R_x - R_y) \cup (R_y - R_x)| \leq 2(rs + 1)$.

If $n \geq R\Big(2(rs+1); (rs+1)^2 + rs + 3\Big)$ then there must be a set $G \subset \{1, \ldots, n\}$ of size $(rs+1)^2 + rs + 3$ and a constant $C \leq 2(rs+1)$ such that any pair R_x and R_y with $x, y \in G$ have $|R_x \Delta R_y| = C$.

Theorem. *Deza (1974): Any member of* $\{1, \ldots, n\}$ *is in either none, one, all but one or all of the* $R_x, x \in G$.

Let R be the elements in all or all but one of the $R_x, x \in G$. If R contained a decreasing sequence of length $r+1$ then that sequence would be in one of the R_x (since $|G| \geq r+1$). So R can be partitioned into r increasing sequences.

Now $\mathcal{P}$ has no (r, s)-partition, so $\mathcal{P} - R$ is not a union of s decreasing sequences, so it must contain an increasing sequence Q of length $s + 1$.

No member of Q is in more than one of the $R_x, x \in G$, so there must be $s + 2$ elements x of G such that $Q \subset \mathcal{P} - R_x$.

At least one of these x is not a member of Q, so that $Q \subset \mathcal{P} - R_x - \{x\}$. But $\mathcal{P} - R_x - \{x\}$ is a union of s decreasing sequences by definition, which is a contradiction.

This proof shows that a member of $\mathcal{F}_{r,s}$ cannot be a permutation on more than $R\left(2(rs+1); (rs+1)^2+rs+3\right)$ elements; which is of course nowhere near the polynomial size that seems likely.

On the other hand, $\mathcal{F}_{r,s}$ does grow fairly quickly: 102 members of $\mathcal{F}_{2,1}$ are known, and although this is believed to be the complete list it has not been proven.

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