# Notes on one-dimensional coverage problem

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## 1 One-dimensional case

This paper discusses the one-dimensional case of a line of research that was previously considered in [1]. The notation of that paper is used.

We consider infinitely many transmitters of equal power equally spaced along the line; that is, the coverage region $\Omega = \mathbf{R}$ and the set of transmitters $T = \mathbf{Z}$. We assume the propagation law to satisfy the condition $(*)$ of [1]; so the points with worst noise-to-signal ratio are $n + 1/2$, $n \in \mathbf{Z}$. We need to impose another condition on the function $N(r)$ without which the problem would become meaningless (the noise at each point of $\Omega$ would become infinitely big):

$(**)$ $\sum_{n=1}^{\infty} N(n) < \infty$.

In addition, we assume that only a finite set of equally spaced channels is used — as it is always so in practice. Let $C = \{0, 1, \ldots, k-1\}$, and $\lambda = e^{-\beta}$ — the interference fall between neighbouring channels. It is natural to measure the total span by the number $sp = \lambda^{1-k}$; one of our aims will be to compare channel assignments with different values of $k$ but with equal spans.

Define a function $f : \mathbf{N} \to \mathbf{R}$ by the formula $f(n) = \frac{N(n-1/2)}{N(1/2)}$. We have $f(1) = 1$, and $f$ monotonically decreases. Then the value of NSR at every

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point $n + 1/2$ is a polynomial in $\lambda$ of degree at most $k-1$ with non-negative coefficients; and all coefficients are sums of certain values of the function $f$. The value of NSR for a receiver at a point $n+1/2$ tuned to the frequency $c(n)$ (resp. $c(n+1)$) will be denoted by $NSR(n,-)$ (resp. $NSR(n,-)$). The coefficient at $\lambda^i$ will be denoted respectively by $a_i^{(n,-)}, \quad a_i^{(n,+)}$.

An assignment $c : T \to C$ is called *periodic* if for $c(n) = c(n+L)$ for some constant value $L$. An assignment is called *strictly periodic* if all frequencies within a period are different, and *permutational* if there exists a permutation $\sigma$ of $\{0, 1, \dots, k-1\}$ such that $c(n) = \sigma(n \bmod k)$. For any strictly periodic assignment of period $L$, and for any point $n+1/2$ where $n \in \mathbf{Z}$, the constant term in NSR has the same value

$$\sum_{i\geq 1}(f(iL) + f(iL+1)) = M^{(L)}.$$

**THEOREM 1** *For any periodic assignment $c$ with given value of $k$,*

$$\max(a_0^{(n,\varepsilon)} \mid n \in \mathbf{Z},\ \varepsilon \in \{+,-\}) \geq M^{(k)},$$

*and the inequality is strict unless the assignment is permutational.*

The following lemma, which will be used in the proof of the theorem, looks like a special case of some more general result from theoretical mechanics. We shall give it an elementary proof which is more in the spirit of the present paper.

Denote by $\mathbf{R}_L$ the metric space $\mathbf{R}/L\mathbf{Z}$: the interval $[0, L)$ viewed as the circle of length $L$, with the natural metric $d(x,y) = \min(|x-y|, L-|x-y|)$ $(x, y \in [0, L))$. The maximum distance between two points in $\mathbf{R}_L$ is $L/2$.

**LEMMA 1** *Let $f : \mathbf{R}^+ \to \mathbf{R}^{\geq 0}$ be any strictly convex decreasing function.*

*(i). For any positive numbers $x_1, \dots, x_n, S$ such that $x_1 + \dots + x_n \leq S$, we have*

$$f(x_1) + \dots + f(x_n) \geq nf(S/n),$$

*with the equality only when $x_1 = \dots = x_n = S/n$.*

*(ii). Let $(M, d)$ be the metric space $R_L$ defined above. For every finite set $X \subset M$ define*

$$f(X) = \sum_{\{x,y\}\subseteq X,\ x\neq y} f(d(x,y)).$$

*Then the minimum value of $f(X)$ over all $n$-element sets $X \subset M$ is achieved only on the set $X_0 = \{0, L/n, 2L/n, \dots, (n-1)L/n\}$, and on its cyclic shifts.*

PROOF. $(i)$ is a standard inequality about convex functions.

