Colouring the Square of a Planar Graph

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Abstract

We prove that for any planar graph G with maximum degree Δ it holds that the chromatic number of the square of G satisfies $\chi(G^2) \leq 2\,\Delta + 25$. We generalise this result to integer labellings of planar graphs involving constraints on distances one and two in the graph.

Keywords: planar graph, chromatic number, labelling of a graph.

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1 Introduction

Throughout this paper, $V = V(G)$ and $E = E(G)$ shall denote the set of vertices and the set of edges, respectively, of a graph G. For vertices u and v in G, we let $dist_G(u,v)$ denote the *distance* between u and v, which is the length of the shortest path joining them. For integers $p, q \geq 0$, a labelling of a graph $\varphi :\longrightarrow \{0, 1, \ldots, n-1\}$, for a certain $n \geq 1$, is called an $L(p,q)$*-labelling* if it satisfies:

$$\begin{aligned} |\varphi(u) - \varphi(v)| \ &\geq \ p, \qquad \text{if } dist_G(u,v) = 1; \\ |\varphi(u) - \varphi(v)| \ &\geq \ q, \qquad \text{if } dist_G(u,v) = 2. \end{aligned}$$

The p,q*-span* of a graph G, denoted $\lambda(G; p, q)$, is the minimum n for which an $L(p,q)$-labelling exists. The problem of determining $\lambda(G; p, q)$ for certain graphs or classes of graphs (or at least finding good lower or upper bounds) has been studied before, see e.g. [3, 4, 5, 6, 10].

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The main inspiration for $L(p,q)$-labellings in those papers comes from problems related to the *Frequency Assignment Problem* on large-scale telecommunication networks.

Determining $\lambda(G;1,0)$ amounts to finding the chromatic number $\chi(G)$ and for planar graphs we have the famous 4-Colour Theorem.

1.1 Theorem (Appel & Haken [1], Appel *et al* [2], Robertson *et al* [9])
If G is a planar graph, then $\chi(G) \le 4$.

For general p, the above is easily seen to yield the following upper bound.

1.2 Corollary
If G is a planar graph, then $\lambda(G;p,0) \le 3\,p+1$.

Now we shall look at the case when $q \ge 1$. The problem of finding an $L(1,1)$-labelling amounts to finding a proper colouring of the *square of G*. The square of a graph G (denoted G^2) is defined such that $V(G^2) = V(G)$, and two vertices u and v are adjacent in G^2 if and only if $dist_G(u,v) \in \{1,2\}$. The question of finding the best possible upper bound for the chromatic number of the square of a planar graph seems to first have been put forward in Wegner [11] in 1977. Wegner conjectured the following.

1.3 Conjecture (Wegner [11])
Let G be a planar graph with maximum degree Δ, then

$$\chi(G^2) \;\le\; \begin{cases} \Delta+5, & \text{if } 4 \le \Delta \le 7; \\ \lfloor \frac{3}{2}\,\Delta \rfloor + 1, & \text{if } \Delta \ge 8. \end{cases}$$

Wegner also gave examples illustrating that these upper bounds are best possible and proved that the square of a planar graph with $\Delta = 3$ can be coloured with 8 colours. He conjectured that in fact 7 colours should suffice. More information and problems relating colouring and distances in graphs can be found in Jensen & Toft [8, Section 2.18].

As a special case of Theorem 1.5, to be formulated later, we obtain the following upper bound:

1.4 Theorem
If G is a planar graph with maximum degree Δ, then $\chi(G^2) \le 2\,\Delta + 25$.

A straightforward argument shows that if G is a graph with maximum degree Δ, then we must have $\lambda(G;p,q) \ge q\,\Delta + p - q + 1$. It is not too hard to construct planar graphs G with $\lambda(G;p,q) = \frac{3}{2}\,q\,\Delta + c_1(p,q)$ where $c_1(p,q)$ is a constant depending only on p and q. As far as upper bounds for $\lambda(G;p,q)$ are concerned, in Chang & Kuo [3] it is shown that $\lambda(G;2,1) \le 2\,\Delta^2 + \Delta$. This suggests that for graphs in general, the best possible upper bound for $\lambda(G;p,q)$ will be of the order $2\,q\,\Delta^2$. Our main result shows that for the case when G is planar, we can reduce the order of the upper bound.

1.5 Theorem
If G is a planar graph with maximum degree Δ and p, q are positive integers with $p \geq q$, then

$$\lambda(G; p, q) \ \leq\ (4\,q-2)\,\Delta + 10\,p + 38\,q - 23.$$

Theorem 1.4 follows immediately from Theorem 1.5 by setting $p = q = 1$ and using the observation $\lambda(G; 1, 1) = \chi(G^2)$.

