Chromatic polynomials for twisted bracelets

1. Introduction

This paper is concerned with the chromatic polynomials of graphs that can be constructed by taking n copies of a given graph and linking them together to form a 'bracelet'. (Formal definitions are given in Section 2.) Although this may appear to be a rather special construction, it does have important applications. That is because theoretical physicists study models based on infinite graphs with a 'lattice-like' structure, and our bracelets can be thought of as finite approximations to such graphs. By considering the behaviour as n tends to infinity, we can obtain useful insights into physical phenomena.

One specific question that will be considered here is the influence of boundary conditions. The underlying problem can be illustrated by reference to some of the oldest results in this area [4]. If we take n copies of K_2 (a single edge) and add edges that link corresponding vertices in successive copies, we obtain an ordinary 'ladder', whose chromatic polynomial is

$$z(z-1)(z^2-3z+3)^n.$$

If we do the same thing, but also link the nth copy to the first copy (so that we have a bracelet), we obtain a 'cyclic ladder'. Here the chromatic polynomial is

$$(z^2-3z+3)^n + (z-1)\Big((3-z)^n + (1-z)^n\Big) + (z^2-3z+1).$$

Clearly the dominant term is $(z^2-3z+3)^n$ in both cases. But the cyclic ladder has also has two 'subdominant' terms that depend on n, and these affect the limiting behaviour [6].

Furthermore, if we give the ladder a 'half-twist' before linking the last copy to the first one, we obtain a Möbius ladder, whose chromatic polynomial is

$$(z^2-3z+3)^n + (z-1)\Big((3-z)^n - (1-z)^n\Big) - 1.$$

So the result of the half-twist is that the coefficient of one of the subdominant terms has been altered, and the last term has apparently almost disappeared. In this paper we shall obtain a general result that explains the alteration in the coefficient. (In fact the change in the last term can be explained in a similar fashion, but that will not be done here.)

The main result is an explicit formula, giving the dominant and subdominant terms in the chromatic polynomial of a twisted bracelet. If the bracelet is constructed from n copies of K_b, with arbitrary twists between the copies, the formula takes the form

$$F^n + (z-1)\Big(G^n + (f-1)H^n\Big) + \dots .$$

Here F, G, H are polynomials, of degrees $b, b-1, b-1$ respectively. We shall obtain formulae for F, G and H in terms of an explicit polynomial $c_b(z)$, and these expressions are independent of the twists. The only term dependent on the twists is the integer coefficient $f-1$, in which f is the number of fixed points of the total twist.

2. Notation

Throughout this paper we deal with two natural numbers b and k. We regard b as 'fixed' and take $V=\{1,2,\ldots,b\}$ to be the vertex-set of the complete graph K_b. On the other hand, we shall think of k as 'variable' and regard an injection $\alpha: V \to \{1,2,\ldots,k\}$ as a (proper) k-colouring of K_b. We denote by P_k the set of all such k-colourings, and by $\mathcal{V}_k$ the vector space of real-valued functions defined on P_k.

The *standard basis* for $\mathcal{V}_k$ contains, for each α in P_k, the function $[\alpha]$ defined by

$$[\alpha](\beta)=\begin{cases}1, & \text{if } \alpha=\beta;\\ 0, & \text{otherwise.}\end{cases}$$

We shall consider families of graphs constructed as follows. Given an integer $n \geq 3$ and n permutations $\sigma_1,\sigma_2,\ldots,\sigma_n$ of V, take n disjoint copies of K_b and link them by extra edges as follows. For $i=1,2,\ldots,n$ join vertex v in the ith copy to vertex $\sigma_i(v)$ in the $(i+1)$th copy, with the convention that $n+1=1$. We obtain a *bracelet*, consisting of n copies of K_b linked by edges in the manner prescribed. The resulting graph will be denoted by

$$\mathcal{C}_{n,b}(\sigma_1,\sigma_2,\ldots,\sigma_n).$$

In fact, as we shall see in Section 3, this construction is unnecessarily general.