$(ii)$ For any $x, y \in [0, L)$, define oriented distance $d'(x, y)$ as

$$d'(x,y) = \begin{cases} y - x & when \quad y \geq x \\ L + y - x & when \quad y < x \end{cases}.$$

Obviously, $d(x, y) \leq d'(x, y)$. Denote elements of $X$ by $0 \leq x_0 < x_1 < \ldots < x_{n-1}$. Regroup the summands in the formula for $f(X)$ as follows:

$$f(X) = \sum_{1 \leq i < n/2} \sum_{k=0}^{n-1} f(d(x_k, x_{k+i}))$$

when $n$ is odd, and

$$f(X) = \sum_{1 \leq i < n/2} \sum_{k=0}^{n-1} f(d(x_k, x_{k+i})) + \sum_{k=0}^{n/2-1} f(d(x_k, x_{k+n/2}))$$

when $n$ is even. Here indices are considered modulo $n$.

Now, for each $i < n/2$ we have

$$\sum_{k=0}^{n-1} d'(x_k, x_{k+i}) = iL.$$

Therefore $\sum d(x_k, x_{k+i}) \leq iL$, and by $(i)$ we have:

$$\sum_{k=0}^{n-1} f(d(x_k, x_{k+i})) \geq nf(iL/n). \tag{1}$$

The last sum in the formula for $f(X)$ for even $n$ is estimated as follows:

$$\sum_{k=0}^{n/2-1} f(d(x_k, x_{k+n/2})) \geq (n/2)f(L/2). \tag{2}$$

Note that only when $X = X_0$, or $X$ is a cyclic shift of $X_0$, all inequalities (1) and (2) become equalities. Adding together all these inequalities, we get the desired inequality $f(X) \geq f(X_0)$. $\square$

PROOF OF THEOREM. Let $c$ be any periodic colouring of $\mathbf{Z}$ by $k$ colours. Taking, if necessary, $k$ periods, we can ensure that the period length $L = kn$ is a multiple of $k$.

First we replace the real line $\mathbf{R}$ with $\mathbf{R}_L$. As in the original problem, the signal from a transmitter located at $x$ (which now represents an infinite set of transmitters, all of the same frequency) in a point $y$ depends on $d = d(x, y)$ only; and is given by the formula

$$N'(d) = \sum_{i \geq 0} N(d + iL) + \sum_{i \geq 0} N(L - d + iL).$$

The function $N'$ is defined for all $d \neq iL$ (we set $N'(iL) = \infty$); it is monotonely decreasing on $(0, L/2]$, and convex on $(0, L)$, with a minimum at $L/2$.

The colouring $c$ is obtained by periodically repeating the sequence

$$(c(0), \ldots, c(L-1));$$

we shall consider it as a colouring of $L$ integer points of the circle. What we need to show is that there exist a colour $c$, an integer $t \in \{0, \ldots, L-1\}$ such that $c(t) = c$, and a receiver position $x \in \{t - 1/2, t + 1/2\}$ (operations modulo $L$) such that

$$\sum_{i:\, c(i)=c} N'(d(x, i)) \geq \sum_{0 \leq i \leq n-1} N'(d(1/2, ki)) = N_0.$$

Let $c$ be any colour which appears in the period at least $n$ times. Let $X \subseteq \{0, \ldots, L-1\}$ be any set of $n$ points coloured by $c$. Also, let $X_0 = \{0, k, 2k, \ldots, (n-1)k\}$.

For every point $z = x_0 \pm 1/2$, $x_0 \in X_0$, the total signal from $X_0$ at $z$ is the same, and is equal to $N_0$. We shall prove that

$$\sum_{x \neq y \in X} \sum_{z = x \pm 1/2} N'(d(z, y)) \geq \sum_{x \neq y \in X_0} \sum_{z = x \pm 1/2} N'(d(z, y)) = 2n(n-1)N_0; \quad (3)$$

this will suffice to prove the theorem.

Define a function $f : (0, L/2] \to \mathbf{R}^{\geq 0}$ as follows:

$$f(d) = 2N'(d + \frac{1}{2}) + 2N'(d - \frac{1}{2}).$$

The function $f$, like $N'$, is monotonely decreasing and convex on $(0, L/2]$. The inequality (3) which we want to prove can now be written as

$$\sum_{\{x,y\} \subseteq X,\, x \neq y} f(d(x, y)) \geq \sum_{\{x,y\} \subseteq X_0,\, x \neq y} f(d(x, y)),$$

and it immediately follows from Lemma 1$(ii)$. $\Box$

Thus, asymptotically, when $\lambda \to 0$, permutational assignments are the best possible among periodic assignments. It is very likely that the condition of periodicity in Theorem 1 may be replaced by some weaker condition, or altogether dropped.

## 2 Permutational assignments

In this section we shall find optimal permutational assignments for small values of $k$. We shall compare the effectiveness of these assignments having the same total span $sp = \lambda^{1-k}$. We assume that the propagation function $N(d)$ satisfies $(*)$ and $(**)$; and use the notation from the previous section.