The remainder of this paper will form the proof of Theorem 1.5. In the next section we will prove Lemmas 2.2 and 2.3 that describe certain "unavoidable configurations" in planar graphs. The existence of these unavoidable configurations will then be used to prove the main theorem in Section 3.

2 Discharging and unavoidable configurations

For problems involving colouring of planar graphs, the usual method of attack is to establish the existence of certain small, so-called, *unavoidable configurations.* For instance, it is well-known that a simple planar graph has a vertex of degree at most five. This was used in HEAWOOD [7] to prove the 5-Colour Theorem, namely that any planar graph has a proper colouring using at most 5 colours. A new proof of the 4-Colour Theorem recently found by ROBERTSON *et al* [9] establishes the existence of a large number of unavoidable configurations for planar triangulations satisfying a certain connectivity constraint. For our proof of Theorem 1.5 we need two structural results. These results are based on a certain method of *discharging*, similar to methods found in the literature.

Let G be a graph. For a vertex $v \in V$, we let $N_G(v)$ denote its neighbour set, use $d_G(v) = |N_G(v)|$ for its degree, and let E_v denote the set of edges incident to v (we omit the subscript G in most cases). For simple graphs, an edge e with end vertices u and v will often be denoted as uv.

Now let G be a simple planar graph with a fixed embedding in the plane. Let F denote the set of faces of G. For each $f \in F$ let $d(f)$ be the number of edges belonging to f, where cut-edges are counted twice. For an edge $e \in E$ let $t(e)$ denote the number of triangular faces containing e, and for a vertex $v \in V$ let $t(v)$ be the number of triangular faces containing v. Using Euler's formula, one easily obtains that (see, e.g., JENSEN & TOFT [8, Section 2.9])

$$\sum_{v\in V}(d(v)-4) \ +\ \sum_{f\in F}(d(f)-4) \ =\ -8. \tag{1}$$

We shall associate a *charge* $\varphi(v)$ to each vertex $v \in V$ where $\varphi(v) = d(v) - 4$. Similarly, we associate a charge $\varphi(f) = d(f) - 4$ to each face $f \in F$. According to (1), the sum of the charges taken over all vertices and faces will be negative. We shall transfer the charge of vertices and faces to the edges of the graph, in such a way that the total charge remains constant. There are three steps to perform:

(2.1) For an edge $e = uv \in E$, we give e a *basic charge* $\varphi_b(e)$ where, given e belongs to faces $f, g \in F$, we set

$$\varphi_b(e) \;=\; \frac{\varphi(u)}{d(u)} + \frac{\varphi(v)}{d(v)} + \frac{\varphi(f)}{d(f)} + \frac{\varphi(g)}{d(g)}. \tag{2}$$

If e belongs to only one face $f \in F$, then we give e a basic charge as in the above taking $g = f$.

(2.2) For each triangular face with vertices u, v, w, where $3 \le d(u) \le 5$, $d(v) \ge 6$, and $d(w) \ge 6$, do the following:

Transfer a charge of $\frac{1}{2}\left(\frac{\varphi(v)}{d(v)} - \frac{1}{3}\right)$ from vw to uw.

Transfer a charge of $\frac{1}{2}\left(\frac{\varphi(w)}{d(w)} - \frac{1}{3}\right)$ from vw to uv.

(2.3) For each triple u, v, v' in V, with $uv, uv' \in E_u$, $d(u) = 5$, $d(v) \ge 6$, $d(v') \ge 6$, $t(uv) = 2$, and $t(uv') = 0$ transfer a charge of $\frac{1}{6}$ from uv' to uv.

After doing all possible charge transfers once, let $\varphi(e)$ be the resulting charge on each edge $e \in E$. Since the total charge on the edges is seen to be equal to the total charge on the vertices and faces, we have from (2) that

$$\sum_{v \in V} \sum_{e \in E_v} \varphi(e) \;=\; \sum_{e \in E} 2\,\varphi(e) \;=\; \sum_{e \in E} 2\,\varphi_b(e) \;=\; -16. \tag{3}$$

The following properties, whose proofs follow by following the two discharging methods given above, are used at numerous places in the sequel, although usually implicitly.

2.1 Proposition

Let G be a simple planar graph with a fixed embedding and let $e = uv$ be an edge in G.

(i) *If $\varphi_b(e) < 0$, then $d(u) \le 5$ or $d(v) \le 5$, and $\varphi(e) \ge \varphi_b(e)$.*

(ii) *If $\varphi_b(e) \ge 0$, then $\varphi(e) \ge 0$.*

Let v be a vertex and vu an edge in a simple planar graph with a fixed embedding. If the edge vw is an edge which directly precedes vv_i, counting the edges of E_v moving clockwise around v, then we shall denote w by u^-. If vw directly succeeds vu, then we denote w by u^+.

