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A matrix method for chromatic polynomials

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Abstract

The chromatic polynomials of certain families of graphs can be expressed in terms of the eigenspaces of a linear operator. The operator is represented by a matrix, which is referred to here as the compatibility matrix. In this paper complete sets of eigenfunctions are obtained for several related families, and the results are used to provide information about the location of the zeros of the associated chromatic polynomials. A number of uniform features are observed, and these are explained in terms of general properties of the underlying construction.

1. The chromatic polynomials of the cyclic ladders

The *cyclic ladder* L_n is the graph with $2n$ vertices $v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n$, joined by edges in the following way. Each of the sets $v_1, v_2, \ldots, v_n$, and $w_1, w_2, \ldots, w_n$ forms a cycle of length n, and in addition v_i is joined to w_i for $i = 1, 2, \ldots, n$.

The *chromatic polynomial* of a graph G is the polynomial function $P(G; z)$ of the complex variable z such that, for each non-negative integer k, $P(G; k)$ is the number of proper colourings of G when k colours are available.

A formula for the chromatic polynomial of L_n was published [5] in 1972. It is

$$P(L_n; z) = (z^2 - 3z + 3)^n + (z - 1)\{(1 - z)^n + (3 - z)^n\} + (z^2 - 3z + 1).$$

This formula was obtained by using the deletion-contraction method. The chromatic polynomial of L_n was expressed in terms of that for L_{n-1} and other closely-related graphs, and a recurrence relation was obtained. The technique was formalised and extended in [6]. The key idea is to obtain a 'transfer matrix' $M(z)$ that represents the effect on the chromatic polynomial of adjoining one copy of a basic graph. Iterating this process n times corresponds to taking the nth power of $M(z)$. The chromatic polynomial of the resulting graph can be expressed as a linear combination of the eigenvalues of $M(z)^n$, which are the nth powers of the eigenvalues of $M(z)$. The coefficients can be found by comparison with the chromatic polynomials for small values of n. Thus, for the graph L_n, the eigenvalues are $z^2 - 3z + 3$, $1 - z$, $3 - z$, and 1, and the coefficients are 1, $z - 1$, $z - 1$, and $z^2 - 3z + 1$.

In [2] an alternative method was used to compute the chromatic polynomials of any family $\{G_n\}$ of graphs that can be constructed in a manner similar to the cyclic ladders. It is based on a different kind of transfer matrix, which we shall refer to as the *compatibility matrix* (the details will be given in Section 3). The main result is that $P(G_n; k)$ an be written as the trace of the nth power of a matrix $T(k)$, and hence as the sum of the nth powers of its eigenvalues, with 'coefficients' equal to the multiplicities of the eigenvalues. Comparing this result with the formula for $P(L_n; k)$ we conclude that in this case the eigenvalues and multiplicities of $T(k)$ are

$$\begin{aligned} k^2 - 3k + 3 &: \text{ multiplicity } 1 \\ 1 - k &: \text{ multiplicity } k - 1 \\ 3 - k &: \text{ multiplicity } k - 1 \\ 1 &: \text{ multiplicity } k^2 - 3k + 1. \end{aligned}$$

Recently there has been a renewal of interest in these matters; see for example [8, 12, 13, 14]. This has highlighted the fact that the compatibility matrix method has not been fully explored: indeed, there has not even been a study of the of the eigenfunctions of $T(k)$ for the cyclic ladders. In this paper complete sets of eigenfunctions will be found, not only for the cyclic ladders, but for other related families. Although it may not be feasible to do the same calculations in full for more complicated families, it is nevertheless possible to identify several important general features, and these throw light on the location of the zeros of the chromatic polynomial.

2. Zeros of chromatic polynomials

In [5] two properties of $P(L_n; z)$ were noted. The first was the observation that the zeros of $P(L_n; z)$ appear to cluster around some simple curves in the complex plane. This observation has been explained by Read and Royle [10], using a general result of Beraha, Kahane and Weiss [1].

Another property of $P(L_n; z)$, proved in reference [5], is that all its zeros lie in the disc $|z| < 3$. It is now possible to calculate explicitly the chromatic polynomials of graphs of moderate size [9,10], and the experimental evidence lends weight to the conjecture, made in reference [5], that a wide-ranging generalisation of Brooks' theorem may hold: specifically, there may be constants c_d such that the zeros of the chromatic polynomial of any regular graph with degree d lie in the disc $|z| < c_d$. A proof of this conjecture has recently been announced [11], with $c_d = 8d$ approximately.