Let $c$ be a permutational assignment; i.e. a $k$-colouring with period of length $k$. In addition to

$$M^{(k)} = \sum_{i\geq 1}(f(ik) + f(ik+1)),$$

we introduce the numbers

$$M_j^{(k)} = \sum_{i\geq 0}(f(ik+j) + f(ik+k+1-j))$$

for $1 \leq j \leq k$. We have: $M_1^{(k)} = M^{(k)} + 1$, and $M_j^{(k)} = M_{k+1-j}^{(k)}$, and $M_1 > M_2 > \ldots > M_{(k+1)/2}$.

Each frequency is assigned to a set of transmitters at equal distances $k$; the total signal from such set at every point $n + 1/2$ is equal to $M_j^{(k)}$ where $j/2$ is the distance from the point to the set. Thus, every value of $NSR(n, \varepsilon)$ is the sum of $M^{(k)}$ (co-channel interference) and a linear combination of powers of $\lambda$ with coefficients $M_1^{(k)}$, $M_2^{(k)}$, $M_2^{(k)}$, etc., $M_{k/2}^{(k)}$. Let us consider small values of $k$; for simplicity of notation we shall omit the upper index $(k)$ whenever possible.

For $k = 2$ the assignment is unique: $\ldots 010101\ldots$, and NSR at all points is equal to $M + \lambda M_1 = (1+\lambda)M - 1$.

For $k = 3$ the assignment is also unique up to symmetry: $\ldots 012012\ldots$, and NSR is equal to $M + M_1\lambda + M_2\lambda^2$ or to $M + (M_1 + M_2)\lambda$. The latter is the worst value of NSR.

For $k=4$ there are three essentially different assignments,
$\ldots 01230123\ldots$,
$\ldots 01320132\ldots$, and
$\ldots 03120312\ldots$: the worst value of NSR in all three cases is $M+(M_1+M_2)\lambda+M_2\lambda^2$.

For $k=5$ there are 12 essentially different assignments, but we don't need to consider all of them to choose the best one. Indeed, consider a receiver between two transmitters one of is assigned the channel 2, tuned to the channel 2 (the channels being 0,1,2,3,4). The value of NSR for this receiver is the sum of $M$ and a linear combination of $\lambda,\lambda,\lambda^2,\lambda^2$ with coefficients $M_1,M_2,M_2,M_3$ in a certain order. The minimum possible value of such combination is $(M_2+M_3)\lambda+(M_1+M_2)\lambda^2$, so a lower bound for NSR is

$$M+(M_2+M_3)\lambda+(M_1+M_2)\lambda^2.$$

But the assignment $\ldots 0241302413\ldots$ has exactly this value of the worst NSR. So, this is an optimal assignment. Note that it is also the best solution of the binary constraints problem with constraints
$|c(i)-c(i+1)|\geq 2$,
$|c(i)-c(i+2)|\geq 1$.

Given any choice of the propagation law, it is easy to calculate the values of $M_i^{(k)}$, and to compare optimal assignments with different number of channels but with equal values of total span (that is, $\lambda_1^{1-k_1}=\lambda_2^{1-k_2}$).

As an example, let us compare assignments of period 2 and period 3 with the same span: $\ldots 020202\ldots$ and $\ldots 012012\ldots$.

The worst values of NSR are equal respectively to $N_2=M_1^{(2)}-1+\lambda^2 M_1^{(2)}$, and $N_3=M_1^{(3)}-1+\lambda(M_1^{(3)}+M_2^{(3)})$. We have:

$$N_3-N_2=M_1^{(3)}-M_1^{(2)}+\lambda(M_1^{(3)}+M_2^{(3)})-\lambda^2 M_1^{(2)}=$$

$$=M_1^{(2)}(1-\lambda)(\lambda-B),$$

where

$$B=\frac{M_1^{(2)}-M_1^{(3)}}{M_1^{(2)}}.$$

Using obvious identities $M_1^{(1)}=2M_1^{(2)}=2M_1^{(3)}+M_2^{(3)}$, we get $B=M_2^{(3)}/M_1^{(1)}$. The assignment of period 2 is better than that of period 3 with the same span if and only if the interference fall $\lambda$ is greater than $B$. In the particular case of the propagation function $d^{-\alpha}$, one can find that the threshold value $B = 3^{-\alpha}$.

# References

[1] N.Biggs, D.Fon-Der-Flaass. Optimizing the signal-to-noise ratio, *CDAM Research Report Series*, LSE-CDAM-98-16 (1998), 4 pp.