We need two structural lemmas which give specific unavoidable configurations for planar graphs. The first lemma is sufficient to prove the main theorem for graphs with maximum degree $\Delta \ge 12$.

2.2 Lemma

Let G be a simple planar graph. Then there exists a vertex v with k neighbours $v_1, v_2, \ldots, v_k$ with $d(v_1) \le \cdots \le d(v_k)$ such that one of the following is true:

(i) $k \le 2$;

(ii) $\quad k = 3$ *with* $d(v_1) \leq 11$*;*

(iii) $\quad k = 4$ *with* $d(v_1) \leq 7$ *and* $d(v_2) \leq 11$*;*

(iv) $\quad k = 5$ *with* $d(v_1) \leq 6$*,* $d(v_2) \leq 7$*, and* $d(v_3) \leq 11$*.*

To be able to prove the main result for graphs with maximum degree less than twelve, we need a second lemma, giving a different collection of unavoidable configurations.

2.3 Lemma
Let G be a simple planar graph with maximum degree Δ. Then there exists a vertex v with k neighbours $v_1, v_2, \ldots, v_k$ with $d(v_1) \leq \cdots \leq d(v_k)$ such that one of the following is true:

(i) $\quad k \leq 2$*;*

(ii) $\quad k = 3$ *with* $d(v_1) \leq 5$*;*

(iii) $\quad k = 3$ *with* $t(vv_i) \geq 1$ *for some* i*;*

(iv) $\quad k = 4$ *with* $d(v_1) \leq 4$*;*

(v) $\quad k = 4$ *with* $t(vv_i) = 2$ *for some* i*;*

(vi) $\quad k = 5$ *with* $d(v_i) \leq 4$ *and* $t(vv_i) \geq 1$ *for some* i*;*

(vii) $\quad k = 5$ *with* $d(v_i) \leq 5$ *and* $t(vv_i) = 2$ *for some* i*;*

(viii) $\quad k = 5$ *with* $d(v_1) \leq 7$ *and* $t(vv_i) \geq 1$ *for all* i*;*

(ix) $\quad k = 5$ *with* $d(v_1) \leq 5$*,* $d(v_2) \leq 7$*, and for each* i *with* $t(vv_i) = 0$ *it holds that* $d(v_i) \leq 5$*.*

First common steps in the proofs of Lemmas 2.2 and 2.3 Both lemmas are proved by contradiction. So let G be a simple, planar graph with a fixed embedding in the plane, and suppose that G is a counterexample to one of the lemmas. According to equation (3) there is a vertex $v \in V$ where $\sum_{e \in E_v} \varphi(e) < 0$. Suppose w is such a vertex and suppose w has m neighbours $w_1, \ldots, w_m$ where $d(w_1) \leq \cdots \leq d(w_k)$. Since (i) does not hold, we know $m \geq 3$.

For $j = 1, 2, \ldots, m$, let T_j be the set of edges between vertices in $\{v_j, \ldots, v_m\}$ belonging to a face containing v, and set $t_j = |T_j|$.

Proof of Lemma 2.2 In this case, we may assume that G is a 2-connected triangulation, for otherwise, we could add edges to G obtaining a triangulation G'. If none of (i)–(iv) holds for G, then clearly none of (i)–(iv) holds for G'.

Claim 1 $m \neq 3$.

Proof Suppose $m = 3$. Because (ii) does not hold, $d(w_j) \geq 12$ for all j, hence $\varphi_b(ww_j) \geq -\frac{1}{3} + \frac{2}{3} - 2 \cdot \frac{1}{3} = -\frac{1}{3}$. According to the procedure for transferring charge, for each j a charge of at least $\frac{1}{6}$ units will be transfered from both $w_j w_j^-$ and $w_j w_j^+$ to ww_j. This gives $\varphi(ww_j) \geq \varphi_b(ww_j) + 2 \cdot \frac{1}{6} \geq 0$, and thus $\sum_{e \in E_w} \varphi(e) \geq 0$, contradicting the choice of w. $\square$

Claim 2 $m \neq 4$.

Proof Suppose $m = 4$. Suppose first that $d(w_j) \geq 8$ for all j. Then $\varphi_b(ww_j) \geq 0+\frac{1}{2}-2\cdot\frac{1}{3} = -\frac{1}{6}$. According to the procedure for transferring charge, for each j a charge of at least $\frac{1}{12}$ units will be transfered from both $w_jw_j^-$ and $w_jw_j^+$ to ww_j. This gives $\varphi(ww_j) \geq \varphi_b(ww_j)+2\cdot\frac{1}{12} = 0$ for all j, and thus $\sum_{e\in E_w} \varphi(e) \geq 0$, contradicting the choice of w.

