# Powers of Roy spaces of weight $\aleph_1$

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**Abstract**

A Roy space is a zero-dimensional metric space having covering dimension at least 1 . Using the technique introduced by Mrówka [4], we prove the following.

**Theorem** The two known Roy spaces of weight $\aleph_1$ (Kulesza [2] and Ostaszewski [7]) have squares of covering dimension 1.

We also show how to generalize the result to arbitrary *finite* powers by indicating the changes needed in the case of cubing. Assuming the Continuum Hypothesis ($CH$), the corresponding theorem for the square was proved by Mrówka [4] in the case of the other then known Roy spaces, which are of weight $2^{\aleph_0}$. In our case $CH$ is unnecessary (since one does not need to map $\aleph_1$ onto the line). There is resurgent interest in the problem of constructing Roy spaces with dimensional discrepancy larger than 1, so the current contribution presents what is hoped to be a more readable account of [4] by elucidating some of the finer details.

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# 1 Introduction

By a Roy space we shall mean a metrizable space which is zero-dimensional but not strongly zero-dimensional i.e has small inductive dimension zero but covering dimension at least 1 (see Engelking [1] for standard dimension-theoretic definitions). Such a space is necessarily non-separable, since it is known that the small inductive dimension agrees with the covering dimension in separable metric spaces (more generally, equality holds in the class of strongly paracompact, strongly hereditarily normal spaces). Using the technique introduced by Mrówka [4], we prove the following.

**Theorem** The two known Roy spaces of weight $\aleph_1$ (Kulesza [2] and Ostaszewski [7]) have squares of covering dimension 1.

We show how to generalize the result to arbitrary $finite$ powers by indicating the changes needed in the case of cubing. Assuming the Continuum Hypothesis ($CH$), the corresponding theorem for the square was proved by Mrówka [4] in the case of the other then known Roy spaces, which are of weight $2^{\aleph_0}$, namely that originally constructed by Prabir Roy in [8] (see also Pears [6]) and the space constructed by Mrówka.

Whilst this theorem disappoints hopes that in the cases mentioned squaring will provide a metrizable space where the small inductive dimension and the covering dimension differ by more than one, it has been recently shown by Mrówka in a break-through advance [5] that, assuming a large cardinal axiom, it is consistent with ZFC to construct a Roy space of covering dimension 1 whose square has covering dimension 2; Kulesza [3] also shows how to extend the consitency argument to provide a Roy space of covering dimension 1 whose $n$ -th power has covering dimension $n$, again assuming the large cardinal axiom. It is however still an open problem whether such spaces can be constructed without strong additional set-theoretic assumptions.

In view of renewed interest in such dimension discrepancies, the current paper serves two purposes: first, to show that $CH$ is unnecessary in the weight $\aleph_1$ cases (since one does not need to map $\aleph_1$ onto the real line), and second: to present, one hopes, a more readable proof than Mrówka's by elucidating some of the details. It currently seems appropriate to make more accessible what is a very well crafted tool for carrying the information coded in the points of a topological space across to efficient covers of the space by a process of geometric coding. Roy's construction may be thought of as a method for discretizing the topology of the line $R$ so that it becomes zero-dimensional (and necessarily of weight $2^{\aleph_0}$) but maintaining enough topological cohesion, by adding many copies of the rational points, to keep the global appearance of a one-dimensional line. Mrówka showed how to carry the discretization idea into coverings. It is hoped that our clarifications will help in the determination of the so far unknown covering dimension of a countable power of any of the known Roy spaces.

The paper is organized as follows: review of the structure of a space $S$ of weight $\aleph_1$; review of the principal coding ideas, the Mrówka coding map $\Phi$ and the intersection diagram $\Phi(S^2)$; representation of codes by intervals on the real line and their use in determining the dimension of the intersection diagram; proof that the coding map is continuous. We close with a mere sketch of the construction of the self-refining sequence of locally finite closed covers of order 1 needed to verify the covering dimension of the square - the intuition at this point is evident.

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## 2 The two Roy spaces

We first recall the definition of the space constructed in [7]. For this we need to introduce signed countable ordinals $\pm\alpha$ (which enable us to map by a 'mirror image' from $+\alpha$ to $-\alpha$), which are to be distinguished from ordinary (unsigned) ordinals. $\{$Note that $+0 \neq -0.\}$ For coding purposes we let

$$sgn(+\alpha) = sgn(\alpha) = 1, sgn(-\alpha) = 0, | \pm \alpha| = \alpha.$$

Our space $S$ has underlying set

$$S = S_I \cup S_F$$

where

$$S_I = I^\omega,\ S_F = X \times (\omega_1 \backslash \omega) \times Z,$$

and where

$$\begin{aligned} I &= \{-1, +1\} \times \omega_1 = I_- \cup I_+, \\ Z &= \omega_1^\omega, \\ X &= I^{<\omega}; \end{aligned}$$

also, for $k = 1, 2, ...,$ we put

$$S_k = I^{k-1} \times (\omega_1 \backslash \omega) \times Z.$$

Now $x \in X$ encodes not only a Baire interval $B(x) = \{y \in I^\omega : y|n = x\}$ where $n = length(x)$ (so that $x = (x_1, ..., x_n)$) for $n \geq 0$, but also labels the additional points $P(x) = \{x\} \times (\omega_1 \backslash \omega) \times Z$, to be thought of as forming a **shell** for $B(x)$. There are thus two types of points in the space, firstly those lying in the sets $P(x)$, which will be referred to as shell points, or points of Type F (which are of finite order equal to $length(x)$), and secondly the points of Type I (of infinite order). The Type F points are there to ensure that large enough disjoint open sets have intersecting closures.

Correspondingly there are two types of basic open set. Type I sets, to be called **blobs**, are the original Baire intervals together with the union of all the shells associated to all the sub-blobs, namely

$$\tilde{B}(x) = B(x) \cup \smile \{P(x') : x' = x \text{ or } x' \text{ extends } x\}.$$

Thus if $B(x') \subseteq B(x)$ then $P(x') \subseteq \tilde{B}(x)$. We say that the **depth** of this set is $length(x)$. To define Type II sets we need to choose for each infinite countable ordinal $\alpha$ a **bijection** from $N = \{1, 2, ...\}$ to $\alpha$ which we denote by $\bar{\alpha}$. With this notation the basic open sets of Type II take the form for $\alpha \in \omega_1 \backslash \omega$ and $n \geq 1$

$$\begin{aligned} U(x, \alpha, z_1, ..., z_{n-1}) \ &= \ \smile_{i \geq n} \tilde{B}(x, \pm\bar{\alpha}(i), \mp\alpha, \mp z_1, ..., \mp z_{n-1}) \cup \\ &\quad\ \{x\} \times \{\alpha\} \times Z(z_1, ..., z_{n-1}) \end{aligned}$$

where $Z(z_1, ..., z_{n-1}) = \{z \in Z : z|n-1 = (z_1, ..., z_{n-1})\}$ is again a Baire interval. Note that $\pm z$ describes an ordinal which has been given an appropriate sign. This basic open set should be visualised as a co-final twin stream of mirror-imaged bubbles boiling off the bottom of the interval

$\{x\} \times \{\alpha\} \times Z(z_1, ..., z_{n-1})$. Note that $Z(<>) = Z$. The **depth** of this set is $length(x) + n + 1$. Note that since $\bar{\alpha}(i) \in \alpha$ we have

$$\bar{\alpha}(i) \neq \alpha. \tag{2.1}$$

**Comments.** We note that the Type II sets are constructed as Baire-like neighbourhoods of shell points $(x, \alpha, z)$ equipped with a memory: i.e there is coding going on so that the constituent blobs remember $\alpha$ by virtue of the signature constancy prefixed by a sign change (we call this the **sign flip**), and also they remember their position in the stream (sequence) by reference to information stored in the two-bit $\pm\bar{\alpha}(i), \mp\alpha$ which starts at position $length(x)$. This position may be called **the flip position** and of course it recreates the depth of $x$. Notice that, by definition, in the case of of a shell point $(x, \alpha, z)$ the term of the sequence $(x, \alpha, z)$ at the flip position is distinct in absolute value from the corresponding term on the nearby type I points $(x, \pm\bar{\alpha}(i), \mp\alpha, \mp z)$; this is because $\bar{\alpha}(i) \neq \alpha$.