3. The compatibility operator

Suppose that a permutation π of V is given. We say that an ordered pair (α,β) of members of P_k is *compatible with* π if

$$\alpha(v) \neq \beta\pi(v) \quad \text{for all } v \in V.$$

The idea is as follows. Suppose two disjoint copies of K_b are linked by edges, each vertex v in the first copy being joined to the vertex $\pi(v)$ in the second. Then, if the colourings α and β are given to the first and second copy respectively, these colourings combine to form a (proper) colouring of the linked graph. If (α,β) is compatible with the identity permutation, we simply say that (α,β) is a *compatible pair.*

For each $k \geq b$ we define a *compatibility matrix* T_π (and an associated *compatibility operator* on $\mathcal{V}_k$) as follows. The rows and columns of T_π correspond to the elements of P_k, and the entries are given by:

$$(T_\pi)_{\alpha\beta} = \begin{cases}1 & \text{if } (\alpha,\beta) \text{ is compatible with } \pi;\\ 0 & \text{otherwise.}\end{cases}$$

There is a simple relationship between the general compatibility operator T_π and the specific operator $T=T_\iota$ corresponding to the identity permutation. Define the linear operator X_π on $\mathcal{V}_k$ by the permutation matrix

$$(X_\pi)_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha(v) = \beta\pi(v) \text{ for all } v \in V; \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1 The operators T_π and $T = T_\iota$ are related as follows:

$$T_\pi = TX_\pi = X_\pi T.$$

Proof We have

$$(TX_\pi)_{\alpha\beta} \;=\; \sum_\gamma T_{\alpha\gamma}(X_\pi)_{\gamma\beta}.$$

The only nonzero term in the sum is the one for which $\gamma = \beta\pi$. So the sum reduces to $T_{\alpha,\beta\pi}$, which is 1 if and only if $\alpha(v) \neq \beta(\pi(v))$ for all vertices v. By the definition of $T\pi$, this is the same as $(T_\pi)_{\alpha,\beta}$.

Similarly

$$(X_\pi T)_{\alpha\beta} \;=\; \sum_\gamma (X_\pi)_{\alpha\gamma} T_{\gamma\beta} \;=\; T_{\alpha\pi^{-1},\beta} = (T_\pi)_{\alpha,\beta}.$$

□

This result enables us to make a substantial simplification in the general case of an arbitrarily twisted bracelet. Intuitively, the reason for this is that if we remove one set of links the bracelet becomes an open chain, and all the remaining twists can be untwisted. We can now close the bracelet again, so that all the links correspond to identity permutations except for the last one.

The formal justification for this argument is contained in the following lemma, which establishes the connection between the compatibility matrix and the chromatic polynomial. It is a generalisation of an old result [1,2].

Lemma 2 Let $\mathcal{C}_{n,b}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ be the bracelet formed from n copies of K_b, as in Section 2. The chromatic polynomial of this graph is

$$\mathrm{trace}\big(T^n X_\pi\big),$$

where π is the composite permutation $\sigma_1\sigma_2 \ldots \sigma_n$.

Proof Let

$$X_i = X_{\sigma_i}, \quad T_i = T_{\sigma_i} = TX_{\sigma_i}, \quad (i = 1, 2, \ldots, n).$$

Consider the effect of assigning the k-colourings $\alpha, \beta, \ldots, \nu$ to the n copies of K_b, in the order given. The product

$$(T_1)_{\alpha\beta}\,(T_2)_{\beta\gamma}\,\cdots\,(T_n)_{\nu\alpha}$$

is 1 if and only if we obtain a proper colouring of the entire bracelet. Thus the total number of proper colourings is equal to

$$
\begin{aligned}
\sum_{\alpha,\beta,\ldots,\nu} & (T_1)_{\alpha\beta}\,(T_2)_{\beta\gamma}\,\cdots\,(T_n)_{\nu\alpha} \\
&= \sum_{\alpha}(T_1T_2\cdots T_n)_{\alpha\alpha} \\
&= \sum_{\alpha}(TX_1TX_2\cdots TX_n)_{\alpha\alpha} \\
&= \sum_{\alpha}(T^nX_1X_2\cdots X_n)_{\alpha\alpha} \\
&= \sum_{\alpha}(T^nX_\pi)_{\alpha\alpha} \\
&= \text{trace}\,\big(T^nX_\pi\big).
\end{aligned}
$$

□

We shall refer to $\pi = \sigma_1\sigma_2\cdots\sigma_n$ as the *total twist* of the bracelet, and use the abbreviation $\mathcal{C}_{n,b}(\pi)$ for any bracelet having the given total twist π. We may think of $\mathcal{C}_{n,b}(\pi)$ as a bracelet in which all the links except one are untwisted, and that one has the twist π.