We conclude $d(w_1) \leq 7$. Since G does not satisfy condition (ii) in the lemma (with $v = w_1$), we know $d(w_1) \geq 4$ and hence $\varphi(ww_1) \geq \varphi_b(ww_1) \geq 0 + 0 - 2 \cdot \frac{1}{3} = -\frac{2}{3}$. It also follows that $d(w_j) \geq 12$ for all $j \geq 2$, hence $\varphi_b(ww_j) \geq 0 + \frac{2}{3} - 2 \cdot \frac{2}{3} = 0$ for all $j \geq 2$. According to the procedure for transferring charge, for each edge $w_jw_\ell \in T_2$ a charge of at least $\frac{1}{6}$ units will be transferred from w_jw_ℓ to both ww_j and ww_ℓ. Observing that $t_2 = 2$, we have

$$\sum_{e\in E_w} \varphi(e) \;\geq\; \varphi_b(ww_1) + \sum_{j\geq 2} \varphi_b(ww_j) + t_2 \cdot 2 \cdot \frac{1}{6} \;\geq\; 0,$$

again contradicting the choice of w. □

Claim 3 $m \neq 5$.

Proof Suppose $m = 5$. First suppose that $d(w_j) \geq 7$ for all j. Then $\varphi_b(ww_j) \geq \frac{1}{5}+\frac{3}{7}-2\cdot\frac{1}{3} = -\frac{4}{105}$. According to the procedure for transferring charge, for each j a charge of at least $\frac{1}{21}$ units is transferred from both $w_jw_j^-$ and $w_jw_j^+$ to ww_j. This gives $\varphi(ww_j) \geq \varphi_b(ww_j)+2\cdot\frac{1}{21} = \frac{2}{35} > 0$, and thus $\sum_{e\in E_w} \varphi(e) \geq 0$, contradicting the choice of w.

So we have that $d(w_1) \leq 6$. Again we know that $d_1(w) \geq 4$ and hence $\varphi_b(ww_1) \geq \frac{1}{5} + 0 - 2 \cdot \frac{1}{3} = -\frac{7}{15}$. If $d(w_j) \geq 8$ for all $j \geq 2$, then $\varphi_b(ww_j) \geq \frac{1}{5} + \frac{1}{2} - 2 \cdot \frac{1}{3} = \frac{1}{30}$. According to the procedure for transferring charge, for each edge $w_jw_\ell \in T_2$, we transfer a charge of at least $\frac{1}{12}$ units from w_jw_ℓ to both ww_j and ww_ℓ. Observing that $t_2 = 3$, we have

$$\sum_{e\in E_w} \varphi(e) \;\geq\; \varphi_b(ww_1) + \sum_{j\geq 2} \varphi_b(ww_j) + t_2 \cdot 2 \cdot \frac{1}{12} \;\geq\; -\frac{7}{15} + 4 \cdot \frac{1}{30} + \frac{1}{2} \;=\; \frac{1}{6} \;>\; 0,$$

contradicting the choice of w.

This means that we know $d(w_1) \leq 6$ and $d(w_2) \leq 7$, hence $d(w_j) \geq 12$ for all $j \geq 3$. Since certainly $d(w_1) \geq 4$ and $d(w_2) \geq 4$, we have $\varphi_b(ww_1) \geq -\frac{7}{15}$ and $\varphi_b(ww_2) \geq -\frac{7}{15}$. Also, $\varphi_b(ww_j) \geq \frac{1}{5} + \frac{2}{3} - 2 \cdot \frac{1}{3} = \frac{1}{5}$ for $j \geq 3$. According to the procedure for transferring basic charge, for each $w_jw_\ell \in T_3$, a charge of at least $\frac{1}{6}$ units will be transferred from w_jw_ℓ to both ww_j and ww_ℓ. Observing that $t_3 \geq 1$ we have

$$\begin{aligned}\sum_{e\in E_w} \varphi(e) \;&\geq\; \varphi_b(ww_1) + \varphi_b(ww_1) + \sum_{j\geq 3} \varphi_b(ww_j) + t_3 \cdot 2 \cdot \frac{1}{6}\\ &\geq\; -2 \cdot \frac{7}{15} + 3 \cdot \frac{1}{5} + \frac{1}{3} \;=\; 0,\end{aligned}$$

again contradicting the choice of w. □

We now know that $m \geq 6$. Since the vertex w is chosen such that $\sum_{e \in E_w} \varphi(e) < 0$, there must be an edge $e \in E_w$ such that $\varphi(e) < 0$. Let $ww_a \in E_w$ be such an edge. By Lemma 2.1 (ii) this must mean that $\varphi_b(ww_a) < 0$ also. Since $d(w) = m \geq 6$, by Lemma 2.1 (i) we have that $d(w_a) \leq 5$.

