THE RUBINSTEIN GAME WITH PRIOR CLAIMS & FIXED CHARGE BARRIER PENALTIES FOR REVOCATIONS

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Abstract

We determine the multiple sub-game perfect Nash equilibria arising in the Rubinstein bargaining game when both players make prior claims on the so-called cake i.e. pre-commit to accept at least a certain percentage of the cake subject to forfeiting a fixed percentage c of the cake (to a third party) if, setting aside their claim, they accept below the prior claim level. This is a study in the treatment of discontinuity, which requires the expanded apparatus of pseudo-fixed points and quasi Nash equilibria (permitting infinitesimal tolerance). We show how to interpret the smallest and largest equilibrium as divisions in a long-term bargaining game with the termination condition that Player-2 (respectively Player-1) is free to choose the final division. It is shown that under the prior claims regime it is not optimal to claim too much; moreover to sustain the standard equilibrium requires a limited penalty, and otherwise the Nash equilibrium shifts upwards for Player-1 to

$$\frac{1 - c - \delta + 2c\delta}{1 - \delta^2}$$

where δ is the common discount factor.

1. Rubinstein's Bargaining Game : an amendment

We describe an amendment to the Rubinstein bargaining model involving prior claims and a cost associated with accepting a proposal below the prior claim as a possible answer to a problem posed to us by A. Muthoo (in c.1989). See Muthoo (1996) for a discussion of the motivation which amounts to modelling a situation where the bargainers seek to influence each other's strategies by making it costly for themselves to set aside their prior claims. Our results differ from those of Muthoo in that we are faced with multiple Nash Equilibria. We begin with some terminology by way of recalling the Rubinstein (1987) model of bargaining with its standard equilibrium calculated via a fixed point; we then introduce the amended game and discuss natural limitations on the fixed charge c. We next relate the occurrence of multiple equilibria to discontinuous present value functions and discuss their fixed points and pseudo-fixed points connecting these to quasi Nash equilibria. Our main results are then presented and followed by an analysis of compensatory payments and an interpretation of the largest and smallest equilibrium. We close with a summary calculation of fixed points; from these we derive a series of propositions proving our main results. In this final argument we exploit generic aspects of the relation of the fixed (percentage) charge c to the discount factor δ (for large enough δ exceeding the 'golden ratio' $(\sqrt{5}-1)/2$) thus justifying computer derived figures (the initial source of inspiration, by unashamed admission).

1.1. The Rubinstein model: a review

Players 1 and 2 take turns in proposing the division of the "cake" into a proportion x to be awarded to the player currently bidding and a proportion $1-x$ to the opponent. The opponent may either agree the division or refuse it and in so doing close the current round and become in turn the bidder of the new round. At the end of each round the "cake" shrinks, i.e. loses its value (ever the bird in the hand worth two in the bush), just as a dollar gained at a future date is less than one obtained now and earning interest. Denoting by (x,n) the contract agreed in round n awarding a proportion x of the cake to Player-1 and $1-x$ to Player-2, the value (utility) of their cut of the cake may be modelled for the two players as follows. In the case of Player-1 when $n=0$ the utility is:

$$u_1(x,0) = x$$

and more generally

$$u_1(x,n) = \delta_1^n \cdot x,$$

where $0 < \delta_1 < 1$ denotes the amount by which Player-1 perceives the value to have shrunk at the end of one round. A similar formula may be written for Player-2 (viz. $(u_2(x,0) = 1-x, \quad u_2(x,n) = \delta_2^n \cdot (1-x))$. This is just one possible modelling of the utility evaluation.

Under "normal" circumstances there is for each Player-i a well-defined "present value" function $d_i(x)$ relating i's utility of a contract awarding x to Player-1 in the next round to an equivalent contract awarding $d_i(x)$ to Player-1 in the current round, so that:

$$u_i(d_i(x),0) = u_i(x,1), \qquad (i = 1,2). \tag{1.1}$$

Under the normal circumstances, it will also be true that

$$u_i(d_i(x),n) = u_i(x,n+1), \qquad (i = 1,2; n = 0,1,...).$$

Thus a pair of numbers x, y satisfying the equations:

$$y = d_1(x), \quad x = d_2(y),$$

prescribes an equilibrium in the game. Why? Suppose that Player-1 bids for more than x, say for $z > x$, then Player-2 would receive $1-z < 1-x = 1-d_2(y)$; so Player-2 prefers to bid y in the next round, hence rejects the bid. Similarly, if in the next round Player-2 offers less to Player-1 than y, then Player-1 prefers to bid x in the next round (since $u_1(d_1(x),1) = u_1(x,2)$). The existence of a unique equilibrium is guaranteed in general models by way of an intersection point

$$(x, d_1(x)) = (x,y) = (d_2(y),y)$$

common to the graphs of the two functions d_1, d_2, appropriately presented say as in Figure 1. (Note that $d_1(x) \le x$ and is increasing in this example. Similarly $1-d_2(y)$ is increasing and $1-d_2(y) \le 1-y$. Evidently continuity assumptions are needed as well as monotonicity of utility. See [Ru 1].)