In the totally untwisted case (when π is the identity permutation ι), $T^nX_\iota = T^n$, and the eigenvalues of T^n are the nth powers of the eigenvalues of T. The trace is the sum of the eigenvalues, counted according to their multiplicities, so if the eigenvalues of T are $\lambda_1(k), \lambda_2(k), \ldots, \lambda_s(k)$, with multiplicities $m_1(k), m_2(k), \ldots, m_s(k)$ respectively, we have

$$P(\mathcal{C}_{n,b}(\iota); k) \;=\; \sum_{r=1}^{s} m_r(k)\lambda_r(k)^n.$$

As a consequence of this formula, the calculation of $P(\mathcal{C}_{n,b}(\iota); k)$ is reduced to the calculation of the eigenspaces of T. This will be discussed in Sections 4, 5 and 6. In Sections 7 and 8 we shall explain how the results can be extended to the twisted case.

4. The untwisted case

We shall obtain a formula for the action of T (in the untwisted case) that can be used to determine its eigenvalues and eigenfunctions. The formula is essentially a version of the sieve principle. Some preparatory machinery is needed.

Let X be a non-empty subset of V and let ξ be an injection from X to $\{1, 2, \ldots, k\}$. For any $\gamma \in P_k$, denote by γ_X the restriction of γ to X. We define $[X|\xi]$ to be the element of $\mathcal{V}_k$ given by

$$[X|\xi] = \sum_{\gamma_X=\xi} [\gamma].$$

In other words, it is the function that takes the value 1 on the colourings that agree with ξ on X, and 0 otherwise.

To cover the case $X = \emptyset$ we define $[-|-] = \sum_\gamma [\gamma]$, where the sum is taken over all $\gamma \in P_k$. So $[-|-]$ is the function which takes the constant value 1 on every element of P_k. We shall also denote $[-|-]$ by the symbol u.

Theorem 1 Suppose that b and k are given, and let T be the associated compatibility operator, with respect to the identity permutation. Then for any given $\alpha \in P_k$

$$T[\alpha] \;=\; \sum_{X \subseteq V} (-1)^{|X|}\,[X|\alpha_X].$$

Proof Denote the expression on the right-hand side by Σ. We have to show that $\Sigma(\beta)$ is 1 when β is such that (α,β) is a compatible pair, and 0 otherwise.

By definition,

$$[X|\alpha_X](\beta) = 1 \quad\iff\quad \alpha(x) = \beta(x) \quad \text{for all } x \in X.$$

Considering α as given, let

$$U_\beta \;=\; \{v \in V \mid \alpha(v) = \beta(v)\}.$$

Then $[X|(\alpha_X](\beta) = 1$ if and only if X is a subset of U_β. Thus

$$\Sigma(\beta) = \sum_{X \subseteq V} (-1)^{|X|}[X|(\alpha_X](\beta) = \sum_{X \subseteq U_\beta} (-1)^{|X|}.$$

If (α,β) is a compatible pair, U_β is empty, and the sum has one term, equal to 1. On the other hand, if (α,β) is not compatible, then U_β is not empty and the sum is $(1+(-1))^{|U_\beta|}$, which is 0, as claimed. □

In order to apply Theorem 1 it is convenient to use a more explicit notation for the functions $[X|\xi]$. We shall write $[v|h]$ for the function which takes the value 1 on the colourings that assign the colour h to vertex v, and the value 0 otherwise; that is

$$[v|h] \;=\; \sum [\gamma],$$

where the sum is taken over all basis elements $[\gamma]$ with $\gamma(v) = h$. Similarly, we write $[v,w|h,i]$ for the sum of all $[\gamma]$ with $\gamma(v) = h$ and $\gamma(w) = i$; and so on. If α is the colouring defined by $\alpha(1) = h, \alpha(2) = i, \ldots, \alpha(b) = l$ then, in this notation, $[\alpha] = [1,2,\ldots,b|h,i,\ldots,l]$. Recall also that $u = [-|-]$ is the sum of all $[\gamma]$ without restriction, that is, the function which takes the value 1 on all colourings.