Claim 4 $m \neq 6, 7$.

Proof Suppose $m = 6$ or $m = 7$. We certainly can assume $d(w_a) \geq 4$, otherwise (i) or (ii) would hold. If $d(w_a) = 4$, then $d(w_a^-) \geq 12$ and $d(w_a^+) \geq 12$, otherwise (iii) holds with $v = w_a$. Then we have $\varphi_b(ww_a) \geq \frac{1}{3} + 0 - 2\frac{1}{3} = -\frac{1}{3}$. Also, according to the procedure for transferring charge, at least $\frac{1}{6}$ units are transferred from both $w_a w_a^-$ and $w_a w_a^+$ to ww_a. This means $\varphi(ww_a) \geq \varphi_b(ww_a) + 2 \cdot \frac{1}{6} \geq 0$, contradicting the choice of ww_a.

Now suppose $d(w_a) = 5$, and thus $\varphi_b(ww_a) \geq \dfrac{m-4}{m} + \dfrac{1}{5} - \dfrac{2}{3} = \dfrac{m-4}{m} - \dfrac{7}{15}$. Since G does not satisfy (iv) with $v = w_a$, we have that either $d(w_a^-) \geq 14 - m$ and $d(w_a^+) \geq 14 - m$, or $\max\{d(w_a^-), d(w_a^+)\} \geq 12$. In the former case we have that a charge of at least $\dfrac{1}{2}\left(\dfrac{10-m}{14-m} - \dfrac{1}{3}\right)$ is transfered from both $w_a w_a^-$ and $w_a w_a^+$ to ww_a. In the latter case a charge of at least $\frac{1}{6}$ is transfered from $w_a w_a^-$ or $w_a w_a^+$ to ww_a. So we obtain

$$\begin{aligned} \varphi(ww_a) &\geq \varphi_b(ww_a) + \min\{\, 2 \cdot \frac{1}{2}\left(\frac{10-m}{14-m} - \frac{1}{3}\right), \frac{1}{6}\,\} \\ &\geq \frac{m-4}{m} - \frac{7}{15} + \min\{\left(\frac{10-m}{14-m} - \frac{1}{3}\right), \frac{1}{6}\} \;\geq\; 0, \end{aligned}$$

again contradicting the choice of ww_a. □

Claim 5 $m \neq 8, 9, 10, 11$.

Proof Suppose $8 \leq m \leq 11$. We can assume $d(w_a) \geq 4$, since otherwise (ii) would hold with $v = w_a$. It suffices to show $\varphi(ww_a) \geq 0$ when $d(w_a) = 4$, as $\varphi_b(ww_a) \geq \frac{1}{2} + \frac{1}{5} - 2 \cdot \frac{1}{3} > 0$ if $d(w_a) \geq 5$. Suppose $d(w_a) = 4$. Then $d(w_a^-) \geq 8$ and $d(w_a^+) \geq 8$, and a charge of at least $\frac{1}{12}$ is transfered from both $w_a w_a^-$ and $w_a w_a^-$ to ww_a. Hence

$$\varphi(ww_a) \;\geq\; \varphi_b(ww_a) + 2 \cdot \frac{1}{12} \;\geq\; \frac{1}{2} + 0 - 2 \cdot \frac{1}{3} + \frac{1}{6} \;=\; 0,$$

contradicting the choice of ww_a. □

To complete the proof of Lemma 2.2, we need to show that $m \geq 12$ also leads to a contradiction. Suppose $m \geq 12$. Then $d(w_a) \geq 3$, otherwise (i) would hold with $v = w_a$. It suffices to show $\varphi(ww_a) \geq 0$ when $d(w_a) = 3$, for otherwise $\varphi(ww_a) = \varphi_b(ww_a) \geq \frac{2}{3} + 0 - 2 \cdot \frac{1}{3} = 0$. If $d(w_a) = 3$, then $d(w_a^-) \geq 12$ and $d(w_a^+) \geq 12$, and a charge of at least $\frac{1}{6}$ is transfered from both $w_a w_a^-$ and $w_a w_a^-$ to ww_a. Thus we find

$$\varphi(ww_a) \;\geq\; \varphi_b(ww_a) + 2 \cdot \frac{1}{6} \;\geq\; \frac{2}{3} - \frac{1}{3} - 2 \cdot \frac{1}{3} + \frac{1}{3} \;=\; 0,$$

the final contradiction in this proof. ■

Proof of Lemma 2.3 We use the notation and definitions from the part common with the proof of Lemma 2.2. In fact, the proof follows a line similar to the proof of the previous lemma, although the arguments are different.

Claim 1 $m \neq 3$.