In particular two distinct type II sets of equal depth cannot meet; one refers to the final sign change and works backwards to establish equality of the intersecting blobs. This is the **most important feature** that is exploited in Mrówka's Theorem and coding procedure.

On the other hand a type I set and a type II set of equal depth may meet by way of inclusion (of a type II in a type I set) and for this reason $Ind(S) \leq 1$. Of course equality requires combinatorial work.

## 2.1 Variant space

As was pointed out in the Postscript to [7], the space constructed by Kulesza [2] may be described in the current context as a variant construction in which the basic open sets of Type II are of the form for $n \geq 1$

$$\begin{aligned} U(x, \alpha, z_1, ..., z_{n-1}) &= \smile_{i \geq n} \smile_{\gamma \in S_i(\alpha)} \tilde{B}(x, \pm\gamma, \mp\alpha, \mp z_1, ..., \mp z_{n-1}) \cup \\ &\cup (\{x\} \times \{\alpha\} \times Z(z_1, ..., z_{n-1})) \end{aligned}$$

where $S_i(\alpha) = [\lambda_i(\alpha), \lambda_{i+1}(\alpha))$ and $\lambda_i(\alpha)$ is a strictly increasing sequence of ordinals with limit $\alpha$. We shall present our material in terms of the other space and so will not need any further details of the Kulesza space beyond the basic open set form just quoted; this will be enough for us to explain how to amend our proof of the continuity theorem.

## 2.2 Signature function

In some sense any part of space $S$ looks like any other. So its intersection properties are held in the **signature** function defined by

$$\varphi(x) = \sum_{i=1}^{\infty} \frac{sgn(x(i))}{2^i},$$

where we agree also to take $sgn(Z) = sgn(-Z) = 0$ and $sgn(+Z) = 1$; thus for $x \in S_k = I^{k-1} \times \omega_1 \times I_+^{\omega}$ we have in binary $\varphi(x) = .....1_{k-1}00000$. Thus the terminating zeros form is the preferred coding for a point in $S_k$. Evidently it is the signature function that permits one to relate $S$ to the real line of which it is a discretized copy.

## 2.3 Signature of the points of shell-depth $k$

The points in $S_k$ take the form

$$\begin{aligned} x &= (x_1, x_2, ..., x_{k-1}, \alpha, z_1, ...) \\ &= (x, \alpha, z) \end{aligned}$$

and they have signature

$$(sgn(x_1), sgn(x_2), ..., sgn(x_{k-1}), 1, 0, 0, 0, ...)$$

and are approached through a sequence of points in $S_I$

$$\bar{z}_i = (x, +\bar{\alpha}(i), -\alpha, -z) \text{ and } z_i = (x, -\bar{\alpha}(i), +\alpha, +z)$$

with signatures

$$(sgn(x_1), sgn(x_2), ..., sgn(x_{k-1}), 1, 0, 0, 0, ...)$$

and

$$(sgn(x_1), sgn(x_2), ..., sgn(x_{k-1}), 0, 1, 1, 1, ...)$$

exhibiting a sign flip followed by sign constancy.

The signature function thus records the fact that $(x, \alpha, z)$ in $S_k$ is in the joint closure of $\{z_i : i = 1, 2, ...\}$ and $\{\bar{z}_i : i = 1, 2, ...\}$. Inevitably as the points of $S_k$ have a special role we shall see the set $D_k$ of dyadic rationals of order $k$ (those of form $(2m + 1)/2^k$) playing an equally important corresponding role.

## 3 Primary tool: Partitioning $n$-tuples of ordinals

The covering argument for $S^2$ depends heavily on the fact that we may partition the set $\omega_1^4$ of quadruples into four sets such that each set is **finite in a specified direction** i.e the set of points with three specified co-ordinates fixed and the fourth one variable is finite.

**Definition**

If $B$ is a subset of $A^I$ we say that $B$ is finite in direction $j$ if for any $(a_i')_{i\in I}$ in $A^I$ the set

$$\{(a_i)_{i\in I} \in B : a_i = a_i' \text{ for all } i \neq j\}$$

is finite.

**Theorem**. (Sierpiński, [10]) For each $m \geq 3$ the set $\omega_1^m$ may be partitioned into $m$ sets $\{A_i : i = 0, 1, 2, ..\}$ such that for for each $j$ the set $A_j$ is finite in direction $j$.

We will refer to such a family $\{A_i : i = 0, 1, 2, ..\}$ as a **Sierpiński partitioning** of the product. Although this theorem is normally quoted for $m = 3$ the same easy proof immediately gives the stronger form which we desire (see the antecedent paper of Sierpiński [9] and also Simms [11] for a history of this partitioning theorem). Indeed it is enough to take for each ordinal $\alpha \in \omega_1$ a well-ordering $\leq_\alpha$of $\alpha$ of finite type or of type $\omega$ (as appropriate) and to set

$$A_j = \{(a_i)_{i\in m} : \alpha = \sup\{a_i : i \in m\} \;\&\; a_j = \text{the } \leq_\alpha \text{-min of } \{a_i : i \in m\}\}.$$

These sets partition $\omega_1^n \backslash \Delta$ where $\Delta = \{(a_i)_{i\in I} : a_i = a_j \text{ for all } i, j\}$ is the diagonal (which is finite in all directions and so may be arbitrarily distributed to obtain a partition of $\omega_1^n$). The partitioning sets so defined are finite in all but one direction.

The partitioning result generalizes the idea that eg $\omega^2$ can be partitioned into two sets (say $\{(\alpha, \beta) : \alpha \leq \beta\}$ and complement) one of which is finite on each vertical and the other finite on each horizontal axis.

For $m \geq 3$ given a point $(x^0, x^1, ..., x^{m-1})$ in $S^m$ and a partition of $\omega_1^m$ we have in mind to associate with the given point and any integer $n$ the unique $j$ such that

$$(|x^0(n)|, |x^1(n)|, ..., |x^{m-1}(n)|) \in A_j.$$

We are prevented from doing quite this in the case $m = 2$ andn we are forced into a little complicating subterfuge; instead, we consider the quadruple:

$$(|x^0(2n)|, |x^0(2n-1)|, |x^1(2n)|, |x^1(2n-1)|).$$

It therefore becomes convenient to index the quadruples in $\omega_1^4$ in the form

$$(\xi_0^0, \xi_1^0, \xi_0^1, \xi_1^1) = (..., \xi_j^i, ...)$$

and likewise the partitioning sets $\Lambda_j^i$ which will be said to be finite in direction $(i, j)$.

We let

$$\begin{aligned} \Lambda_j^i(\eta) &= \Lambda_j^i((..., \eta_{j'}^{i'}, ...)) \\ &= \{\xi : \exists(..., \xi_{j'}^{i'}, ...) \in \Lambda_j^i \text{ and } \xi = \xi_j^i, \forall (i', j') \neq (i, j)[\xi_{j'}^{i'} = \eta_{j'}^{i'}]\} \end{aligned}$$

denote the finite set in direction $(i, j)$ corresponding to $\eta$.

Given a point $(x_0, x_1)$ in the square $S^2$ we shall exploit the fact that for any $n$ the canonical quadruple

$$\bar{\xi}(n) = (|x^0(2n)|, |x^0(2n-1)|, |x^1(2n)|, |x^1(2n-1)|)$$

uniquely determines $i, j$ such that

$$\bar{\xi}(n) \in \Lambda_j^i.$$

If we regard as fixed the three co-ordinates in $\bar{\xi}(n)$ other than the $(i, j)$ just determined, we can use topological separation to avoid the remaining finite number of points of $\Lambda_j^i$. The construction of an order-one cover which is locally finite will take advantage of the finitary character of the 'codes'$\bar{\xi}(n), i, j$. We shall wish to encode the $i, j$ and the signature of $x_0$ and $x_1$.