SEE FIGURE 1 APPENDIX A

[Fig. 1 Graphs of d_1, d_2– obtained by way of the command:

ParametricPlot[$\{\{x,x\},\{x,d_1[x]\},\{d_2[x],x\}\},\{x,0,1\}$]]

taking $d_1[x] := \delta(1-x)$ and $d_2[x] := 1-\delta(1-x)$.

Note that the equilibrium may also be computed as the fixed-point $x = d_2(d_1(x))$ here seen to obey $x = 1-\delta(1-\delta x)$.]

1.2. Standard Example of Equilibrium Division

For illustration and for referencewe calculate the equilibrium point from the defining equations when both Players have the same constant discounting factor in their utility evaluation given earlier.

$$y = d_1(x) = u_1(d_1(x), 0) = u_1(x, 1) = \delta x$$

and

$$1 - x = 1 - d_2(y) = u_2(d_2(y), 0) = u_2(y, 1) = \delta(1 - y)$$

lead to

$$x = \frac{1}{(1+\delta)}, \quad y = \frac{\delta}{(1+\delta)}.$$

It is usual to note that as $\delta \to 1$ we arrive in the limit at a division into equal parts. Otherwise Player 1, who is the first to bid, receives a slightly greater share of the cake.

1.3. The amended game with prior claims

Now consider the variant of the Rubinstein bargaining game in which Players-1 and -2 are first required to announce simultaneously their prior claims on the cake in the form of a commitment not to accept a cut of the cake below a certain (percentage) level. (We denote the respective commitment levels by z_1, z_2.) Thereafter the Players negotiate a division of the cake in the usual way, but with the proviso that if a player accepts a percentage π of the cake below his/her pre-announced level, then a penalty is incurred by way of a payment, made to a third party, equal to a fixed percentage c of the cake . An alternative (continuous) model was suggested by us wherein the penalty is proportional to the "offence" (= accepted offer – commitment level); details of this model were calculated in Muthoo (1996) following our preliminary calculations for the model presented here.

Assuming the same constant discounting factor δ applied to the cake by both players, the contract (x, n) agreed on day n giving a proportion x of the cake to Player-1 and $1 - x$ to Player-2 has the following utility for the two players: in the case of Player-1

$$\begin{aligned} u_1(x, 0; z_1) \quad &= x - c \quad \text{if } x < z_1 \\ &= x \qquad \text{otherwise} \end{aligned}$$

and, more generally,

$$u_1(x, n) = \delta^n u_1(x, 0),$$

and in the case of Player 2:

$$u_2(x, n; z_2) = u_1(1 - x, n; z_2).$$

Here the penalty is modelled by c which denotes the fixed proportion of cake subtracted from a Player who accepting a division of the cake must set aside the prior claim. We assume to begin with that

$$0 < c < 1.$$

In our case the functions relating a next-round ("tomorrow") contract to a current-round ("to-day") contract are NOT continuous, in fact even their definition via (1) may need adjustment as we shall indicate after a discussion of the relation between the value of c and δ.

1.4. Funding compensations: natural limitations on c

The standard dominance argument for intertemporal comparison of contracts also needs adjustment. At its simplest, the argument requires that in order to tempt Player-1 away from a payoff of, say, x with $x \geq z_1\delta^{m+1}$,(or $x - c\delta^{m+1}$ if $x < z_1\delta^{m+1}$), arising from the contract $(x, m+1)$ to a pay-off no worse than this arising from, say, (X, m) it may be necessary for Player-2 to propose either $X = x+\varepsilon$ (for an infinitesimal ε) or even $X = x+\varepsilon+c\delta^m$ in case $x < z_1\delta^m$. Player-2 must fund this enticement from the complementary pay-off $\delta^m - X$ and still be no worse off than when receiving $\delta^{m+1} - x$; this requires that the enticement be less, at the very least, than the growth in the cake size

$$c\delta^m < \delta^m - \delta^{m+1} \tag{L1}$$

i.e. $c < 1 - \delta$.