The motivation for this work is the fact that the location of the zeros of $P(G; z)$ determines the chromatic number of G. Since there is no good way of finding the chromatic number, any information provided by a general theory is potentially useful. In Section 7 of this paper it will be shown that our results about the eigenspaces of $T(k)$ provide the basis for results about the location of zeros of certain families of graphs. Finally, we shall discuss why some of our results have uniform features that may be capable of generalisation.

3. The compatibility matrix

A *graph scheme* [3] is a pair (B, J), where B is a graph with vertex-set V_B, and J is a subset of $V_B \times V_B$. Let P_k denote the set of proper vertex-colourings of B with k colours. We say that two colourings α, β in P_k are *compatible* with respect to J if

$$(v, w) \in J \quad \Longrightarrow \quad \alpha(v) \neq \beta(w).$$

The idea is that if two copies of B are linked by edges, joining a vertex v in the first copy to w in the second copy whenever $(v, w) \in J$, then using α to colour the first copy of B and β to colour the second copy results in a proper colouring of the entire graph.

For each integer $n \geq 2$ let $\mathcal{C}_n(B, J)$ denote the graph formed as follows. Take n disjoint copies of B and, for $i = 1, 2, \ldots, n$ join vertex v in the ith copy to w in the $(i+1)$th copy if and only if $(v, w) \in J$. (Here, by convention, $n + 1 = 1$). For example, the cyclic ladder L_n is obtained by taking B to be the complete graph K_2 with $V_B = \{a, b\}$, and $J = \{(a, a), (b, b)\}$.

For a given graph scheme (B, J) and a given k, the *compatibility matrix* $T = T(k)$ is the matrix with rows and columns indexed by the elements of P_k, and

$$T_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are compatible;} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem Let $T(k)$ be the compatibility matrix for the graph scheme (B, J). Then

$$P(\mathcal{C}_n(B, J); k) \;=\; \mathrm{trace}\,[T(k)^n].$$

Proof Let $\alpha, \beta, \ldots, \nu$ be n colourings belonging to P_k. The product

$$T_{\alpha\beta}\, T_{\beta\gamma} \, \cdots \, T_{\nu\alpha}$$

is 1 precisely when we obtain a proper colouring of $\mathcal{C}_n(B,J)$ by colouring the first copy of B using α, the second copy using β, and so on. Thus the total number of proper colourings is

$$\begin{aligned} P(\mathcal{C}_n(B,J);k) &= \sum_{\alpha,\beta,\ldots,\nu} T_{\alpha\beta}\, T_{\beta\gamma} \, \cdots \, T_{\nu\alpha} \\ &= \sum_{\alpha} (T^n)_{\alpha\alpha} \\ &= \operatorname{trace} T^n. \end{aligned}$$

□

Corollary Suppose that the matrix $T(k)$ has eigenvalues $\lambda_1(k), \lambda_2(k), \ldots, \lambda_s(k)$, with multiplicities $m_1(k), m_2(k), \ldots, m_s(k)$. Then

$$P(\mathcal{C}_n(B,J);k) \;=\; \sum_{r=1}^{s} m_r(k)\lambda_r(k)^n.$$

Proof This is a consequence of the fact that the trace of a matrix is the sum of its eigenvalues with appropriate multiplicities. □

We note that the individual expressions $m_i(k)$ and $\lambda_i(k)$ are not necessarily polynomial functions of k. Furthermore, the matrix $T(k)$ need not be symmetric, so its eigenvalues may be not be real. But as we shall see, in some cases these expressions are amenable to simple analysis.

Since $T(k)$ is a non-negative matrix, the well-known Perron-Frobenius theory asserts that in certain circumstances it will have a unique eigenvalue of largest modulus. For example, consider the cyclic ladder L_n, constructed as $\mathcal{C}_n(B,J)$ in the manner described above. Here $T(k)$ has $k(k-1)$ rows and columns, and each row and column contains $k^2 - 3k + 3$ 1's and $2k - 3$ 0's. This ensures that $k^2 - 3k + 3$ is the unique eigenvalue of largest modulus, and it has multiplicity 1. In the next section we shall compute the remaining eigenvalues of $T(k)$ and their multiplicities.