# 4 Encoding the product

We are about to define the Mrówka coding function $\Phi$, mapping $S^2$ into a triple lying in a set $R^2 \times \tilde{Z}$ where $\tilde{Z} = (\{0,1\}^3)^\omega = 8^\omega$, using the signatures of the two component points and a third item. Recall that

$$S_k = I^{k-1} \times (\omega_1 \backslash \omega) \times \omega_1^\omega, \; S_\infty = I^\omega.$$

When $x_0 \in S_{k_0}, x_1 \in S_{k_1}$ we **associate** an **index** $z \in \tilde{Z} = (\{0,1\}^3)^\omega$ with the pair $(x^0, x^1)$ as follows. Let us we write

$$k_i = 2n_i - j_i$$

even when $k_i = \infty$ (in which case $n_i = \infty$ with $j_i$ undefined). Introduce the shorthand equivalence that $(a,b,c) \simeq (A,B,C)$ iff $a = A, b = B$. We shall define for each $n$ the term $z(n) = (z(n,0), z(n,1), z(n,2)) \in \{0,1\}^3$ where the intention is that $z(n,0)$ selects an axis, $z(n,1)$ chooses between the dyadic order $2n$ and $2n-1$, and $z(n,2)$ describes the parity of a sign difference. We now indicate our intentions more fully.

## 4.1 Coding Intentions

Case 1. If $k_0, k_1$ are infinite we wish to arrange for each $n$ that

$$z(n) = (i, j, k)$$

where $i, j$ is selected uniquely so that

$$(x^0(2n), x^0(2n-1), x^1(2n), x^1(2n-1)) \in \Lambda_j^i$$

and where

$$k = |sgn(x^i(2n-j)) - sgn(x^i(2n-j-1))|.$$

Remaining Cases: This paradigm requires amendment in the other three cases when $k_0, k_1$ are not both infinite. We repeat the coding above at each $n$ except when $n$ is $n_0$ or $n_1$ and note how to fix the third co-ordinate when $z(n) \simeq (i,j)$: we select the third co-ordinate $z(n,2)$ so as to record sign changes in the sequence

$$(x^i(1), x^i(2), ..., x^i(k_i - 1), \alpha, z)$$

at all positions except $k_i$. This is in view of the fact that a dyadic rational of order $k_i$ cannot be uniquely coded arithmetically in a certain way (see a later section). We therefore put

$$z(n) = (i, j, |sgn(x^i(2n-j)) - sgn(x^i(2n-j-1))|), \text{ if } 2n-j < k_i,$$

and in order to include the later sign differences: $1-0, 0-0, 0-0, ...$ we require

$$\begin{aligned} z(n) &= (i, j, 1), \text{ if } 2n-j = k_i+1, \\ &= (i, j, 0), \text{ if } 2n-j > k_i+1. \end{aligned}$$

Note that this leaves $z(n,2)$ undefined when $2n-j = k_i$, which only occurs when we have both of $n = n_i$and $j = j_i$, where $n_i$ is finite, i.e. in at most two exceptional cases (two if $n_0, n_1$ are both finite). As we said, these require separate treatment. We now make sure that we are still able to follow this later paradigm of recording the sign difference just as before, and the idea is to require that at $n_s$ there is avoidance of $(s, j_s, 0)$ and $(s, j_s, 1)$ as index values.

Case 2. if $n_0$ is finite and $n_1 \neq n_0$ then

$$z(n_0) \notin \{(0, j_0, 0), (0, j_0, 1)\};$$

Case 3. if $n_1$ is finite and $n_1 \neq n_0$ then

$$z(n_1) \notin \{(1, j_1, 0), (1, j_1, 0)\}$$

(which might be termed an avoidance choice) and again we fix the third co-ordinate when $z(n_s) \simeq (i, j)$ by applying the preceding rules, which this time define $z(n_s, 2)$ since $(i, j) \neq (s, j_s)$ (i.e. we still select the third co-ordinate $z(n, 2)$ so as to record the sign changes at $2n-j$ in the sequence

$$(x^i(1), x^i(2), ..., x^i(k_i - 1), \alpha, z)$$

at all positions except $k_i$). Whilst it appears that we have not selected $z(n)$ uniquely in this case, we shall make amends by introducing an equivalence relation which makes any two avoidance choices equivalent. The avoidance choice falls into one equivalence class which thereby encodes the information that $k_i = 2n_i - j_i$.

Case 4. if $k_0, k_1$ are in some order the successors $2n-1, 2n$ then $z(n) \in \{0,1\}^3$ is arbitrary. As this is the only place where we might be accused of giving up we comment that any two choices for $z(n)$ will be regarded as falling into one equivalence class thereby signalling that $n_0 = n_1 = n$.

**Comment**: We will write in somewhat sloppy fashion

$$z(n) = (i, j, flip(i, j, n)),$$

where

$$flip(i, j, n) = |sgn(x^i(2n-j)) - sgn(x^i(2n-j-1))|.$$

# 5 An equivalence relation

We consider the index $z$ in association with $t^0$ and $t^1$ co-ordinate by co-ordinate. So consider a fixed $n$. Following our discussion of coding intentions, we place no restrictions on $z(n)$ when $t^0$ and $t^1$ are both dyadic of order either $2n$ or $2n-1$, i.e they are signatures of points from consecutive $S_k$ with $k \in \{2n, 2n-1\}$; thus so as far as $n$ is concerned any other $z'$ is as good as $z$; also if $t^s$ codes $S_k$ with $k \in \{2n, 2n-1\}$ say with $k = 2n-j$ but $z(n)$ is not $(s, j, 0)$ or $(s, j, 1)$ then any other $z'$ which is not $(s, j, 0)$ or $(s, j, 1)$ is as good as $z$. We treat them alike since neither contains the true order information at $n$. This corresponds to an intention to avoid this kind of truthful information in general.

Hence let us write

$$ord(t) = n \text{ if } t \text{ is constant from } n \text{ onwards.}$$

i.e. arithmetically $t$ is dyadic of order $n$. Now we have

$$z \sim z'$$

iff

$$\forall n[\text{either}(\{ord(t^0), ord(t^1)\} \subseteq \{2n, 2n-1\}) \text{ or}$$

$$(\forall s)[\exists j((ord(t^s) = 2n-j) \ \& \ z(n) \notin \{(s,j,0),(s,j,1)\}] \Rightarrow$$

$$\Rightarrow \ [z'(n) \notin \{(s,j,0),(s,j,1)\}]].$$

A rough translation: for each $n$ if $z$ has nothing to say at $n$ then $z'$ has nothing to say at $n$, and if $z$ is a liar at $n$ then $z'$ is a liar at $n$. Here $z$ is a liar at $n$ if $z(n) \neq (s, j, anything)$ when $ord(t^s) = 2n - j$.

### 5.1 Illustration of the equivalence classes

Fix $t^0, t^1$. Choose $n_s, j_s$ so that $t^s \in D_{2n_s - j_s}$. We illustrate the more interesting case $n_0 \neq n_1$. Define the truth set $T_0$ to be the set of codes $z(n_0)$ truthfully recording that $t^0 \in D_{2n_0 - j_0}$, i.e.

$$T_0 = \{a_0, a_1\} = \{(0, j_0, k) : k \in \{0, 1\}\}.$$

Let $F_0$ be the complementary set of codes. Denote by $T_1, F_1, b_k$, the corresponding sets and points in relation to $z(n_1)$.

Now consider the set of codes in $(t^0, t^1) \times \tilde{Z}$ projected onto the $(n_0, n_1)$-subspace i.e. down to the set of all $\{(z(n_0), z(n_1))\}$. This set contains $8 \times 8$ pairs $z$.