At the other extreme: it may be that in both rounds Player-2 would also be setting aside its prior claim and have to pay a penalty (i.e. $y = \delta^{m+1} - x < z_2\delta^{m+1}$ and $Y = \delta^m - X < z_2\delta^{m+1}$) as well as recompensing Player-1, in which case for dominance we would require

$$\delta^{m+1} - x - c\delta^{m+1} < \delta^m - X - c\delta^m$$

or, since $X - x = c\delta^m$

$$c\delta^m + c(\delta^m - \delta^{m+1}) < \delta^m - \delta^{m+1}$$

or

$$c + c(1 - \delta) < 1 - \delta$$

i.e.

$$c < \frac{1-\delta}{2-\delta} \ (< 1-\delta)\,. \tag{L2}$$

We call this last upper bound the **natural limitation** and will adopt this condition as an assumption on c by way of maximum simplification so as henceforth not worry about the funding issue. In practice it is easier to restrict further so that $c < \frac{1-\delta}{2}$.

We note that a more general argument would consider enticements between round m and round $m+k$, in this case our first limitation for funding requires that

$$c\delta^m < \delta^m - \delta^{m+k}$$

i.e.

$$c < 1 - \delta^k$$

and so for k large enough this is satisfied when $c < 1$. This amounts to requiring that all the dominance arguments be repeated every k rounds. Our simplifying assumption involves no great loss of generality.

We may now return to an earlier issue.

1.5. Present value functions

Let us follow through the arguments establishing present value:

1. Tomorrow's x, if $\geq z_1$, is valued to-day at δx; thus if also $\delta x \geq z_1$ then today's equivalent contract is δx, i.e. under these circumstances $d_1(x) = \delta x$.

But, if $\delta x < z_1$ then $u_1(\delta x, 0) = \delta x - c$; if moreover $\delta x + c < z_1$ then $u_1(\delta x + c, 0) = \delta x$. Thus if $z_1 \leq x < \frac{z_1 - c}{\delta}$ then $d_1(x) = \delta x + c$.

However, if $\frac{z_1 - c}{\delta} \leq x < z_1/\delta$ (which presumes that $c < z_1$) then $d_1(x)$ is left undefined by the so-called defining equation. Our analysis in Section 4 shows that a definition that fills the gap is:

$$d_1(x) = z_1,$$

as being the least inducement leaving the player no worse off than in the next round.

2. Tomorrow's x, if $< z_1$, has utility $\delta(x-c)$ which is also $< z_1$. Today's equivalent contract must award $\delta(x-c)+c$ provided $\delta(x-c)+c < z_1$. Thus, if $0 \leq x < z_1$, we have $d_1(x) = \delta x + c(1-\delta)$. It should be noted that when $z_1 \geq c$ then $\delta(x-c)+c < z_1$. [Because $z_1(1-\delta) \geq c(1-\delta)$, hence $z_1 \geq \delta z_1 + c - c\delta > \delta x + c - c\delta = \delta(x-c)+c$.]

Again, if $\delta(x-c)+c \geq z_1$, then we should take in lieu of the present value z_1.

This result is illustrated in Figure 2.

1.6. Multiple equilibria

We may now generate an illustrative example presenting multiple solutions to the equation

$$x = d_2(d_1(x)).$$

This is done most simply by computer and we give our Mathematika implementation details for convenience:

$d[x,z]$:=Which$[z/\delta \leq x, \delta \cdot x, z \leq x \&\& \; x < (z-c)/\delta, \; \delta \cdot x + c, \; (z-c)/\delta \leq x$ $\&\& \; x < z/\delta \; \&\& \; z < x, z, \; x < z \; \&\& \; \delta(x-c)+c < z, \delta \cdot x + c \cdot (1-\delta), x < z \; \&\&$ $\delta(x-c)+c < z, z]$

$$d1[x] := d[x, z1]$$

$$d2[x] := 1 - d[1-x, z2]$$

1.7. Pseudo-fixed points and Nash equilibria

It is to be noted that though the present-value function in our example is discontinuous, it is,as well may be expected, monotonic increasing, albeit weakly. Generally speaking if $d_i(x)$ are bounded, increasing functions of x we may form the limits:

$$d_i(x+) = \lim_{\varepsilon \to 0+} d_i(x+\varepsilon), \;\; d_i(x-) = \lim_{\varepsilon \to 0+} d_i(x-\varepsilon).$$

Then a pair (x,y) with

$$x = d_2(y), y = d_1(x-)$$

is still an equilibrium by an amended argument: Indeed if Player-1 bids $z > x$ we have

$$1 - z < 1 - x = 1 - d_2(y)$$

Player-2 will thus prefer to bid y in the next round. If Player-2 bids $z < y = d_1(x-) \leq d_1(x)$, then Player-1 is better-off bidding x in the next round and rejects the offer.