Proof Suppose $m = 3$. Since (ii) and (iii) do not hold for G, we have $d(w_j) \geq 6$ for all j, and $t(w) = 0$. Thus $\varphi(ww_j) \geq \varphi_b(ww_j) \geq -\frac{1}{3} + \frac{1}{3} = 0$ for all $ww_j \in E_w$. It follows that $\sum_{e \in E_w} \varphi(e) \geq 0$, contradicting the choice of w. □

Claim 2 $m \neq 4$.

Proof Suppose $m = 4$. Since (iv) does not hold for G, we have $d(w_j) \geq 5$ for all j. If $t(w) \leq 1$, then we find $\sum_{e \in E_w} \varphi(e) \geq 4 \cdot 0 + 4 \cdot \frac{1}{5} - 2 \cdot \frac{1}{3} > 0$. Thus $t(w) \geq 2$. If $t(w) \geq 3$, then $t(ww_i) = 2$ for some i, in which case (v) holds. Consequently, $t(w) = 2$ and in fact $t(ww_j) = 1$ for all j.

If $d(w_j) = 5$ for some j, then setting $v = w_j$ and $v_i = w$ we find that (vi) holds, contradicting the choice of G. Thus $d(w_j) \geq 6$ for all j, and hence $\sum_{e \in E_w} \varphi(e) \geq 4 \cdot 0 + 4 \cdot \frac{1}{3} - 4 \cdot \frac{1}{3} = 0$, contradicting the choice of w. □

Claim 3 $m \neq 5$.

Proof Suppose $m = 5$. We first note that $d(w_j) \geq 4$ for all j, otherwise (i) or (ii) would hold. Also, $d(w_1) \leq 7$, for otherwise $\varphi(ww_j) \geq \varphi_b(ww_j) \geq \frac{1}{5} + \frac{1}{2} - 2 \cdot \frac{1}{3} > 0$ for all j. If $t(w) \leq 1$, then we find $\sum_{e \in E_w} \varphi(e) \geq 5 \cdot \frac{1}{5} + 5 \cdot 0 - 2 \cdot \frac{1}{3} > 0$. Thus $t(w) \geq 2$. Furthermore, since (vi) and (vii) do not hold, if $t(ww_j) = 1$ for some j, then $d(w_j) \geq 5$; and if $t(ww_\ell) = 2$ for some ℓ, then $d(w_\ell) \geq 6$.

If $t(w) = 2$, then there are at least three neighbours w_j of w with $t(ww_j) \geq 1$, and hence $d(w_j) \geq 5$. This means $\sum_{e \in E_w} \varphi(e) \geq 5 \cdot \frac{1}{5} + 3 \cdot \frac{1}{5} + 2 \cdot 0 - 4 \cdot \frac{1}{3} > 0$.

If $t(w) = 3$, then, since (viii) does not hold, there must be at least one neighbour w_j with $t(ww_j) = 0$. This means that in fact there are two neighbours w_j with $t(ww_j) = 1$, and hence $d(w_j) \geq 5$; and two neighbours w_ℓ with $t(ww_\ell) = 2$, and hence $d(w_\ell) \geq 6$. This gives $\sum_{e \in E_w} \varphi(e) \geq 5 \cdot \frac{1}{5} + 0 + 2 \cdot \frac{1}{5} + 2 \cdot \frac{1}{3} - 6 \cdot \frac{1}{3} > 0$.

If $t(w) \geq 4$, then for all j we find $t(ww_j) \geq 1$, which means that (viii) holds. So in all cases we contradict the choice of G or the choice of w. □

We now know that $m \geq 6$. Since the vertex w is chosen such that $\sum_{e \in E_w} \varphi(e) < 0$, there must be an edge $e \in E_w$ such that $\varphi(e) < 0$. Let $ww_a \in E_w$ be such an edge. By Lemma 2.1 (i) this must mean that $\varphi_b(ww_a) < 0$ also, and hence

$$0 \;>\; \varphi_b(ww_a) \;\geq\; \frac{m-4}{m} + \frac{d(w_a)-4}{d(w_a)} - t(ww_a) \cdot \frac{1}{3}. \tag{4}$$

Claim 4 $m \neq 6, 7$.

Proof Suppose $m = 6$ or $m = 7$. From equation (4) it follows that the only possibilities for $d(w_a)$ and $t(ww_a)$ are:

$$d(w_a) \leq 2;$$
$$d(w_a) = 3 \quad \text{and} \quad t(ww_a) \geq 1;$$
$$d(w_a) = 4 \quad \text{and} \quad t(ww_a) = 2;$$
$$d(w_a) = 5 \quad \text{and} \quad t(ww_a) = 2.$$

In the first three options we see that (i), (iii), and (v), respectively, hold, where we take $v = w_a$.