Working in this set suppose $z \sim z'$. Assume that $z(n_0) \in T_0$, i.e. $z(n_0)$ truthfully records that $t^0 \in D_{2n_0 - j_0}$. Then we must have $z(n_0) = z'(n_0)$. If $z(n_1) \in F_1$ then we must have $z'(n_1) \in F_1$. This argument shows that the equivalence classes are the following finite sets:

$\{a_0\} \times F_1, \{a_1\} \times F_1, F_0 \times F_1, F_0 \times \{b_0\}, F_0 \times \{b_1\}, \{(a_i, b_j)\}$ for $i, j \in \{0, 1\}$.

Thus the equivalence classes traced on $(t^0, t^1) \times \tilde{Z}$ are finite. See Figure 1.

**Remark.** The case $n_0 = n_1 = n$ leads to a single equivalence class in the $n$-subspace consisting of 8 points. Here again the equivalence classes traced on $(t^0, t^1) \times \tilde{Z}$ are finite.

# 6 The Mrówka coding function

### 6.1 The Intersection Diagram

Given the definition of an index, whereby for each $n$ we have

$z(n) = (i, j, flip(i, j, n))$, and intending to encode both the signatures and the sign flip we are interested in a subset of the product of the unit square with the space of indices, $[0,1]^2 \times (\{0,1\}^3)^\omega$ consisting of those $(t^0, t^1, z)$ such that

$$\forall n[t^i \text{ codes up at } 2n - j \text{ the flip } k \text{ when } z(n) = (i, j, k)].$$

A sequence space setting in logic books notation would want us to consider sequences $(t^0, t^1, z)$ in $2^\omega \times 2^\omega \times 8^\omega$ such that

$$\forall n(\exists i \in \{0,1\})(\exists j \in \{0,1\})(\exists k \in \{0,1\})(\exists l \leq 2n)$$

$$\{[i = z(n,0)] \ \&\ [j = z(n,1)] \ \&\ [l = 2n{-}j] \ \&\ [t^i(l){-}t^i(l{-}1) = k] \ \&\ [k \equiv z(n,2)]\},$$

where $\equiv$ denotes equality $(\mathrm{mod}\, 2)$,and the dependence on only a finite number of restrictions gives immediate proof that this defines a compact set. However, we are caught out by the need to include dyadic rationals of order $l$. This is rectified by taking an alternative route.

**Definition** The subset $F'$ of $[0,1]^2 \times (\{0,1\}^3)^\omega$ consists of those $(t^0, t^1, z)$ such that

$$\forall n \exists p[|t^{z(n,0)} 2^{\{2n - z(n,1)\}} - 2p| \leq 1 \text{ and } p = z(n,2)(\mathrm{mod}\, 2)].$$

This will be the domain for the intersection diagram proper. We explain this definition in the next section.

**Definition**

$$F = F' / \sim .$$

We shall see in a moment that this is also a compact metric space and shall show that $\dim F' \leq 1$.

**Definition** The **Mrówka coding function** is defined by reference to the equivalence relation as follows:

$$\Phi(x, y) = [\varphi(x), \varphi(y), z].$$

Thus $\Phi(S^2)$ records the signature for each pair in the Cantor space as well as sign flips.That $\Phi$ is continuous is proved later.

# 7 From codes to intervals: the arithmetic of coding

Before we obtain a cover of $S^2$ we construct a cover of $\Phi(S^2)$. This is done by covering a larger set, namely the set $F$, which we continue to think of as lying in $[0,1]^2 \times \tilde{Z}$. We therefore need to consider real intervals covering sets like $\{t \in 2^\omega : t(l+1) - t(l) = z(l,2)\}$.

The following lemma establishes a connection with coding the sign-flip. Recall that the intersection properties hinge on knowing where the flip occurs.

**Lemma**. Let $t = \sum_{k\geq 1} a_k 2^{-k}$ and $t_l = 2^l \sum_{k\geq l+1} a_k 2^{-k}$ with each $a_k \in \{0,1\}$. The following inequality for the integer $p$

$$|2^l t - 2p| \leq 1$$

has a unique solution iff $t$ is not dyadic of order $l$, i.e. not of the binary form $....1_l 0000$ or $....0_l 1111....$ (equivalently $a_l + t_l \neq 1$), given by

$$\begin{aligned} p &= a_{l-1} \text{ if } a_l = 0, \\ &= 1 + a_{l-1} \text{ if } a_l = 1, \end{aligned}$$

so that $p = a_l - a_{l-1} (\text{mod } 2)$. Otherwise, when $t$ is dyadic of degree $l$ the inequality has exactly two consecutive solutions.

We omit the proof which is routine.

**Comment 1**. When $p$ is uniquely defined, the parity of $p$ records whether or not there was a sign change from $a_{l-1}$ to $a_l$.

**Comment 2.** The value $m$ such that $p = 2m + i$ in both cases (whether $p$ be unique or not) is itself **unique;** the covering construction uses this. Of course this amounts to saying that the sequence $(a_1, a_2, ..., a_{l-2})$ is unique.

**Covering and coding: the key**. We can rewrite the above in the equivalent format without the (mod 2) notation

$$|2^l t - 4m - 2i| \leq 1,$$

by taking $p = 2m + i$. An important connection with dyadic points is given by the observation that when $i = 0$ the interval has dyadic end-points of order $l$, namely $(4m \pm 1)/2^l$, and contains dyadic points of order less than $l-1$ of form $m/2^{l-2}$; when $i = 1$ this interval contains a dyadic point of order $l-1$, namely $(2m+1)/2^{l-1}$.

Thus knowing $m, i$ as a unique code at level $l$ for the non-dyadic $t$ is equivalent to knowing the sequence of terms previous to $l$ in one of the two forms: $(a_1, ..., a_{l-2})$ and $a_{l-1}$ if $a_l = 0$, or $(a_1, ..., a_{l-2})$ and $1 + a_{l-1}$ if $a_l = 1$.

We shall need the suggestive terminology that $m$ **codes** $t$ **via** $i$ at **level** $k$ if

$$|2^k t - 2(2m + i)| \leq 1.$$

## 7.1 Metrization of the quotient $F$

If $\rho$ is any one of the canonical metrics on $[0,1]^2 \times \tilde{Z}$, then we can define a corresponding metric on the quotient space such that when $w, v$ are inequivalent elements the distance is

$$\rho([w],[v]) = \inf \sum_{i=1}^{n} \rho(w_{2i-1}, w_{2i}) \geq ||(t^0, t^1)_w - (t^0, t^1)_v||_2.$$

Here $(t^0, t^1)_w$ denotes the projection of $w$ onto $[0,1]^2 \times \tilde{Z}$ and similarly for $v$; $||.||_2$ is the usual norm in $R^2$, and the infimum is taken over all finite chains $\{w = w_1, w_2, ..., w_{2n} = v\}$ where alternate members of the chain are equivalent, that is

$$w_{2i-1} \sim w_{2i} \text{ for } i = 1, ..., n.$$

Evidently if $(t^0, t^1)_w = (t^0, t^1)_v$ we are able to verify that $\rho([w],[v]) > 0$ by looking at the projection onto $\tilde{Z}$. It is routine to verify the triangle inequality.

## 7.2 Why $\dim F \leq 1$.

Fix an integer $l$. This parameter will ultimately determine the diameter of a cover. Although initially $z(l)$ keeps track of sign flips at levels $2l$ and $2l-1$ on one of $t^0$ or $t^1$, we shall later constrain co-ordinates $z(n)$ for $n < l$ in order to include the $z$ indices in sets of small diameter.