A similar argument can be constructed for a pair (x,y) with

$$x = d_2(y-), y = d_1(x).$$

Remark We note in passing that if $d_1(x)$ is continuous at x then since $d_1(x)$ is increasing one is tempted to say that $d_2(d_1(x)-) = d_2(d_1(x-))$. Indeed if $y_n = d_1(x_n)$ is increasing with limit $y = d(x)$ then x_n is increasing and bounded above by x with limit $\xi \leq x$; then $y = d(x) \geq d(\xi) \geq d(x_n)$ so $d(\xi) \geq y$ i.e. $d(x) = d(\xi)$. If $x = \xi$ we have $x = d_2(d_1(x-))$, but if $x > \xi$ then, ξ is the smallest member of $d^{-1}(y)$ and $y = d(\xi) = \lim d(x_n)$ and we cannot construct y_n.

Thus the equilibria are among what may be termed a left pseudo-fixed points i.e. points x solving:

$$x = d_2(d_1(x-)).$$

If x solves

$$x = d_2(d_1(x+))$$

we almost have a Nash Equilibrium at x. Let $\varepsilon > 0$ be arbitrary. If $y = d_1(x+)$ so that $x = d_2(y)$ it is clear that if Player-1 offers $z > x$ then as before Player-2 rejects since $1 - z < 1 - d_2(y)$. If Player-2 offers $z < y$ then there is $x' < x + \varepsilon$ such that $y' = d_1(x') < y + \varepsilon$. Thus Player-1 is better off in the next round if an infinitesimal adjustment to x is tolerated at that stage and so prefers to wait his turn and rejects the offer. Let us call a value x which may be interpreted up to an infinitesimal adjustment a **quasi Nash equilibrium.**

1.8. Fixed and pseudo-fixed points of an increasing function

Let $F(x)$ be an increasing function defined for $0 \leq x \leq 1$ with values in $[0,1]$ with $F(0) > 0$ and $F(1) < 1$.

Definition. Let us call x a right- or left- **pseudo-fixed point** (of F) if respectively

$$x = F(x+) \text{ or } x = F(x-)$$

where e.g. $F(x+) = \lim_{z \to x+} F(z)$ denotes the limit from above. Evidently $F(x-) \leq F(x) \leq F(x+)$. An ordinary fixed point is both a left- and a right-pseudo-fixed point.

We note some easy facts connecting these fixed points (see in this connection R.O. Davies *et al.*(1971)) and the iteration $x_{n+1} = F(x_n)$.

Suppose y is a right-pseudo-fixed point but not a fixed point then $F(y-) < F(y+) = y$; we may thus select $x_1 < y$ with $F(x_1) < x_1$ whereupon $z = \lim x_n$ is a right-fixed point (since $x_3 = F(F(x_1)) < F(x_1) = x_2$ and the sequence is bounded, etc) and is evidently to the left of y. Hence:

The least right-pseudo-fixed point to the left of a strict right-pseudo-fixed point y is thus a fixed point.

Similarly, the greatest left-pseudo-fixed point to the right of a strict left-pseudo-fixed point is a fixed point.

(ii) If $x_1 \leq F(x_1)$ then the iteration increases monotonically and the nearest pseudo-fixed point at, or to the right of x_1, is the limit, and is evidently a left-pseudo-fixed point.

Indeed suppose $x_1 < F(x_1)$ and let $y = F(y-)$ be the nearest with $x_1 < y$. As $F(x_1) \leq F(y)$, either $F(x_1) = F(y)$ and so since the function is constant on $[x_1, y]$ we have $y = F(y-) = F(y)$ whereupon y is the nearest pseudo-fixed point; or, else, $F(x_1) < F(y)$, so $F(x_1) \leq F(y-) = y$. If $F(x_1) = F(y-)$, then again the function is constant on $[x_1, y)$ and y is again the nearest pseudo-fixed point. Finally, if $x_2 = F(x_1) < F(y-) = y$ then the previous argument may be repeated inductively. If the nearest pseudo-fixed point is $y = F(y+)$, then $x_2 = F(x_1) \leq F(y) \leq F(y+) = y$ etc). Similarly:

(iii) If $x_1 \geq F(x_1)$ then the iteration decreases monotonically and the nearest pseudo-fixed point at, or to the left of x_1, is the limit, and is evidently a right-pseudo-fixed point.