So the only possibility left is $d(w_a) = 5$ and $t(ww_a) = 2$. Let the neighbours of w_a be $\{w_a^-, w, w_a^+, u_1, u_2\}$. Then certainly $t(w_a w) \geq 1$, $t(w_a w_a^-) \geq 1$ and $t(w_a w_a^+) \geq 1$. Hence if $t(w_a u_1) \geq 1$ and $t(w_a u_2) \geq 1$, then (viii) holds with $v = w_a$.

So for at least one $p \in \{1, 2\}$, $t(w_a u_p) = 0$. Moreover, since (ix) does not hold, for at least one $p \in \{1, 2\}$ we have that $t(w_a u_p) = 0$ and $d(u_p) \geq 6$. Without loss of generality we can assume that u_1 has these properties. Combining everything we find that $d(w_a) = 5$, $d(w) \geq 6$, $d(u_1) \geq 6$, $t(w_a w) = 2$, and $t(w_a u_1) = 0$. This means that in the final step of the discharging process a charge of $\frac{1}{6}$ is transferred from $w_a u_1$ to $w_a w$. We find that the final charge for the edge ww_a satisfies

$$\varphi(ww_a) \;\geq\; \frac{1}{3} + \frac{1}{5} - 2 \cdot \frac{1}{3} + \frac{1}{6} \;>\; 0,$$

contradicting the choice of ww_a. $\square$

To complete the proof of Lemma 2.3 we only need to show that $m \geq 8$ also leads to a contradiction. Suppose $m \geq 8$. From equation (4) it follows that the only possibilities for $d(w_a)$ and $t(ww_a)$ are:

$$d(w_a) \leq 2;$$
$$d(w_a) = 3 \quad \text{and} \quad t(ww_a) \geq 1;$$
$$d(w_a) = 4 \quad \text{and} \quad t(ww_a) = 2.$$

If the first possibility holds, then (i) follows; if the second holds, then (iii) holds; and the third possibility gives that (v) holds, every time taking $v = w_a$. This gives the final contradiction against the existence of a counterexample G. $\blacksquare$

3 Proof of Theorem 1.5

Let G be a planar graph and let Δ be its maximum degree. If $\Delta \leq 5$, then Theorem 1.5 can be proven using a straight-forward "greedy" colouring method. In fact, in this case the theorem holds even when the planarity condition is removed. The only essential observations are that for any vertex in a graph H with maximum degree Δ, the number of vertices at distance one from v is at most Δ and the number of vertices at distance two is at most $\Delta\,(\Delta-1)$. Moreover, if we assign a certain label to a vertex at distance one from v, then this reduces the number of labels available to v with at most $2\,p-1$, whereas assigning a label to a vertex at distance two from v can "forbid" at most $2\,q-1$ labels for v. We leave the verification of the further details to the reader.

In the remainder, we are solely interested in the case $\Delta \geq 6$. We shall prove Theorem 1.5 by induction on the number of vertices and edges. Let G be a planar graph such that for all planar graphs H with $|V(H)|+|E(H)|<|V(G)|+|E(G)|$ the theorem is true. We note first that can assume that G is simple and $\Delta \geq 6$.

For an edge $e \in E$ let G/e denote the graph obtained from G by contracting e. For a vertex $v \in V$ let $G*v$ denote the graph obtained by deleting v and for each $u \in N(v)$ adding an edge between u and u^- and between u and u^+ if these edges do not exists in G already. We will use Lemmas 2.2 and 2.3 to show that there is a vertex $v \in V$ such that $d(v) \leq 5$, the number of vertices at distance 2 from v is at most $2\,\Delta+19$, and at least one of the following is true:

(a) $\Delta(G/e) \leq \Delta$ for some $e \in E_v$;

(b) $\Delta(G*v) \leq \Delta$.

The following proposition formulates the essential properties of the vertex degrees and distances after the operations G/e and $G*v$ have been performed.

3.1 Proposition

Let G be a simple graph, v a vertex and $e=vu$ an edge in G.

(i) Let $H=G/e$, and let v' be the vertex in H corresponding to the edge vu. Then for each $w \in V(H)\backslash\{v'\}$ we have $d_H(w) \leq d_G(w)$, and $d_H(v')=d_G(v)+d_G(u)-2-t_G(vu)$.

(ii) Let $H=G*v$. Then for each $w \in V(H)$ we have $d_H(w)=d_G(w)$ if $w \notin N_G(v)$, and $d_H(w)=d_G(w)+1-t_G(vw)$ if $w \in N_G(v)$.

(iii) Let $H=G/e$, and let v' be the vertex in H corresponding to the edge vu. Then for any two vertices $w,w' \in V(H)\setminus\{v'\}$ it holds that $dist_H(w,w') \leq dist_G(w,w')$ and $dist_H(w,v') \leq dist_G(w,u)$.