We construct a covering the elements of which take the form

$$P_0(m) \cap P_1(m')$$

where $P_s(m)$ is a basic open set in $[0,1]^2 \times 8^\omega$ roughly speaking in the form of a strip with its narrow side on the $t^s$-axis and long side parallel to the $t^{1-s}$-axis and depending in the third direction only on $z(l)$. Ignoring the $z$-direction the covering elements $P_0(m) \cap P_1(m')$ are almost squares; whilst $P_s(m)$ will be contiguous to $P_s(m \pm 1)$, the presence of the third axis allows, by reference to $z(l)$, sufficient dispersion for there to be no points in the intersection

$$P_0(m) \cap P_0(m+1) \cap P_1(m') \cap P_1(m'+1). \tag{7.1}$$

This guarantees an order-one cover.

To bring about this dispersion we construct $P_s(m)$ from four pieces $P_s(m, i_0, i_1)$ for $(i_0, i_1) \in \{0,1\}^2$, themselves basic open sets, by considering whether $z(l)$ codes for $t^s$ or for $t^{1-s}$. In this connection it is helpful to think of the set $[0,1]^2 \times 8^\omega$ as consisting after transposition of the four disjoint pieces

$$[0,1]^2 \times \{z : z(l) \simeq (s, i_1)\} \times 8^\omega$$

$$[0,1]^2 \times \{z : z(l) \simeq (1-s, i_1)\} \times 8^\omega$$

for $i_1 \in \{0,1\}$; then it will be clear that we shall have constructed a series of four strips in the $t^0, t^1$-plane.

At this stage it is important to recall how we may convert a code $m$ and $y$ to a covering interval.

a) At level $2l$ $(for\ j = 0)$ the members $t^s$ of the interval

$$|2^{2l}t^s - 4m - 2z(l,2)| \leq 1.$$

may be regarded as coding binary sequence $t^s$ for which

$$z(l,0) = s, z(l,1) = 0, z(l,2) = a_{2l} - a_{2l-1} \bmod 2.$$

so we will code these two possibilities for $z = 0$ and $y = 1$ on two separate strips; each carrying a label $(s, 0, y)$ for $y = 0$ or $y = 1$; note that for $y = 0$, as $m$ varies, the $t$ intervals

$$|2^{2l}t - 4m| \leq 1$$

are pairwise disjoint and similarly for $y = 1$, as $m$ varies, the intervals

$$|2^{2l}t - 4m - 1| \leq 1$$

are disjoint, so we are combining book-keeping with disjointness.

b) Similarly at level $2l - 1$ (for $j = 1$) the members $t^s$ of the interval

$$|2^{2l-1}t^s - 2m| \leq 1, \text{ and } z(l, 2) = m (\mathrm{mod}\, 2).$$

may be regarded as coding binary sequence $t^s$ for which

$$z(l, 0) = s, z(l, 1) = 1, z(l, 2) = m (\mathrm{mod}\, 2).$$

This may be reformulated in terms of an interval, centered on $4m$, as a level $2l$ interpretation again but with the double the width used above

$$|2^{2l}t^s - 4m| \leq 2$$

and we shall use these coding intervals on the two strips $(s, 1, y)$ with $y = 0$ and $y = 1$. Note that, as $m$ varies but with *fixed* parity $y$, although the intervals are twice as wide they remain disjoint.

See Figure 2.

**Geometric coding: the intuition**

We should perhaps explain our coding intuition so far. An analogy with computers might be helpful. If we regard these four strips as magnetic tape we have

$$\cup_m (t^s\text{-interval coded by } m \text{ via } z(l, 2)\ ) \times [0, 1]$$

polarised 'plus' and its complement 'minus'; if a reading head finds any plus it reads that $t^s$ lies specifically in the interval encoded by $m$. It is important to notice that $m$ is decoded because the corresponding area of 'plus' polarised information is delimited (as a function of $m$ and and $z(l, 2)$) and the limits detectable because the 'plus' area is separated by 'minus' polarised areas. We regard $z(l)$ as a byte of information coming from a set of eight possibilities.

But now we have used up four possibilities, namely those with $z(l, 0) = s$, for a geometric coding which has dispersed the $m$-coded intervals of the $t^s$ axis onto four strips in the form

$$(t^s\text{-interval coded by } m \text{ via } z(l, 2)\ ) \times [0, 1].$$

So far the magnetic tape carries no $t^{1-s}$ information. To promote symmetry, the remaining bytes with $z(l,0) = 1-s$ cannot therefore contain $m$-coded information about $t^s$ unless we put $m$-coded information about $t^{1-s}$ into the first four bytes. So we go back to our first four strips and must now devise a way to make use of the $t^{1-s}$ axis without harming the detectability of the 'plus' area (or of decoding the relevant $m$). The simplest idea is to **superimpose** a second channel as computer scientists would say, say by splitting the tape down the middle, parallel to the $t^s$-axis, into a **front** and **back half**, so to speak; both halves need to be closed if we are to achieve a covering by closed sets. If the front half preserves all the delimiting 'minus' polarisation needed to decode $m$, we can use the back half to superimpose additional information by changing the polarisation, and of course this information allows us to classify almost all the points of the $t^{1-s}$-axis as falling into one of two sets (possibly to both sets when we are at the common boundary points of the two sets). The superimposition idea now entitles us to place intersection-diagram information on the other four strips with $z(l,0) = 1-s$. It is not obvious at first how to do this, as symmetry considerations require us to encode information about *all* values for $t^s$ (because we included all values of $t^{1-s}$ in the splitting idea). So all we can do is intelligently choose closed intervals $I_m$ covering the $t^s$-axis in such a way that points in the intersection $I_m \cap I_{m+1}$ (themselves delimiters)give useful information, i.e. if $I_m = [4m-k, 4m+4-k]/2^{2l}$ we need an inspired choice for $k$, and it is intuitively clear that we want $k$ to be rational because this gives dyadic rationals as points in the intersection. This we now explain.

**To sum up.** So far we know that if

$$(t^0, t^1, z) \in P_s(m) \cap P_s(m+1)$$

then for $z(l,0) = s$ the point $t^{1-s}$ will be in the back half-strip because the $m$-coded interval is disjoint from any $(m+1)$-coded interval in the front half-strips and for $z(l,0) = 1-s$ the point $t^s$ lies in $I_m \cap I_{m+1}$, i.e. is of form $4m-k$.

Our first aim is to *not* have any points with

$$(t^0, t^1, z) \in P_0(m) \cap P_0(m+1) \cap P_1(m) \cap P_1(m+1);$$

if this happened, suppose without loss of generality that $z(l,0) = 0$ : then we would have

$$t^0 \text{ is in the back-half and } t^0 = (4m-k)/2^{2l}. \tag{7.2}$$

Thus our first aim is to derive a contradiction from this.

Our second aim is to ensure that the covering satisfies:

$$(t^0, t^1, z') \sim (t^0, t^1, z) \in P_s(m) \Rightarrow (t^0, t^1, z') \in P_s(m),$$

i.e. closure under equivalence. Our attention focuses therefore on the use of the information in $z(l)$ when $z(l)$ codes information about $t^s$ and $z'(l) \neq z(l)$.

If the truth of $t^s \in I_m$ is independent of $z$ (i.e. $k$ is selected without reference to $z(l)$) then we need consider only the situation when $(t^0, t^1, z)$ is in the front strip. If we could conclude from that circumstance that $t^{1-s}$ is not dyadic of order $2l$ or $2l-1$ then from $z'(l) \neq z(l)$ we immediately infer that $t^s$ is dyadic of order $2l$ or $2l-1$. Our second aim will be achieved if we ensure that whenever $z(l) \notin \{(s,0,0),(s,0,1)\}$ a specific dyadic rational $d$ of order $2l$ or $2l-1$ is in the $t^s$-interval coded by $m$ and $z(l,2)$ then $d$ is in the $t^s$-interval coded by $m$ and $z'(l,2)$.