4. Eigenfunctions for the cyclic ladders

Throughout this section we deal with a fixed integer $k \geq 3$. Denote a proper colouring of K_2, with k colours available, by (i,j), where $1 \leq i,j \leq k$ and $i \neq j$, and let V denote the vector space of real-valued functions defined on the set of such colourings. The *standard basis* for V consists of the functions e_{ij} which are defined, for i and j in the given range, by

$$e_{ij}(r,s) = \begin{cases} 1 & \text{if } (r,s) = (i,j) \; ; \\ 0 & \text{otherwise.} \end{cases}$$

We shall find the eigenspaces of the transformation of V which is represented (with respect to the standard basis) by $T = T(k)$. Here T is the transfer matrix for the scheme (K_2, J) with $J = \{aa, bb\}$, (for convenience we write aa instead of (a, a), and so on.)

In fact, it is simpler to work with $S = A - T$, where A is the matrix each of whose entries is 1. Thus S corresponds to the *incompatibility* of colourings of K_2, with respect to J. The operation of S on the standard basis can be described as follows. Define

$$a_i = e_{i1} + e_{i2} + \cdots + e_{ik}, \quad b_j = e_{1j} + e_{2j} + \cdots + e_{kj}.$$

Note that a_i and b_j both contain $k-1$ summands (that is, there are no terms e_{ii} and e_{jj}), and that if $i \neq j$ both contain a term e_{ij}. Thus the action of S is given by

$$S(e_{ij}) = a_i + b_j - e_{ij}.$$

Clearly

$$\sum_{i=1}^{k} a_i = \sum_{j=1}^{k} b_j = u,$$

where u is the function that takes the value 1 on every colouring. It follows from the formula for $S(e_{ij})$ that

$$\begin{aligned} S(a_i) &= S(e_{i1} + e_{i2} + \cdots + e_{ik}) \\ &= (k-1)a_i + (b_1 + b_2 + \cdots + b_k) - (e_{i1} + e_{i2} + \cdots + e_{ik}) \\ &= (k-1)a_i + (u - b_i) - a_i \\ &= u + (k-2)a_i - b_i. \end{aligned}$$

Similarly

$$S(b_i) = u - a_i + (k-2)b_i.$$

We also have $S(u) = (2k-3)u$, in agreement with the fact that any given colouring is incompatible with exactly $2k-3$ others. Thus u is an eigenfunction with eigenvalue $2k-3$.

It is easy to construct some more eigenfunctions using the equations given above. For future use, it is convenient to introduce the notation

$$d_i = k(a_i - b_i), \quad s_i = k(a_i + b_i) - 2u.$$

Then it follows that

$$S(d_i) = (k-1)d_i, \quad S(s_i) = (k-3)s_i.$$

In other words, we have k eigenfunctions d_i with eigenvalue $k-1$, and k eigenfunctions s_i with eigenvalue $k-3$. In each case the dimension of the eigenspace is $k-1$, because the sum of all k eigenfunctions is identically zero.

Since $S = A - T$, and $k(k-1)$ is an eigenvalue of A with eigenfunction u, it is clear that the eigenvalue $2k-3 = k(k-1) - (k^2-3k+3)$ corresponds to the eigenvalue k^2-3k+3 of T. We have already observed that the multiplicity of this eigenvalue is 1. The remaining eigenvalues of A are all zero, and so an eigenvalue μ of S corresponds to an eigenvalue $-\mu$ of T. Thus we have accounted for the eigenvalues k^2-3k+3, $1-k$, and $3-k$ of T, with multiplicities 1, $k-1$, $k-1$, as listed in Section 1.

The remaining eigenfunctions also arise from simple combinations of the e_{ij}. Define

$$\begin{aligned} p_{ij} &= k^2(e_{ij} - e_{ji}) - (d_i - d_j) \\ q_{ij} &= k(k-1)(k-2)(e_{ij} + e_{ji}) - (k-1)(s_i + s_j) - 2(k-2)u. \end{aligned}$$

Then it follows from the equations already obtained that

$$S(p_{ij}) = -p_{ij} \quad \text{and} \quad S(q_{ij}) = -q_{ij}.$$

Both p_{ij} and q_{ij} are eigenfunctions with eigenvalue -1. (The reason for distinguishing between them will become clear in the next section.) In the space spanned by the p_{ij}'s we have the linear relationships

$$\sum_{j \neq i} p_{ij} = 0 \quad (1 \leq i \leq k), \qquad \sum_{i \neq j} p_{ij} = 0 \quad (1 \leq j \leq k).$$

Thus we can express all the functions of the form p_{ik} and p_{kj} as linear combinations of the rest. Also $p_{ij} = -p_{ij}$, so the space is spanned by the p_{ij} with $1 \leq i < j \leq k-1$. It is easy to check that these eigenfunctions are linearly independent, so we have a space of dimension $(k-1)(k-2)/2$.