We can illustrate these definitions in the case of the graph $\mathcal{C}_{n,3}(\iota)$, the *toroidal square lattice*, with 3 squares around one circumference of the torus and n squares around the other. In this case, according to Theorem 1, the action of T on a standard basis element $[\alpha] = [1,2,3|h,i,j]$ is given by

$$T[1,2,3|h,i,j] = u - \Big([1|h]+[2|i]+[3|j]\Big) + \Big([1,2|h,i]+[2,3|i,j]+[1,3|h,j]\Big) - [1,2,3|h,i,j].$$

The complete chromatic polynomial of $\mathcal{C}_{n,3}(\iota)$ can be derived from this formula [3,5]. The resulting polynomial has the form

$$\begin{aligned} P(\mathcal{C}_{n,3}(\iota); z) \ &= \ (z^3 - 6z^2 + 14z - 13)^n \\ &\quad + (z-1)\Big((-z^2 + 7z - 13)^n + 2(-z^2 + 4z - 4)^n\Big) \\ &\quad + r_n(z), \end{aligned}$$

where $r_n(z)$ is a polynomial of degree $n+2$. Thus the explicit terms give the correct coefficients for all degrees from $3n$ down to $n+3$, and a great deal of useful information can be deduced from these terms alone. In the following sections we shall obtain the analogous result for the general case $\mathcal{C}_{n,b}(\iota)$, $b > 3$.

5. Application of the sieve formula

We begin by using the sieve formula to calculate the action of T on functions like $[v|h]$. As we shall see, it follows from these calculations that certain subspaces of $\mathcal{V}_k$ are invariant under T.

Using Theorem 1 and the definition of $[v|h]$ we have

$$\begin{aligned} T[v|h] \ &= \ T\left(\sum_{\alpha(v)=h} [\alpha]\right) \ = \ \sum_{\alpha(v)=h} T[\alpha] \\ &= \ \sum_{\alpha(v)=h} \sum_{X\subseteq V} (-1)^{|X|}\, [X|\alpha_X] \\ &= \ \sum_{X\subseteq V} (-1)^{|X|}\, \sigma(X,v,h), \end{aligned}$$

where

$$\sigma(X,v,h) = \sum_{\alpha(v)=h} [X|\alpha_X].$$

In Lemma 3 we present a formula for $\sigma(X,v,h)$. This involves the 'falling factorial' function, defined for any complex number z and non-negative integer p as follows:

$$(z)_0 = 1, \qquad (z)_p = z(z-1)\cdots(z-p+1) \quad (p \geq 1).$$

Lemma 3 If $|X| = r$ then

$$\sigma(X,v,h) = \begin{cases} (k-r)_{b-r}\,[v|h] & \text{if } v \in X; \\ (k-r-1)_{b-r-1}\Big(u - \sum_{x\in X}[x|h]\Big) & \text{if } v \notin X. \end{cases}$$

Proof Without loss of generality we can take $v = b$.

We have to calculate a sum taken over all colourings α such that $\alpha(b) = h$. Writing $\alpha(v) = \alpha_v$, this means that $\alpha_1, \alpha_2, \ldots, \alpha_{b-1}$ are distinct colours, different from $h = \alpha_b$. So the sum can be written as follows,

$$\sum_{\alpha(b)=h} [X|\alpha_X] = \sum_{(\alpha_1,\ldots,\alpha_{b-1})\in N(h)} [X|\alpha_X],$$

where $N(h)$ is defined to be the set of $(b-1)$-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_{b-1})$, the elements being distinct and not equal to h.

Let $X = \{x_1, x_2, \ldots, x_r\}$, so that the required sum is

$$\sum_{(\alpha_1,\ldots,\alpha_{b-1})\in N(h)} [x_1, x_2, \ldots, x_r|\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_r)].$$

Suppose that $b \in X$, say $b = x_r$. In this case the sum can be written as

$$\sum_{(\alpha_1,\ldots,\alpha_{b-1})\in N(h)} [x_1, x_2, \ldots, b|\alpha(x_1), \alpha(x_2), \ldots, h].$$