Now if $t^{1-s}$ is dyadic of order $2l$ or less then $2^{2l+2}t^{1-s}$ is an integer divisible by 4. So the simplest way of defining the back-half set is

$$\begin{aligned} B_s &= \{t : (\exists h)|2^{2l+2}t - 4h| \leq 1\} \\ &= \cup_h \{t : |2^{2l+2}t - 4h| \leq 1\}, \end{aligned}$$

which is closed. The closure of its complement will be denoted by $P_s$. Thus

$$P_s = \{t^s : \forall h(|2^{2l+2}t^{1-s} - 4h| \geq 1\}.$$

Referring to (??) we see that if $t^0$ is not in the back-half then for some $h$

$$|2^{2l+2}t^0 - 4h| \leq 1,$$

so if $2^{2l}t^0 = 4m - k$ is an integer we have

$$16m - 4k = 4h \text{ or } 4h \pm 1.$$

Modulo 4 we want to avoid $0, 1, 3$ as values for $4k$ so we take $4k = 2$ i.e. $k = 1/2$. We are now ready for the formal definitions.

**Definition of $P_s(m, i_0, i_1)$**

Case 1: $i_0 = s, i_1 = 0$.Thus

$$z(l,0) = s, z(l,1) = 0,$$

i.e. we think of $z$ as a coding for the signature $t^s$ of a point with order $2l$; thus for $t^s$ non-dyadic

$$z(l,2) = a^s_{2l} - a^s_{2l-1} (\mathrm{mod}\, 2).$$

We put $P_s(m,s,0)$ to be the union for $y \in \{0,1\}$ of

$$\begin{aligned}&(I_m \times B_s) \cup \\ &\cup \left(\{t^s : |2^{2l+2}t^s - 4m - 2y| \leq 1\} \times P_s\right),\end{aligned}$$

crossed with

$$\{z : z(l) \simeq (s,0,y)\}$$

which is clearly a closed subset.

Case 2. $i_0 = s, i_1 = 1$.Thus

$$z(l,0) = s, z(l,1) = 1, z(l,2) = m (\mathrm{mod}\, 2).$$

We put $P_s(m,s,1)$ to be the union for $y \in \{0,1\}$

$$\begin{aligned}&(I_m \times B_s) \\ &\cup \left(\{t^s : |2^{2l-1}t^s - 2m| \leq 1 \;\&\; y \equiv m\} \times P_s\right),\end{aligned}$$

crossed with

$$\{z : z(l) \simeq (s,1,y)\}.$$

Case 3. $i_0 = 1-s, i_1$ arbitrary. We put

$$P_s(m, 1-s, i_1) = \{(t^0, t^1, z) : |2^{2l+1}t^s - 8m - 1| \leq 4\} \times \{z : z(l) \simeq (1-s, i_1)\}.$$

Evidently, to cover the interval $0 \leq t2^{2l+1} \leq 2^{2l+1}$ we need $m \leq 2^{2l-2}$.

It is easy to prove that for $m = 0, \ldots, 2^{2l+1-3}$ the sets

$$P_s(m) = \smile \{P_s(m, i_0, i_1) : (i_0, i_1) \in \{0,1\}^2\}$$

cover $F'$. They also cover $F$ as we shall see.

**Covering Proof.** Fix $s$. Consider $(t^0, t^1, z)$. Let $i_0 = z(l, 0)$ and $i_1 = z(l, 1)$.

First suppose that $i_0 = 1 - s$, select $m$ so that $(t^0, t^1, z) \in I_m$.

Next suppose that $i_0 = s$. If there is zero flip on $t^{1-s}$ at level $2l + 2$ just choose $m$ appropriately. If however there is a positive flip on $t^{1-s}$ argue as follows. Suppose first that $z(l, 1) = i_1 = 0$; then by definition of $F'$ there is an $m$ such that

$$|2^{2l-0}t^s - 4m - 2z(l, 2)| \leq 1,$$

which puts $(t^0, t^1, z)$ in $P_s(m, s, 0)$. Next suppose that $z(l, 1) = i_1 = 1$; then by the same definition we have for some $m$ that

$$|2^{2l-1}t^s - 4m - 2z(l, 2)| \leq 1,$$

or equivalently

$$|2^{2l-1}t^s - 2m'| \leq 1,$$

where

$$m' = z(l, 2) \pmod 2.$$

This puts the point $(t^0, t^1, z)$ in $P_s(m', s, 1)$.

## 7.3 Why the $z$ we construct puts $(sgn(x^0), sgn(x^1), z)$ in $F'$

This is really a formality.

**Lemma**: If $z$ is the index for $x^0, x^1$, then

$$(\varphi(x^0), \varphi(x^1), z) \in F'.$$

**Proof**. Our goal is to verify that if $t^s = \varphi(x^s)$ and $z$ is the corresponding index then

$$\forall n \exists p [|t^{z(n,0)} 2^{\{2n - z(n,1)\}} - 2p| \leq 1 \ \& \ p = z(n, 2) (\bmod 2)].$$

Suppose $x_s \in S_{k_s} = I^{k_s - 1} \times (\omega_1 \backslash \omega) \times Z$. Fix $n$, and suppose $z(n) = (i, j, anything)$. Put $l = 2n - j$.

Now

$$\begin{aligned} a^i_q &= sgn(x_i(q)) \text{ for } q < k_i, \\ &= 1, q = k_i, \\ &= 0, q > k_i. \end{aligned}$$

First suppose that $l < k_i$ then according to our coding scheme,

$$\begin{aligned} z(n,2) &= flip(i,j,n) \\ &= |sgn(x_i(2n-j) - sgn(x_i(2n-j-1))| \\ &= |a^i_l - a^i_{l-1}|, \end{aligned} \tag{7.3}$$

but $|t^{z(n,0)}2^{\{2n-z(n,1)\}} - 2p| \leq 1$ always has a solution with $p = a^i_l - a^i_{l-1} (\bmod 2)$.

If $l = k_i$ then $a^i_l = 1$ and $t_l = 0$, so there are two solutions to

$$|t^{z(n,0)}2^{\{2n-z(n,1)\}} - 2p| \leq 1,$$

and one of these will solve also $p = z(n,2) (\bmod 2)$.

If $l = k_i + 1$ then $a^i_l = 0$ and $a^i_{l-1} = 1$, so $z(n,2) = 1$. Similarly if $l > k_i + 1$ then $a^i_l = a^i_{l-1} = 0$ so $z(n,2) = 0$. In both these cases our requirements will be satisfied with the choice $p = a^i_l - a^i_{l-1} = z(n,2) \pmod 2$.

## 7.4 Why $\Phi$ is continuous

Our goal, given any points $x^0, x^1$, is to find for every $p$ an $m \geq p$, so that we shall have $z_1(n) = z_2(n)$ for all $n \leq p$ provided

$$(y^0, y^1) \in U_m(x^0) \times U_m(x^1),$$

and

$$z_1 \text{ is an index for } \varphi(x^0), \varphi(x^1)$$

where

$$z_2 \text{ is an index for } \varphi(y^0), \varphi(y^1).$$

Without loss of generality we shall suppose that $(y^0, y^1) \in S_I = I^\omega$ (otherwise we have agreement between $y^s$ and $x^s$ up to co-ordinate $p$).

Suppose for greatest generality that $(x^0, x^1) \in S_{k_0} \times S_{k_1}$, with $k_s = 2n_s - j_s$ finite, and $n_0 \neq n_1$. Put

$$\eta_s = (|x^0(2n_s)|, |x^0(2n_s - 1)|, |x^1(2n_s)|, |x^1(2n_s - 1)|).$$

Now $\Lambda^s_{j_s}(\eta_s)$ is finite, so there exists $m_s$ such that we can separate $x^s(k_s) = x^s(2n_s - j_s)$ from all $\xi \in \Lambda^s_{j_s}(\eta_s)$ different from $x^s(k_s)$ by using the depth $m_s$ neighbourhood of $x^s(k_s)$. Let

$$m = \max\{m_s, 2n_s : s = 0, 1\}.$$

So if $(y^0, y^1) \in U_m(x^0) \times U_m(x^1)$ we make the following the assertion.