Slightly different results hold for the space spanned by the q_{ij}. Here we have $q_{ij} = q_{ji}$ and, more significantly, the set of eigenfunctions q_{ij} with $1 \leq i < j \leq k-1$ is not linearly independent, because the sum of all them is identically zero. For example, when $k = 3$ the only function in the range is q_{12}, and it is easy to check that it is zero. (Note that p_{12} is not zero.) Hence the dimension of the space spanned by the q_{ij} is $(k-1)(k-2)/2 \; - \; 1$.

These spaces intersect only in the zero function, so the dimension of the eigenspace corresponding to the eigenvalue -1 is

$$(k-1)(k-2)/2 \; + \; (k-1)(k-2)/2 - 1 = k^2 - 3k + 1.$$

In terms of the matrix T, we have the eigenvalue 1, with the 'right' multiplicity, as listed in Section 1.

It is worth noting that we have assumed $k \geq 3$ throughout. For $k = 1$ the entire set-up is vacuous, since $k(k-1) = 0$. For $k = 2$ S is the identity matrix, with two eigenvalues equal to 1, corresponding to the eigenfunctions u and d_1. The functions s_1, p_{12}, q_{12} are all zero.

5. The twisted ladders

The cyclic ladder graph L_n was defined as $\mathcal{C}_n(B, J)$, where $B = K_2$ and $J = \{aa, bb\}$. In this section and the next one we shall study other families of graphs, also with $B = K_2$, but with a different J. Although the choice of K_2 is obviously very special, there are patterns in the algebra which may provide scope for further generalisation (see Section 8).

Define the graph $L_n^\times$ to be $\mathcal{C}_n(B, J^\times)$, where $J^\times = \{ab, ba\}$. When n is even the graph $L_n^\times$ is isomorphic to L_n, but when n is odd it is a different graph, often called the *Moebius ladder.*

With the same notation as in Section 4, the action of $S^\times$ on the standard basis is given by

$$S^\times(e_{ij}) = a_j + b_i - e_{ji}.$$

It follows that

$$\begin{aligned} S^\times(a_i) &= u - a_i + (k-2)b_i \\ S^\times(b_i) &= u + (k-2)a_i - b_i \\ S^\times(u) &= (2k-3)u. \end{aligned}$$

Thus, with d_i and s_i also as in Section 4, we have

$$S^\times(d_i) = (1-k)d_i, \quad S^\times(s_i) = (k-3)s_i.$$

So we have eigenvalues $2k-3$, $1-k$, and $k-3$, with eigenfunctions u, d_i, and s_i. The eigenfunctions are the same as those for the untwisted case, a fact that will be useful in due course. The respective multiplicities are also the same, that is 1, $k-1$, $k-1$.

For the remaining eigenfunctions, let p_{ij} and q_{ij} be as in Section 4. Then

$$S^\times(p_{ij}) = p_{ij}, \quad S^\times(q_{ij}) = -q_{ij}.$$

The dimensions of the eigenspaces spanned by the p_{ij} and q_{ij} are as before, and so we the following eigenvalues of $S^\times$:

$$\begin{array}{rl} 1 & \text{with multiplicity } (k-1)(k-2)/2 \\ -1 & \text{with multiplicity } (k-1)(k-2)/2 - 1. \end{array}$$

Translating our results about $S^\times$ into results about the eigenvalues of the compatibility matrix $T^\times$ we obtain the following. There is a unique eigenvalue of greatest modulus, $k(k-1) - (2k-3) = k^2 - 3k + 3$; there are eigenvalues $k-1$ and $3-k$ of multiplicity $k-1$; and there are eigenvalues -1 and 1 of multiplicity $m_k, m_k - 1$ respectively, where $m_k = (k-1)(k-2)/2$. It follows that

$$P(L_n^\times; k) = (k^2 - 3k + 3)^n + (k-1)[(k-1)^n + (3-k)^n] + (-1)^n m_k + (m_k - 1).$$

When n is even we have $(k-1)^n = (1-k)^n$ and $(-1)^n = 1$, so the formula is the same as that for L_n (indeed, the graphs are isomorphic). But when n is odd we get the well-known formula [5] for the chromatic polynomial of the Moebius ladder:

$$P(L_n^\times; k) = (k^2 - 3k + 3)^n + (k-1)[(3-k)^n - (1-k)^n] - 1.$$

6. The augmented ladders

Let

$$J^+ = \{aa, ba, bb\}, \quad J^{++} = \{aa, ab, ba, bb\}.$$

The corresponding graphs

$$L_n^+ = \mathcal{C}_n(K_2, J^+) \quad \text{and} \quad L_n^{++} = \mathcal{C}_n(K_2, J^{++})$$

are modifications of the ladder L_n with additional 'rungs', one and two (respectively) in each section.