Summing over the $r-1$ colours $\alpha(x_t)$ $(1 \le t \le r-1)$, gives the result $[b|h]$. If $r = b$ these $r-1$ colours are just a permutation of $\alpha_1, \ldots, \alpha_{b-1}$, and the summation is complete. If $r < b$ there are $(k-r)\cdots(k-b+1)$ choices for the remaining colours, and the result is

$$(k-r)\cdots(k-b+1)\ [b|h] = (k-r)_{b-r}[b|h].$$

Suppose, on the other hand, that $b \notin X$. In this case none of the colours $\alpha(x_t)$ $(1 \le t \le r)$ can be equal to h, since $\alpha(b) = h$ and $x_t \neq b$. Summing over all choices of the r colours $\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_r)$ accounts for all colourings except those in which one of $x_1, x_2, \ldots, x_r$ is assigned the colour h. Hence this partial summation produces the result

$$u - [x_1|h] - [x_2|h] - \cdots - [x_r|h].$$

If $r = b-1$ the colours $\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_r)$ are just $\alpha_1, \alpha_2, \ldots, \alpha_{b-1}$, and the summation is complete, giving the result displayed above. If $r < b-1$, the number of ways of choosing the remaining colours is $(k-r-1)(k-r-2)\cdots(k-b+1)$, and the result is

$$(k-r-1)(k-r-2)\cdots(k-b+1)\Big(u - [x_1|h] - [x_2|h] - \cdots - [x_r|h]\Big)$$
$$= (k-r-1)_{k-b-1}\Big(u - [x_1|h] - [x_2|h] - \cdots - [x_r|h]\Big).$$

□

Let $c_i(z)$ $(i \ge 0)$ be the sequence of polynomial functions defined by the recursion

$$c_0(z) = 1, \quad c_i(z) = (z-i)c_{i-1}(z) - c_{i-1}(z-1) \quad (i \ge 1).$$

The first few polynomials $c_i(z)$ are:

$$c_0(z) = 1, \quad c_1(z) = z - 2, \quad c_2(z) = z^2 - 5z + 7, \quad c_3(z) = z^3 - 9z^2 + 29z - 34.$$

The following lemma is elementary.

Lemma 4 For all $i \geq 1$ the polynomial $c_i(z)$ is given explicitly by the formula

$$c_i(z) \;=\; \sum_{r=0}^{i} (-1)^r \binom{i}{r} (z - r - 1)_{i-r}.$$

These polynomials also satisfy the relation

$$c_i(z) = c_i(z-1) + i c_{i-1}(z-1).$$

□

Theorem 2 Let the integers $k \geq b \geq 2$ be given, and let $c_{b-1}(k)$, $c_{b-2}(k-1)$ be the numbers determined as in Lemma 4. Then

$$T[v|h] = c_{b-1}(k)\Big(u - [v|h]\Big) + c_{b-2}(k-1)\Big(\sum_{w \neq v} [w|h]\Big).$$

Proof We have

$$T[v|h] = \sum_{X \subseteq V} (-1)^{|X|}\, \sigma(X, v, h).$$

Explicit expressions for $\sigma(X, v, h)$ were obtained in Lemma 3, and it remains only to collect up the coefficients of the various terms.

The function u occurs only in the occurs in the terms $\sigma(X, v, h)$ for which X does not contain v, and for each such X of size r the coefficient is $(k - r - 1)_{b-r-1}$. There are $\binom{b-1}{r}$ such subsets, so the coefficient of u is

$$\sum_{r=0}^{b-1} (-1)^r \binom{b-1}{r} (k - r - 1)_{b-r-1}.$$

This is the expression for $c_{b-1}(k)$ given in the lemma.

Similarly, $[v|h]$ occurs only in the terms $\sigma(X, v, h)$ for which X contains v, and for each such X of size r the coefficient is $(k-r)_{b-r}$. There are $\binom{b-1}{r-1}$ such subsets, so the coefficient of $[v|h]$ is

$$\sum_{r=1}^{b} (-1)^r \binom{b-1}{r-1} (k - r)_{b-r}.$$

Writing this as a sum from 0 to $b-1$, we get $-c_{b-1}(k)$, as claimed.

Finally, if $w \neq v$ the function $[w|h]$ occurs in the terms in the terms $\sigma(X, v, h)$ for which X contains w but not v, and for each such X of size r the coefficient is $(k-r-1)_{b-r-1}$. There are $\binom{b-2}{r-1}$ such subsets, so the coefficient of $[w|h]$ is