**Claim.**

$$\zeta_s = (|y^0(2n_s)|, |y^0(2n_s - 1)|, |y^1(2n_s)|, |y^1(2n_s - 1)|) \notin \Lambda^s_{j_s}(\eta_s).$$

**Proof of claim.** Suppose $j_s = 0$. Then since $y^s \in U_m(x^s)$ we have $y^s(k_s - 1) = x^s(k_s - 1)$ and similarly since $y^{1-s} \in U_m(x^{1-s})$ we have $y^{1-s}(k_s) = x^{1-s}(k_s)$ and $y^{1-s}(k_s - 1) = x^{1-s}(k_s - 1)$. This is because the topology usually requires the co-ordinates to agree. Thus $\zeta_s = \eta_s$ in the three directions other than $(s, j_s)$. Finally $y^s(k_s)$ is close enough to $x^s(k_s)$ to avoid all but possibly one point of the set $\Lambda^s_{j_s}(\eta_s)$. (We avoid as many $\bar{\alpha}(i)$ as we need to by selecting $m_s$ large enough; in Kulesza's space we avoid the finite number of sets $S_i(\alpha)$ which meet $\Lambda^s_{j_s}(\eta_s)$.) But the point $y^s$ is not in $x^s \times \{\alpha\} \times Z$, and so $|y^s(k_s)|$ is not equal to $|x^s(k_s)|$ by the relation (2.1). Thus $\zeta_s \notin \Lambda^s_{j_s}(\eta_s)$.

Now it is the case that for all $n$ we have $\{k_0, k_1\} \neq \{2n, 2n - 1\}$, so the coding device $z_2$ for $(y^0, y^1)$ is unique and our definition of association (see Section 4) *requires* that $z_2(n_s) \not\cong (s, j_s)$. Now $\{2n_0, 2n_0 - 1\}$ is disjoint from $\{2n_1, 2n_1 - 1\}$, so $k_s \in \{2n_0, 2n_0 - 1\}$ and $k_{1-s} \notin \{2n_0, 2n_0 - 1\}$; accordingly we are required to have $z_1(n_s) \not\cong (s, j_s) \not\cong z_2(n_s)$. But $(x^0, x^1) \in S_{k_0} \times S_{k_1}$ so the signatures $t^s$ are dyadic of order $2n_s - j_s$. By ambiguity of solutions to the inequality

$$|2^{2n_s - j_s} t^s - 2p| \leq 1$$

we are then able to satisfy the condition $p = z_2(n_s) \pmod 2$ on $F'$, and thus have $z_1(n_s) = z_2(n_s)$ as required.

# 8 Extension to finite product spaces.

## 8.1 Continuity Theorem revisited.

It is clear that when working in $\omega_1^m$ (with $m \geq 3$ a finite integer) the substitute definition in the form

$$\eta_s = (|x^0(n_s)|, |x^1(n_s)|, ..., |x^{m-1}(n_s)|),$$

where for $s = 0, 1, ..., m-1$ the point $x^s$ has order $k_s = n_s$, will work just as well and make it possible to prove that

$$\Phi(x^0, ..., x^{m-1}) = [(\varphi(x^0), ..., \varphi(x^{m-1}), z)]$$

is continuous. Of course now the index $z$ is easier to define (there is no need to worry about the adjacent values $2n, 2n-1$) and has $z(n) = (i, k) \in 2^2$ defined so that

$$\eta_s \in \Lambda_i$$

where $\{\Lambda_i : i = 0, 1, ..., m-1\}$ is a Sierpiński partition of $\omega_1^m$ and
$k = |x^i(n) - x^i(n-1)|$ provided $n \neq n_s$; moreover we require $z(n_s) \neq (s, k)$ for $k = 0, 1$.

## 8.2 Covering finite products

The geometric coding of the last section devised for the square readily transfers to higher powers of $S$ and in fact with somewhat less fuss (in view of Sierpiński's partitioning theorem at the power concerned). It is enough to consider the case of $S^3$; here the index $z$ lies in $(3 \times 2)^\omega$ (as the axis selector $z(l, 0)$ lies in$\{0, 1, 2\}$ and the parity code $z(l, 1)$ is again in $\{0, 1\}$) and the level $l$ now corresponds to the binary coefficient at power $2^{-l}$ rather than one of $2^{-2l}$ or$2^{-2l+1}$.

To define $P_s(m)$ for $s \in \{0, 1, 2\}$ let us agree a simplifying notation that the superscript $t^s$ is to be read modulo 3; we let $!^s$ denote the operation taking the vector $(t^s, t^{s-1}, t^{s+1})$ via the transposition $(t^{s-1}, t^s, t^{s+1})$ to its cyclic reordering in which $t^0$ comes first.

Now proceed just as earlier in the geometric coding, but replace the second-axis (i.e the $t^{1-s}$-axis of that Section) by the $(t^{s-1}, t^{s+1})$-plane. That planar set must carry a superimposed chanel of coding, so we defineo two subsets of the plane: a back piece

$$B_s = \left(\{t^{s-1} : \exists h(|2^l t^{s-1} - 4h| \leq 1)\} \times [0,1]\right) \cup$$

$$\cup \left([0,1] \times \{t^{s+1} : \exists h(|2^l t^{s+1} - 4h| \leq 1)\}\right)$$

with the front piece $P_s$ being the closure of its complement. Then, if $z(l,0) = s$ we take $P_s(m)$ to be the $!^s$-cyclic re-ordering of

$$(\{t^s : t^s \in I_m\} \times B_s) \cup$$

$$\cup \left(\{t^s : |2^l t^s - 4m - 2z(l,1)| \leq 1\} \times P_s\right)$$

crossed with

$$\{z : z(l,0) = s\}$$

and otherwise we take $P_s(m)$ to be

$$!^s \left(\{t^s : t^s \in I_m\} \times [0,1]^2\right) \times \{z : z(l,0) \neq s\}.$$

Now suppose $(t^0, t^1, t^2, z)$ is in the intersection

$$P_s(m) \cap P_s(m+1),$$

then if $z(l,0) = s$, the point $(t^{s-1}, t^{s+1})$ will be in the back set, since the front sets are mutually disjoint. If $z(l,0) \neq s$, then the point $t^s$ lies in $I_m \cap I_{m+1}$.

As usual $\Phi(x^0, x^1, x^2) = [\varphi(x^0), \varphi(x^1), \varphi(x^2), z]$. The covering of $\Phi(S^3)$ this time of is by the sets

$$P_0(m) \cap P_1(m') \cap P_2(m'').$$

Thus if $(t^0, t^1, t^2, z)$ is in the intersection

$$P_0(m) \cap P_1(m') \cap P_2(m'') \cap P_0(m+1) \cap P_1(m'+1) \cap P_2(m''+1),$$

then, as before, assume without loss of generality, that $z(l,0) = 0$; now $(t^0, t^1, t^2, z) \in P_0(m) \cap P_0(m+1)$ implies that one of $t^1, t^2$ obeys the relevant inequalities in the definition of $B_s$, say it is $t^1$. But $(t^0, t^1, t^2, z) \in P_1(m') \cap P_1(m''+1)$, so $t^1$ lies in $I_m \cap I_{m+1}$ and this is the same contradiction.

The covering will be closed under equivalence, since if $(t^{s-1}, t^{s+1})$ is in not in the back set both points $t^{s-1}$ and $t^{s+1}$ are not dyadic of order up to and including $l$. It follows that when $z(l) \neq z'(l)$ the point $t^s$ will be dyadic of order $l$. Now if $(t^0, t^1, t^2, z) \in P_s(m)$ and $z(l) \notin \{(s,0),(s,1)\}$ we shall have that $2^l t^s$ is an odd integer in the interval $[4m - 2 + \frac{1}{2}, 4m + 2 + \frac{1}{2}]$. Thus this integer is equal to $4m \pm 1$. Consequently, if $z'(l) \notin \{(s,0),(s,1)\}$ and $(t^{s-1}, t^{s+1})$ is not in the back set, then $(t^0, t^1, t^2, z') \in P_s(m)$.