For L_n^{++} we do not need any new calculations. This is because J^{++} is the disjoint union of J and $J^\times$, and $S^{++} = S + S^\times$. In Sections 4 and 5 we observed that the eigenspaces of S and $S^\times$ are identical, so the eigenvalues of S^{++} are the sums of the corresponding eigenvalues of S and $S^\times$:

Eigenfunctions :	u	d_i	s_i	p_{ij}	q_{ij}
Eigenvalues of S :	$2k-3$	$k-1$	$k-3$	-1	-1
Eigenvalues of $S^\times$:	$2k-3$	$1-k$	$k-3$	-1	1
Eigenvalues of S^{++} :	$4k-6$	0	$2k-6$	-2	$0.$

Translating these into eigenvalues of T^{++} by the usual rule, we get the chromatic polynomial

$$P(L_n^{++}; k) \;=\; \{(k-2)(k-3)\}^n + (k-1)(6-2k)^n + k(k-3)2^{n-1}.$$

For L_n^+ the calculations are similar to those in Section 4, but not identical. Using the same notation, we have the basic equation

$$S^+(e_{ij}) = a_i + a_j + b_j - e_{ij}.$$

From this we obtain

$$\begin{aligned} S^+(a_i) &= (k-3)a_i - b_i + 2u \\ S^+(b_i) &= (k-2)a_i + (k-2)b_i + u \\ S^+(u) &= (3k-4)u. \end{aligned}$$

Because these equations lead to irrational eigenvalues, it is convenient to use a method slightly different from the one used in Section 4. For each i the subspace spanned by u, a_i, b_i is invariant under S^+, and the action of S^+ is represented by the matrix

$$\begin{pmatrix} 3k-4 & 0 & 0 \\ 2 & k-3 & -1 \\ 1 & k-2 & k-2 \end{pmatrix}.$$

So we have eigenvalues $3k-4$ and the roots $\mu_1(k), \mu_2(k)$ of the equation

$$\mu^2 - (2k-5)\mu + (k-2)^2 = 0.$$

The eigenvalue $3k-4$ has multiplicity 1. The eigenvalues $\mu_1(k), \mu_2(k)$ are irrational, but it is easy to check that when $\mu = \mu_1(k)$ or $\mu = \mu_2(k)$, and $1 \le i \le k$, the function

$$k(k-2-\mu)a_i + kb_i - (k-1-\mu)u$$

is an eigenfunction with the appropriate eigenvalue. As before, the sum of all k eigenfunctions is zero in each case, so the eigenspaces have dimension $k-1$.

The remaining eigenvalues are also simple combinations of the e_{ij}, but here too it is convenient to use a new method. First, define

$$f_{ij} = e_{ij} - e_{i+1,j} - e_{i,j+1} + e_{i+1,j+1},$$

provided all the terms exist (and with the convention that $k+1 = 1$). Then it follows immediately from the formula for $S^+(e_{ij})$ that

$$S^+(f_{ij}) = -f_{ij},$$

so each f_{ij} is an eigenfunction with eigenvalue -1. Similarly, if we define

$$c_l = e_{1,1+l} + e_{2,2+l} + \cdots + e_{k,k+l},$$

then we have $S^+(c_l) = 3u - c_l$, so $c'_l = (k-1)c_l - u$ is also an eigenfunction with eigenvalue -1.