$$\sum_{r=1}^{b-1}(-1)^r\binom{b-2}{r-1}(k-r-1)_{b-r-1}.$$

Writing this as a sum from 0 to $b-1$, we get $c_{b-2}(k-1)$. $\square$

6. The principal representation

The main consequence of Theorem 2 is that, for each $v \in V$, the expression for $T[v|h]$ is a linear combination of the $b+1$ functions u, $[1|h]$, $[2|h]$, $\ldots$, $[b|h]$. In other words, for each colour h, the subspace $\mathcal{W}_k(h)$ of $\mathcal{V}_k$ spanned by u, $[1|h]$, $[2|h]$, $\ldots$, $[b|h]$ is invariant under T. We observe that if $k < b$ the result is vacuous (since the set of k-colourings of K_b is empty). If $k = b$ the spanning set is linearly dependent (u is the sum of the other functions, since every b-colouring of K_b assigns colour h to exactly one vertex). However, if $k > b$ there are k-colourings of K_b that do not use the colour h, so u is linearly independent of the other functions, and the spanning set is a basis for $\mathcal{W}_k(h)$. We shall assume henceforth that this condition holds.

We shall refer to the action of T on a subspace $\mathcal{W}_k(h)$ as a *principal representation* of T.

As yet, we have no explicit formula for $T(u)$. This is easily deduced from the formula for $T[v|h]$, because every coluring assigns exactly one colour h to the vertex v, and so

$$u = \sum_{h=1}^{k}[v|h].$$

It follows that

$$\begin{aligned} T(u) &= \sum_{h=1}^{k} T[v|h] \\ &= \sum_{h=1}^{k} c_{b-1}(k)u - c_{b-1}(k)[v|h] + c_{b-2}(k-1)\big(\sum_{w\neq v}[w|h]\big) \\ &= kc_{b-1}(k)u - c_{b-1}(k)u + (b-1)c_{b-2}(k-1)u \\ &= \big((k-1)c_{b-1}(k) + (b-1)c_{b-2}(k-1)\big)u. \end{aligned}$$

Thus u is an eigenfunction with eigenvalue

$$\begin{aligned} \theta_b(k) &= (k-1)c_{b-1}(k) + (b-1)c_{b-2}(k-1) \\ &= kc_{b-1}(k) - c_{b-1}(k-1). \end{aligned}$$

The second expression is obtained using the formula given in Lemma 4.

An alternative definition of $\theta_b(k)$ is that it is the number of 1's in each row of the matrix T; in other words, $\theta_b(k)$ represents the number of k-colourings β of K_b such that (α,β) is a compatible pair, for any given α. Standard arguments lead to the conclusion that the eigenvalue $\theta_b(k)$ has multiplicity 1.

According to this result and the formula given in Theorem 2, the action of T on $\mathcal{W}_k(h)$ is represented by a matrix of the form

$$\begin{pmatrix} \theta_b(k) & 0 & 0 & . & . & . & 0 \\ x & -x & y & . & . & . & y \\ x & y & -x & . & . & . & y \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ x & y & y & . & . & . & -x \end{pmatrix},$$

where $x = c_{b-1}(k)$ and $y = c_{b-2}(k-1)$. The eigenvalues of this matrix are also eigenvalues of T. Clearly, they are $\theta_b(k)$ and the eigenvalues of the $b \times b$ matrix M in the lower right-hand corner. In terms of the identity matrix I and the all-1 matrix J we have

$$M = y(J - I) - xI.$$

The eigenvalues of a matrix of this form are well known: there are just two distinct ones, and they are:

$$\begin{aligned} \phi_b(k) = (b-1)y - x \ &= (b-1)c_{b-2}(k-1) - c_{b-1}(k) \quad \text{(multiplicity 1);} \\ \psi_b(k) = -y - x \ &= -c_{b-2}(k-1) - c_{b-1}(k) \quad \text{(multiplicity } b-1\text{).} \end{aligned}$$

Alternative expressions for these eigenvalues can be obtained using the formula given in Lemma 4:

$$\phi_b(k) = -c_{b-1}(k-1), \quad \psi_b(k) = -c_{b-1}(k-1) - bc_{b-2}(k-1).$$

In the following sections we need to know the eigenfunctions as well, and these can be obtained directly from the formula in Theorem 2. First, let

$$f^{\phi}(h) = [1|h] + [2|h] + \cdots + [b|h] - (b/k)u.$$

It is easy to check that $Tf^{\phi}(h) = \phi_b(k)f^{\phi}(h)$, so that $f^{\phi}(h)$ spans a 1-dimensional space of eigenfunctions for $\phi_b(k)$.