(iv) Let $H=G*v$ and suppose $d_G(v) \leq 5$. Then for any two vertices $w,w' \in V(H)$ it holds that $dist_H(w,w') \leq dist_G(w,w')$.

Now define a vertex $v \in V(G)$, possibly an edge $e \in E(G)$, and a graph H as follows:

(3.1) If $\Delta \geq 12$, then let v be as described in Lemma 2.2, and set $e=vv_1$ and $H=G/e$.

(3.2) If $6 \leq \Delta \leq 11$ and one of Lemma 2.3 (i), (ii), or (iv) holds, then let v be as described, and set $e=vv_1$ and $H=G/e$.

(3.3) If $6 \leq \Delta \leq 11$ and Lemma 2.3 (iii) holds, then let v be as described, set $e = vv_i$ with $t(vv_i) \geq 1$, and set $H = G/e$.

(3.4) If $6 \leq \Delta \leq 11$ and Lemma 2.3 (v) holds, then let v be as described, set $e = vv_i$ with $t(vv_i) = 2$ and set $H = G/e$.

(3.5) If $6 \leq \Delta \leq 11$ and Lemma 2.3 (vi) holds, then let v be as described, set $e = vv_i$ with $d(v_i) \leq 4$ and $t(vv_i) \geq 1$, and set $H = G/e$.

(3.6) If $6 \leq \Delta \leq 11$ and Lemma 2.3 (vii) holds, then let v be as described, set $e = vv_i$ with $d(v_i) \leq 5$ and $t(vv_i) = 2$, and set $H = G/e$.

(3.7) If $6 \leq \Delta \leq 11$ and Lemma 2.3 (viii) holds, then let v be as described and set $H = G{*}v$.

(3.8) If $6 \leq \Delta \leq 11$ and Lemma 2.3 (ix) holds, then let v be as described and set $H = G*v$.

In the cases (3.1)–(3.6), identify the end vertex of e different from v with the vertex in H corresponding to the contracted edge e. Then using Proposition 3.1, we find that in cases (3.1)–(3.7), $d_H(w) \leq d_G(w)$ for all $w \in V(H)$, hence $\Delta(H) \leq \Delta(V) = \Delta$. In case (3.8) we can have $d_H(w) = d_G(w) + 1$ for a vertex $w \in N(v)$ with $t(vw) = 0$, but then $d_G(w) \leq 5$, and we still find $\Delta(H) \leq \Delta$. By induction, this means

$$\lambda(H; p, q) \ \leq\ (4\,q - 2)\,\Delta + 10\,p + 38\,q - 23.$$

Set $n = (4\,q - 2)\,\Delta + 10\,p + 38\,q - 23$ and let $\varphi_H : V(H) \longrightarrow \{0, 1, \ldots, n-1\}$ be an $L(p,q)$-labelling of H. Again using Proposition 3.1, for any two vertices $w, w' \in V(H)$ it holds that $dist_H(w, w,') \leq dist_G(w, w')$. Therefore, to find an $L(p,q)$-labelling for G, we need only extend φ_H to G by giving v an appropriate colour. For each $w \in V(H)$ let $\varphi(w) = \varphi_H(w)$.

For any vertex $v \in V(G)$, the number of vertices at distance two from v is equal to

$$\sum_{u \in N(v)} d(u) - d(v) - 2\,t(v). \tag{5}$$

Since v was chosen according to (3.1)–(3.8), $d(v) \leq 5$ and equation (5) gives that there are at most $2\,\Delta + 19$ vertices at distance two from v. So, since

$$n \ >\ (4\,q - 2)\,\Delta + 10\,p + 38\,q - 24 \ =\ 5 \cdot (2\,p - 1) + (2\,\Delta + 19) \cdot (2\,q - 1),$$

we can choose a colour $\varphi(v) \in \{0, 1, \ldots, n-1\}$ such that

$$\begin{aligned} |\varphi(u) - \varphi(v)| \ &\geq\ p, \qquad \text{if } dist_G(u,v) = 1; \\ |\varphi(u) - \varphi(v)| \ &\geq\ q, \qquad \text{if } dist_G(u,v) = 2. \end{aligned}$$

Choosing such a colour for v, we see that φ is an $L(p,q)$-labelling for G. It now follows that

$$\lambda(G; p, q) \ \leq\ n \ =\ (4\,q - 2)\,\Delta + 10\,p + 38\,q - 23,$$

which completes the induction step. ■

Remark
By a more elaborate case analysis, it is possible to slightly improve Lemmas 2.2 and 2.3 in such a way that we get a slightly better bound in Theorem 1.5. But this would only improve the additive term, and not the factor $4\,q-2$ in front of Δ. For this reason we haven't tried to push our method to the limit.

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