There are some linear relationships among the various f_{ij} and c'_l. For example

$$f_{1,l+1} + f_{2,2+l} + \cdots + f_{k,k+l} = 2c_l - c_{l-1} - c_{l+1} = 2c'_l - c'_{l-1} - c'_{l+1}.$$

One way to get the 'right' number $k^2 - 3k + 1$ of linearly independent eigenfunctions is to note that, for each of the values $i = 1, 2, \ldots, k-1$, there are $k-3$ linearly independent eigenfunctions f_{ij}, specifically those for which $j = 1, 2, \ldots, i-2, i+2, \ldots, k$. This provides $(k-1)(k-3)$ f_{ij}'s, and the fact that we have excluded the ones with $i = k$ destroys the linear relationships with the c'_l's given above. The only remaining relationship is that the sum $c'_1 + c'_2 + \cdots + c'_{k-1}$ is zero. So we have $k-2$ linearly independent c'_l's, and we are finished, since

$$(k-2) + (k-1)(k-3) = k^2 - 3k + 1.$$

We have now accounted for all $k(k-1)$ eigenvalues of S^+. Translating them into eigenvalues of T^+ the formula for the chromatic polynomial is

$$P(L_n^+; k) \; = \; (k-2)^{2n} + (k-1)[(-\mu_1(k))^n + (-\mu_2(k))^n] + (k^2 - 3k + 1).$$

Of course, the irrationality of $\mu_1(k)$ and $\mu_2(k)$ is not a serious difficulty, because

$$(-\mu_1(k))^n + (-\mu_2(k))^n = \text{trace}\begin{pmatrix} 3-k & 1 \\ 2-k & 2-k \end{pmatrix}^n$$

is clearly a polynomial function of k.

There are clear similarities between the results obtained above for the various families $\mathcal{C}_n(B,J)$ with $B = K_2$. For completeness we should mention also the case $J^* = \{aa, ba\}$, which produces graphs L_n^* that look like cycles with a triangle attached to each edge. The eigenfunction method leads to the results given in the table below, and hence the chromatic polynomial is

$$P(L_n^*; k) = \{(k-1)(k-2)\}^n + (k-1)(2-k)^n.$$

Of course, this formula can also be derived by elementary methods.

The following table summarises all our results about the incompatibility matrix. It shows that the existence of eigenspaces U, W_1, W_2, X_1, X_2, having dimensions $1, k-1, k-1, (k-1)(k-2)/2, (k-1)(k-2)/2 - 1$ respectively, is a feature of this situation, and suggests some general questions that will be discussed in Section 8.

Eigenvalues and multiplicities for the incompatibility matrix

$$m_k = \frac{1}{2}(k-1)(k-2).$$

	Eigenspace :	U	W_1	W_2	X_1	X_2
	Dimension :	1	$k-1$	$k-1$	m_k	$m_k - 1$
J						
aa, ab, ba, bb		$4k-6$	0	$2k-6$	0	-2
aa, ba, bb		$3k-4$	$\mu_1(k)$	$\mu_2(k)$	-1	-1
ab, ba		$2k-3$	$1-k$	$k-3$	1	-1
aa, bb		$2k-3$	$k-1$	$k-3$	-1	-1
aa, ba		$2k-2$	$k-2$	0	0	0

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7. Location of zeros

The general formula for $P(\mathcal{C}_n(B,J); k)$, in the form stated in Section 3, invites us to extend the result to a complex variable z:

$$P(\mathcal{C}_n(B,J); z) \;=\; \sum_{r=1}^{s} m_r(z)\lambda_r(z)^n.$$

This extension requires some care. The left-hand side is certainly a polynomial function, defined in the entire complex plane, but the individual terms on the right-hand side may not be so well-behaved. However, in the case of the graphs discussed in Sections 4, 5, and 6, it is easy to ensure that everything is in order. The only problem is that in the case of L_n^+ two of the eigenvalues of the compatibility matrix (specifically $-\mu_1$ and $-\mu_2$ in the notation of Section 6) are irrational. However, their contribution to the sum can be written as the trace of a 2×2 matrix with polynomial terms, so we have:

$$P(L_n^+; z) = (z-2)^{2n} + (z-1)\,\mathrm{trace}\begin{pmatrix} 3-z & 1 \\ 2-z & 2-z \end{pmatrix}^n + (z^2 - 3z + 1).$$

In this form, it is clear that all the terms are polynomial functions.

In Section 2 it was noted that the zeros of the polynomials under discussion here lie close to certain curves. For the graphs L_n this phenomenon has been explained in detail by Read and Royle [10], and similar results can be obtained for the other families.