Next, for any vertices $v \neq w$ let

$$f^{\psi}_{vw}(h) = [v|h] - [w|h].$$

Again, it is easy to check that $Tf^{\psi}_{vw}(h) = \psi_b(k)f^{\psi}_{vw}(h)$. So we have $b(b-1)/2$ eigenfunctions, one for each edge vw of K_b. These eigenfunctions span a space of dimension $b-1$, a basis for which is obtained by taking the functions corresponding to the edges of any spanning tree of K_b. For example, we may fix $w = b$, and take $v = 1, 2, \ldots, b-1$.

Up to this point we have been considering the action of T on the subspace $\mathcal{W}_k(h)$, for a given colour h. In fact, for each one of the k colours there is an eigenfunction $f^\phi(h)$, and the only linear relationship between them is that their sum is identically zero. Thus in the action of T on the whole space $\mathcal{V}_k$, the multiplicity of the eigenvalue $\phi_b(k)$ is $(k-1)$. Similarly, the multiplicity of $\psi_b(k)$ is $(b-1)(k-1)$.

In summary, we have obtained the following eigenvalues of T, and their associated multiplicities.

Eigenvalue	Eigenfunctions	Multiplicity
$\theta_b(k) = kc_{b-1}(k) - c_{b-1}(k-1)$	u	1
$\phi_b(k) = -c_{b-1}(k-1)$	$[1\|h] + \ldots + [b\|h] - (b/k)u$	$k-1$
$\psi_b(k) = -bc_{b-2}(k-1) - c_{b-1}(k-1)$	$[v\|h] - [b\|h]$	$(b-1)(k-1)$.

These results determine the 'leading terms' of the chromatic polynomial $\mathcal{C}_{n,b}(\iota)$. We have

$$P(\mathcal{C}_{n,b}(\iota); k) \;=\; \theta_b(k)^n + (k-1)\Big(\phi_b(k)^n + (b-1)\psi_b(k)^n\Big) + \ldots\;,$$

where $\theta_b(k)$, $\phi_b(k)$, and $\psi_b(k)$ can all be expressed in terms of the functions c_{b-1} and c_{b-2}.

7. Twisted bracelets

In the foregoing sections we have been considering the case when the graphs K_b that form the bracelet are linked by edges that correspond to the identity permutation of the vertices of K_b. Now we shall consider the 'twisted' case, using the formula obtained in Lemma 2:

$$P(\mathcal{C}_{n,b}(\pi); k) = \operatorname{trace}\big(T^n X_\pi\big).$$

The relationship between the eigenvalues of T^n and $T^n X_\pi$ is as follows. Suppose we can construct an eigenfunction f for T that is also an eigenfunction for X_π:

$$Tf = \lambda f, \quad X_\pi f = \mu f.$$

Then we have

$$T^n X_\pi f = T^n \mu f = \lambda^n \mu f,$$

so that f is an eigenfunction for $T^n X_\pi$ with eigenvalue $\lambda^n \mu$.

Clearly, the eigenfunction u of T is also an eigenfunction for any X_π, with $\mu = 1$. Similarly, $f^\phi(h) = [1|h]+[2|h]+\cdots+[b|h]-(b/k)u$ is an eigenfunction for any X_π, also with $\mu = 1$. So the corresponding eigenvalues $\theta_b(k)$ and $\phi_b(k)$ are unchanged by the introduction of twists. It follows that, in the chromatic polynomial the dominant term and the first subdominant term are unchanged: for any twist π we have

$$P(\mathcal{C}_{n,b}(\pi); k) \;=\; \theta_b(k)^n + (k-1)\phi_b(k)^n + \ldots\;.$$

We now consider the action of X_π on the $(b-1)$-dimensional space spanned by the eigenfunctions $f^\psi_{vw}(h)$, which we denote by $\Psi(h)$. It will be convenient to use the abbreviation

$$[vw] = f^\psi_{vw}(h) = [v|h] - [w|h],$$

and for definiteness we choose the basis for $\Psi(h)$ corresponding to the spanning tree consisting of all edges incident with the vertex b, that is

$$[vb], \quad v = 1, 2, \ldots, b-1.$$

For any vertices x, y we can express $[xy]$ in terms of this basis very simply. If either x or y is b the expression is obvious, otherwise $[xy] = [xb] - [yb]$.

The following simple examples illustrate the action of X_π on $\Psi(h)$, and the resulting effect on the chromatic polynomial. A general result will be given in Section 8.