# 9 Closing arguments: a sketch

The cover of $S^2$ constructed in the last but one section may be made arbitrarily small in the $(t^0, t^1)$-direction by choice of $m$. Given that we are concerned only by the information in $z(l)$, all we need do to obtain smallness in the $z$-direction is to restrict the previous $l-1$ co-ordinates of $z(n)$. This yields a covering by the rectangles

$$R(m_0, m_1, z'|l-1) = R(m_0, m_1) \cap [0,1]^2 \times \{z : z(n) = z'(n) \text{ for } n = 1, 2, ..., l-1\},$$

where $z' \in \tilde{Z}$ is arbitrary. If for some $(t^0, t^1, z)$

$$(t^0, t^1, z) \in (R(m_0, m_1, z'|l-1) \cap R(m_0, m_1, z''|l-1),$$

then in these circumstances we have for all $n < l$ that

$$z'(n) = z(n) = z''(n), \text{unless } n = n_s, \tag{9.1}$$

$$z'(n_s) \not\cong (s, j_s) \not\cong z''(n_s), \tag{9.2}$$

where $m_s/2^{2l-2}$ is dyadic of order $2n_s - j_s$, and conversely; so we may call $z'|l-1, z''|l-1$ equivalent under these circumstances. The following union over all $z''|l-1$ equivalent to $z'|l-1$ is a union of pairwise intersecting sets:

$$\widetilde{R}(m_0, m_1, [z'|l-1]) = \cup_{z''} R(m_0, m_1, z''|l-1).$$

There is a diameter bound on each such set, at most double that for the individual terms (since if $x, y$ are points in the union belonging to two sets with a common point $z$ then $d(x,y) \leq d(x,z) + d(z,y)$). But the individual terms are bounded uniformly as a function of $l$ (since $m_s \leq 2^{2l-2}$).

Finally, if we construct out of our finite, closed, order-one coverings of $F$ a self-refining sequence $\{\mathcal{B}_n\}$ with members of $\mathcal{B}_n$ all of diameter $2^{-n-2}$, we can extract from it another such sequence $\mathcal{A}_n$ refining the basic open sets by taking intersections of basic open sets in $S^2$with $\Phi^{-1}[A]$. To do this first fix $n$; next we find for each $A \in \mathcal{B}_n$ two dyadic rationals $d^0, d^1$ of highest order such that

$$A \subseteq [I[d^0] \times I[d^1] \times \tilde{Z}].$$

The intersections with $\Phi^{-1}[A]$ will be with sets of the form

$$U_{n+1-j_0}(x^0) \times U_{n+1-j_1}(x^1). \tag{9.3}$$

Here $x^s$ with $s \in \{0,1\}$ is arbitrary but subject to the requirements that

**either** $x^s \in S_k$ with $k = ord(d^s)$ and the signature of $x^s$ agrees with $d^s$ (i.e. $\varphi(x^s) = d^s$);

**or** $x^s \in S_\infty$ and

$$|x^s(l)| \in |x^s(l+1)|,$$

where $l = ord(d^s)$. The condition here mathches the situation occurring at the $l$-th term when $x^s$ is a point in the stream associated with the shell $P((x^s(1), x^s(2), ..., x^s(l-1))$ of depth $l$ whose signature agrees with $d^s$.

Finally, to have the correct basic open set depth in (9.3), we want $j_s = 1$ when $ord(d^s) = n$, and $j_s = 0$ when $ord(d^s) < n$. This ensures that the family of such sets $\mathcal{A}_n$ refines the natural basis $\mathcal{U}_n = \{U \times U' : U, U'$ are of depth $n\}$. By construction it is a locally finite, order-one family of closed sets.

Figure 1 Equivalence classes.

Figure 2 The sets $P_s(m)$.

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# 11 Appendix: Routine calculations

## 11.1 Cover closure under equivalence

Suppose that $(t^0, t^1, z') \sim (t^0, t^1, z) \in P_s(m)$. We wish to show that $(t^0, t^1, z') \in P_s(m)$.

So suppose that $z(l) \neq z'(l)$. If $t^{1-s}$ belongs to the superposition interval at level $2l+2$, i.e.

$$|2^{2l+2}t^{1-s} - 4h| \leq 1,$$

then clearly $(t^0, t^1, z') \in P_s(m)$ if and only if $t^s \in I_m$, which is true by assumption.

Now suppose that

$$t^{1-s} \text{ does not belong to the superposition interval at level } 2l+2. \quad (*)$$

Thus by definition of that interval, the number $t^{1-s}$ is not dyadic of order $2l$ or $2l-1$. So, since $z(l) \neq z'(l)$, it must be that $t^s$ is dyadic of order $2l$ or $2l-1$ (see the definition of equivalence).The proof divides into two cases.

(a) $ord(t^s) = 2l$ and $z(l), z'(l) \notin \{(s,0,0),(s,0,1)\}$.

If $z(l,0) \simeq (s,1)$, then referring to the second summand in case 2 of the definition, $2^{2l}t^s$ is an odd integer in the interval $[4m-2, 4m+2]$ and so (see Figure 2) we must have $2^{2l}t^s = 4m \pm 1$.

Next suppose that $z(l,0) = 1-s$. Then, referring to case 3, we have $2^{2l}t^s$ is an odd integer in the interval $[4m-2+\frac{1}{2}, 4m+2+\frac{1}{2}]$ and so we must again have $2^{2l}t^s = 4m \pm 1$.

Either way $2^{2l}t^s = 4m \pm 1$. The argument is reversible in that if $2^{2l}t^s = 4m \pm 1$ and $z'(l) \notin \{(s,0,0),(s,0,1)\}$, then subject to (*) $(t^0, t^1, z') \in P_s(m)$.

(b) $ord(t^s) = 2l-1$ and $z(l), z'(l) \notin \{(s,1,0),(s,1,1)\}$. Here $2^{2l}t^s$ is an even integer.

If $z(l,0) \simeq (s,0)$, then referring to the second summand in case 2 of the definition, $2^{2l}t^s$ is a $2\times$odd integer in the interval $[4m+y-1, 4m+y+1]$ for $y=0$ or $y=1$; so we must have $y=1$ and $2^{2l}t^s = 4m+2$, i.e. $2^{2l-1}t^s = 2m+1$.

Next suppose that $z(l,0) = 1-s$. Then, referring to case 3, we have $2^{2l}t^s$ is a $2\times$odd integer in the interval $[4m-2+\frac{1}{2}, 4m+2+\frac{1}{2}]$ and so we must again have $2^{2l}t^s = 4m+2$, i.e. $2^{2l-1}t^s = 2m+1$.

Either way $2^{2l-1}t^s = 2m+1$. The argument is again reversible in that if $2^{2l-1}t^s = 2m+1$ and $z'(l) \notin \{(s,1,0),(s,1,1)\}$, then subject to (*) $(t^0, t^1, z') \in P_s(m)$.

**Remark** Subject to (*), if $ord(t^s) \leq 2l-2$, we have of course that $2^{2l}t^s$ is divisible by 4 and

case 1: lies in the interval $[4m+2y-1, 4m+2y+1]$ for $y=0$ or $y=1$, so it must be that $2^{2l}t^s = 4m$;

case 2: lies in the interval $[4m+2, 4m-2]$ , so it must be that $2^{2l}t^s = 4m$;

case 3: lies in the interval $[4m+\frac{1}{2}-2, 4m+2+\frac{1}{2}]$, so it must be that $2^{2l}t^s = 4m$.

Thus in all cases $2^{2l-2}t^s = m$, as might be expected, given the encoding role of $m$ in the definition of $P_s(m)$.