We shall focus on the more general problem of locating the zeros of the polynomials within a suitable disc. It is well-known [4, p.76] that if G has N vertices and M edges then the coefficient of z^{N-1} in $P(G; z)$ is equal to $-M$. Equivalently, the centroid of the zeros of $P(G; z)$ is the point $\delta/2$, where δ is the average degree of G. It is therefore reasonable to consider the location of the zeros with respect to a disc of the form

$$D_R = \{z \, : \, |z - \delta/2| \leq R\}.$$

In the case $G = \mathcal{C}_n(B, J)$, the key idea is to write the chromatic polynomial in the form

$$P_n(z) = P(\mathcal{C}_n(B, J); z) = F(z)^n + G_n(z),$$

where $F(z)$ represents the dominant eigenvalue, which has unit multiplicity, and $G_n(z)$ represents the sum of the remaining eigenvalues according to their multiplicities. Suppose we can show that, for some R and all sufficiently large n,

$$|F(z)|^n \; > \; |G_n(z)| \quad \text{whenever } |z - \delta/2| = R.$$

When this condition is satisfied, Rouché's Theorem tells us that if all zeros of $F(z)$ are inside D_R, then so are all zeros of $P_n(z)$. It remains to choose the smallest value of R for which the condition holds.

The strategy outlined in the previous paragraph works well in the case of the graphs $\mathcal{C}_n(B, J)$ with $B = K_2$. For L_n we have $\delta = 3$ and

$$|F(z)| = |z^2 - 3z + 3| = \left| \left(z - \frac{3}{2}\right)^2 + \frac{3}{4} \right| \geq R^2 - \frac{3}{4},$$

on the circle $|z - \frac{3}{2}| = R$.

For $G_n(z)$ we have

$$\begin{aligned} |G_n(z)| &= |(z-1)[(3-z)^n + (1-z)^n] + (z^2 - 3z + 1)| \\ &\leq |z-1|\,[|z-3|^n + |z-1|^n] + |z^2 - 3z + 1|, \end{aligned}$$

and on the circle $|z - \frac{3}{2}| = R$,

$$|z-1| \leq R+1/2, \quad |z-3| \leq R+3/2, \quad |z^2-3z+1| \leq R^2+5/4.$$

The 'largest' term in $|G_n(z)|$ is the one involving $|z-3|^n$, and so we have an estimate of the form

$$|G_n(z)| < c_R \left(R+\frac{3}{2}\right)^n \quad \text{on } |z-\frac{3}{2}| = R.$$

Choose R to be any fixed real number such that

$$R^2 - \frac{3}{4} > R + \frac{3}{2}.$$

Then there is a positive integer $n_0(R)$ such that, for all $n \geq n_0(R)$,

$$\left(R^2-\frac{3}{4}\right)^n > c_R \left(R+\frac{3}{2}\right)^n.$$

It follows that the condition $|F(z)|^n > |G_n(z)|$ holds on $|z-\frac{3}{2}| = R$ for all sufficiently large n. We have established the desired result under the condition that $R > R_0$, where R_0 is the positive root of the equation

$$R^2 - \frac{3}{4} = R + \frac{3}{2}.$$

Explicitly, $R_0 = \frac{1}{2}(1+\sqrt{10}) = 2.081\ldots$. The function $F(z)$ has zeros at the points $(3 \pm i\sqrt{3})/2$, and so all zeros of $F(z)^n$ are in D_R whenever $R > R_0$. It follows from Rouché's Theorem that, for all sufficiently large n, all zeros of $P_n(z)$ also lie in D_R whenever $R > R_0$.

This is a minor improvement on the old result [5] that the zeros lie in the disc $|z| \leq 3$. It suggests that similar results may hold more generally, and in fact the generalisation to L_n^+ and L_n^{++} is almost automatic.

For L_n^+ we have $\delta = 4$, so we need to consider the circle $|z-2| = R$, where clearly $|F(z)| = |z-2|^2 = R^2$. In this case

$$\begin{aligned} |G_n(z)| &= |(z-1)[(-\mu_1(z))^n + (-\mu_2(z))^n] + (z^2-3z+1)| \\ &\leq |z-1|\,[|\mu_1(z)|^n + |\mu_2(z)|^n] + |z^2-3z+1|. \end{aligned}$$

The 'largest' term on the circle $|z-2| = R$ is $|\mu_1(z)|^n \leq (2R+1)^n$, with the appropriate choice of μ_1. Using the same argument as for L_n, we see that $|F(z)|^n > |G_n(z)|$ will hold for all sufficiently large n provided that $R > R_0$, where R_0 is the positive root of the equation

$$R^2 = 2R+1.$$

In this case $R_0 = 1+\sqrt{2} = 2.414\ldots$.