Examples When $b = 3$ the chosen basis of $\Psi(h)$ consists of the two functions [13] and [23]. There are essentially only two nontrivial twists to be considered, corresponding to the permutations

$$\sigma = (12), \quad \tau = (123).$$

For the action of X_σ on the basis of $\Psi(h)$ we have

$$X_\sigma[13] = [23], \quad X_\sigma[23] = [13],$$

so the action is represented by the matrix

$$Y_\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of Y_σ are 1 and -1. The corresponding eigenfunctions are $[13] + [23]$ and $[13] - [23]$, and (trivially) they belong to the space $\Psi(h)$. It follows that the eigenvalue $\psi_b(k)^n$ of T^n gives rises to two eigenvalues of $T^n X_\sigma$: $\psi_b(k)^n$ and $-\psi_b(k)^n$.

Consequently the coefficient of the term $\psi_b(k)^n$ in the chromatic polynomial is $(1 + (-1))(k-1)$, that is, 0. The twist σ corresponds to constructing a Klein bottle instead of a torus (the untwisted case), and the result agrees with that previously obtained by direct calculation [5].

For τ we have

$$\tau[13] = [21] = [23] - [13], \quad \tau[23] = [31] = -[13],$$

and the representative matrix is

$$Y_\tau = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are ω and ω^2, the complex cube roots of unity. Applying the argument used above we deduce that the coefficient of $\psi_b(k)^n$ in this case is $(\omega+\omega^2)(k-1)$, that is, $-(k-1)$.

8. The general case

The examples discussed in the previous section are special cases of the following lemma.

Lemma 5 In the chromatic polynomial of $\mathcal{C}_{n,b}(\pi)$ the coefficient of $\psi_b(k)^n$ is

$$(\nu_\pi - 1)(k-1),$$

where ν_π is the number of fixed points of π.

Proof Let $\mu_1, \mu_2, \ldots, \mu_{b-1}$ be the eigenvalues (not necessarily distinct) of the $(b-1)\times(b-1)$ matrix Y_π, representing the action of X_π on the space $\Psi(h)$, with respect to the chosen basis. This action contributes a term

$$(\mu_1 + \mu_2 + \cdots + \mu_{b-1})\psi_b(k) = (\operatorname{trace} Y_\pi)\psi_b(k)$$

to the chromatic polynomial.

If π has at least one fixed point, we may assume that b is a fixed point. Then the action of π on the $b-1$ functions $[vb]$, $(v \neq b)$ is the same as the action of π on the vertices v, $(v \neq b)$. Thus the trace of Y_π is the number of those vertices that are fixed by π, which is one less than the total number of fixed vertices.

On the other hand, if π has no fixed points, we may suppose that $\pi(b)$ is the vertex labelled 1. Then

$$Y_\pi[1b] = [x1] = \begin{cases} [xb]-[1b] & \text{if } \pi(1) = x \neq b; \\ -[1b] & \text{if } \pi(1) = b. \end{cases}$$

So the matrix Y_π has a term -1 in the diagonal position corresponding to $[1b]$. But if $v \neq 1, b$ we have

$$Y_\pi[vb] = [y1] = \begin{cases} [yb]-[1b] & \text{if } \pi(v) = y \neq b; \\ -[1b] & \text{if } \pi(v) = b. \end{cases}$$

It follows that the matrix Y_π has a term 0 in the diagonal position corresponding to $[vb]$. Thus the only non-zero term on the diagonal of Y_π is -1, and the trace is -1. Since the number of fixed points is zero, the result holds in this case too.

Thus from each $\Psi(h)$ there is a contribution of $\nu_\pi - 1$ to the coefficient. As in the untwisted case, the subspaces $\Psi(h)$, $h = 1, 2, \ldots, k$ contribute $k-1$ linearly independent eigenfunctions for each eigenvalue, and so the total coefficient is $(\nu_\pi - 1)(k-1)$.

We have obtained our final result:

$$P(\mathcal{C}_{n,b}(\pi); k) = \theta_b(k)^n + (k-1)\Big(\phi_b(k)^n + (\nu_\pi - 1)\psi_b(k)^n\Big) + \ldots\ ,$$

where $\theta_b(k)$, $\phi_b(k)$ and $\psi_b(k)$ can all be expressed in terms of the functions c_{b-1} and c_{b-2}.

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