A fractal example in ordinary differential equations

by Anatole Beck

1. Introduction. A friend recently asked me to back up a claim I made in my book [1] to the effect that there exist flows in the plane with no fixed points for which every orbit is a separatrix. In considering this problem, I came up with an example which has an interesting fractal construction.

Definition: Let f be a continuous flow in the plane, i.e. f is a continuous mapping of $\mathbb{R} \times E$ onto E (where $\mathbb{R}$ is the real line and E is the plane) satisfying $f(0,x) = x$ and $f(t, f(s,x)) = f(s+t,x)$ for all x in E and all s, t in $\mathbb{R}$. The orbit of a point x, $\mathcal{O}(x)$, is the set $f(\mathbb{R}, x)$. A sequence $\{\mathcal{O}(x_j)\}$ converges to an orbit $\mathcal{O}(y)$ if there is a sequence (t_j) of reals such that $f(t_j, x_j) \to y$ in E. This concept of convergence does not necessarily have unique limits. An orbit $\mathcal{O}(x)$ is called *regular* if x is contained in an open set in E which is a union of orbits and in which this convergence is unique; otherwise it is a *separatrix.*

2. The fractal construction. We start with a set A_1 to be included in the unit square $Q = [0,1] \times [0,1]$. We take this to consist of the open unit interval $(0,1)$ together will all the points (u,v) where $0 < u < 1$ and $0 < v < c(u)$, where c is the Cantor function. c takes dyadic rational values $n/2^m$ on intervals, and these intervals in the graph of c will be designated as *shelves.* We observe that each shelf is of a length $1/3^k$ for some natural number k, and that the one of that length which is closest to the line $v = 1$ is $[1 - 2/3^k, 1 - 1/3^k]$, where c takes the value $1 - 1/2^k$. Thus, if we construct a square on that interval,

$$[1 - 2/3^k, 1 - 1/3^k] \times [1 - 1/2^k, 1 - 1/2^k + 1/3^k],$$

then that square lies entirely in the interior of Q.

To construct the set A_2, we construct a square in this way on each of the shelves of A_1. We note that each such square lies in the interior of Q and that their closures (they are in fact closed) do not intersect. Into each square, we put a similar copy of A_1. To construct the set A_{n+1} from the set A_n, we erect a square on each shelf of A_n and into each square we put a similar copy of A_1. Clearly A_{n+1} includes A_n, and the union of all of them we call A.

It is an interesting feature of this construction, not needed for this example, that if we put a similar copy of A_m onto each shelf of A_n, we will get A_{n+m}. We do not prove that here.

Finally, we reflect A in the u-axis with the transformation $T(u,v) = (u,-v)$ and define the union of A and $T(A)$ to be the set B. This concludes the fractal construction.

3. The flow f. We define a flow in B by taking for the orbits all the maximal open intervals in B parallel to the u-axis and defining the flow as moving from left to right at each point of each interval with a speed there which is equal to the distance of that point from the boundary of B. This flow is clearly continuous. (See e.g. [2].) For each of these orbits $\mathcal{O}(x)$ which lies in A_1, we can characterize it as the set of all points (u,v) for a fixed v satisfying $0 < u < 1$ and $v < c(u)$. If v is a dyadic rational, then this orbit lies adjacent to a shelf which is an orbit in A_2. Every orbit lying beneath $\mathcal{O}(x)$ lies also beneath the mated orbit which is that shelf, so that convergence to $\mathcal{O}(x)$ is not unique. For each such v, we call the orbit $\mathcal{O}(x)$ a primary separatrix. The primary separatrices are clearly dense in A_1, so that every orbit in A_1 is a separatrix.

When we construct A_2, the images of the primary separatrices also also separatrices by the same argument. We call them secondary separatrices, and observe that all the orbits in A_2 are separatrices. Similarly for A_3, &c.. By this argument, we see that every orbit in B is a separatrix for this flow.

4. Smoothness. It might be imagined that restrictions on the differentiability of the flow f would defeat this remarkable property. So we will improve the flow to make it very smooth. Let h be the conformal mapping carrying B onto the unit disc D in the complex plane $\mathbf{C}$ with $h(0,0) = -1$, $h(1,1) = i$, $h(1,0) = 1$, and $h(1,-1) = -i$. Then h carries each orbit of f onto an analytic curve in D. We note that some of these curves join a single point of the left half of the unit circle K to a single point of the right half and that no two of these curves share either of these endpoints. Other curves join a point in the left half of K to another point in the left half which is the starting point of one of the curves mentioned above. No two curves share a starting point. The flow g is defined in D as moving along these curves as orbits with the speed at any point z being defined as $1 - |z|^2$. Then the flow g has infinitely many derivatives at each point, and every orbit is a separatrix. If we now transform D into E by taking each re^{it} into $\tan(\pi r/2)e^{it}$, then the flow defined on the images of the orbits of g which move from left to right at speed 1 will again be infinitely differentiable and every orbit will be a separatrix.

References

[1] Anatole Beck, *Continuous Flows in the Plane*, Grundlehren der Mathematischen Wissenschaften, Vol. 201, Springer-Verlag (Heidelberg) 1974.

[2] Anatole Beck, On invariant sets, Annals of Math. 2, 67 (1958) pp 99–103.