For L_n^{++} we have $\delta = 5$, and on the circle $|z-\frac{5}{2}| = R$, $|F(z)| = |z^2-5z+6| \geq R^2 - \frac{1}{4}$.

For $|G_n(z)|$ we have

$$\begin{aligned}|G_n(z)| &= |(z-1)(6-2z)^n + z(z-3)2^{n-1}| \\ &\leq |z-1|\,|2z-6|^n + |z(z-3)|2^{n-1}.\end{aligned}$$

So, on the circle $|z-\frac{5}{2}| = R$ the largest term is $|2z-6|^n \leq (2R+1)^n$ (provided $2R+1 > 2$). In this case the critical value R_0 is the positive root of

$$R^2 - \frac{1}{4} = 2R+1,$$

that is, $R_0 = 2.5$.

8. Concluding discussion

In this section we shall explain why some of the uniform features observed in the preceding discussions are not entirely coincidental. It is possible that some of the following remarks can be applied more generally, but the generalisations will not be attempted here.

If the base graph B has vertex-set V_B and edge-set E_B, the graph $\mathcal{C}_n(B,J)$ has $n|V_B|$ vertices and $n(|E_B|+|J|)$ edges. Hence the average degree of each graph in the family is given by

$$\delta = \frac{2(|E_B|+|J|)}{|V_B|}.$$

In particular, when $B = K_2$ we have $\delta = |J|+1$.

Now, for a given J, the number of proper k-colourings of K_2 that are incompatible with a given one is of the form $\alpha k - \beta$, where α is the size of the set J, and β is a fixed positive integer. The number $\alpha k - \beta$ is the dominant eigenvalue of the incompatibility matrix, and hence the dominant eigenvalue of the compatibility matrix $T(k)$ is

$$\begin{aligned}k(k-1) - (\alpha k - \beta) &= k^2 - (\alpha+1)k + \beta \\ &= k^2 - \delta k + \beta \\ &= (k-\delta/2)^2 + \gamma,\end{aligned}$$

for some real constant γ. Thus, on the circle $|z-\delta/2| = R$, the leading term $F(z)^n$ in the formula for the chromatic polynomial is such that

$$|F(z)|^n \geq (R^2 - |\gamma|)^n.$$

We now discuss $G_n(z)$. Here too there is a simple observation based on the interpretation of $G_n(k)$ in terms of the eigenspaces of $T(k)$.

Suppose that π is any permutation of $\{1,2,\ldots,k\}$. Then π induces a permutation of the set P_k of proper k-colourings of the base graph B. Clearly the property of compatibility is preserved under π, and so the compatibility matrix $T = T(k)$ commutes with the corresponding permutation matrix M_π. It follows that if $Tf = \lambda f$, then $TM_\pi f = M_\pi T f = \lambda M_\pi f$. In other words, if f is an eigenfunction with eigenvalue λ, then so is $M_\pi f$.

If f is constant, then $M_\pi f = f$. But if f is not constant we can obtain new eigenfunctions by this method. For example, if π is a k-cycle, then we get k eigenfunctions

$$f_i = (M_\pi)^i f \quad i = 0, 1, \ldots, k-1.$$

Under quite general conditions the only linear relationship connecting the f_i is the fact that their sum is zero. It follows that the multiplicity of any eigenvalue that contributes to $G_n(k)$ is at least $k-1$. This explains why, in the case $B = K_2$, we obtain expressions of the form

$$G_n(k) = (k-1)g_n(k) + q(k)h_n.$$

Here the term g_n is a polynomial function of degree n, q is a quadratic polynomial, and h_n is independent of k.

The properties of $G_n(z)$ worked out in Section 7 are necessary consequences of this general fact. Specifically, we have an estimate of the form

$$|G_n(z)| \leq c_R\, \theta_n(R) \quad \text{on } |z - \delta/2| = R,$$

where θ_n is a polynomial of degree n.

The method used in Section 7 depends on the fact that we have a lower bound for $|F(z)|^n$ that is of order R^{2n}, and an upper bound for $|G_n(z)|$ that is of order R^n. This leads to the conclusion that there is a critical value R_0 such that, for sufficiently large n, we have $|F(z)|^n > |G_n(z)|$ whenever $|z - \delta/2| = R > R_0$. The values of R_0 obtained in Section 7 do not provide a reliable guide to what may be true more generally. However, it is possible that there may exist a uniform estimate, of the form $R_0 \leq R_B$ for all families $\mathcal{C}_n(B, J)$ with a given base graph B.

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