Of course since there is a left-pseudo-fixed point generated from an iteration with $x_1 = 0$ and a right-pseudo-fixed point generated from $x_1 = 1$, it follows that:

(iv) there exists a fixed point.

This is a special case of Tarski's Fixed Point Theorem (see A. Tarski (1955)).

Thus if the present value functions $d_i(x)$ are monotonic a Nash equilibrium is guaranteed.

1.9. A genericity condition on the prior claims

The most general case illustrated as in Figure 2 requires that

$$z_1 < \frac{z_1 - c}{\delta} < \frac{z_1}{\delta} < 1.$$

The first inequality implies that $c < z_1 - z_1\delta$ so that

$$\frac{c}{1-\delta} < z_1.$$

We regard the condition

$$\frac{c}{1-\delta} < z \tag{L3}$$

to be a **genericity condition on the prior claim**. This leads to some further limitations on c and δ if we are to illustrate genericity and have $z_1 < \delta$. Evidently we require

$$c < \delta(1-\delta) \leq 0.25,$$

so we shall limit $c < 25\%$ and require,

$$\frac{1}{2}\{1 - \sqrt{1-4c}\} < \delta < \frac{1}{2}\{1 + \sqrt{1-4c}\}. \tag{L4}$$

Fortunately (L2) is less stringent that (L4) viz. we have the inequality that for $0 < c \leq \frac{1}{4}$

$$\frac{1+\sqrt{1-4c}}{2} < \frac{1-2c}{1-c}.$$

Thus by (L4) we are guaranteed that $c < \frac{1-\delta}{2-\delta}$.

For illustrative purposes, we will take $c = 0.1$,and need, say $\delta < 0.5(1+0.8)$; we take $\delta = 0.6$ and require $\frac{c}{1-\delta} = \frac{1}{3} < z_i \leq \delta = 0.6$.

In Figure 3i we have selected by way of example $c = 0.1$, $\delta = 0.6$, $z_1 = 0.625$, $z_2 = 0.6$.

[Figure 2. Graph of $d_1(x)$]

SEE FIGURE 2 APPENDIX A

SEE FIGURE 3 APPENDIX A

[Figure 3. Multiple equilibria. Obtained by way of:

(i) ParametricPlot$[\{\{x, d_1(x)\}, \{1 - d_2(1-x), x\}\}, \{x, 0, 1\}]]$

(ii) Plot$[d2[d1[x], \{x, 0, \delta\}]$ with $z1 := 0.625; z2 := 0.6;$

2. Main Result

Our calculations, which as may be appreciated require case by case consideration, are illustrated in Figure 4. Figure 4i shows typical graphs of the Nash or quasi Nash equilibrium pay-offs to Player-1 against possible announced commitment levels z_2 of Player-2. Each such graph arises from a value of z_1. The analysis of inequalities in Section 6.3 shows that for $\delta \geq (\sqrt{5}-1)/2 = 0.618...$ there are really only two cases up to homeomorphism viz.

$$\text{low charge: } c \leq \frac{(1-\delta)^2}{1-\delta+\delta^2} < c$$

and

$$\text{high charge: } \frac{(1-\delta)^2}{1-\delta+\delta^2} < c \leq \frac{1-\delta}{2}.$$

(The upper bound being the natural limitation on c discussed above.) Thus the two cases are generic and illustrated tell the whole story (see Appendix B). The essential difference between them is the presence a flat graph in the low charge case; that is absent in the high charge case. We shall deduce a natural interpretation for the largest and smallest equilibrium positions; they correspond to the negotiation game terminating after a large number of rounds (large relative to the various constants in the model, see Section 4.1), at which stage, depending on parity, one of the Players is entitled to take the whole cake. There are thus two games: TOP - playing for the top outcome (corresponding to Player-1 being free at termination to divide the cake) and BOTTOM - playing for bottom (with Player-2 free to choose at termination). There is an interval of equivalent Nash equilibria in the claims in both games; in the case of TOP if

$$c < \frac{(1-\delta)^2}{1-\delta+\delta^2},$$

they are essentially:

$$z_1 = \frac{1}{1+\delta}; \quad z_2 = \frac{\delta}{1+\delta}$$

with pay-off as per claim (in this case it is a quasi equilibrium pay-off); otherwise, if $\delta > 0.618..$ and

$$\frac{(1-\delta)^2}{1-\delta+\delta^2} < c \leq \frac{1-\delta}{2},$$

the Nash equilibrium is unique

$$z_1 = \frac{1-c-\delta+2c\delta}{1-\delta^2}, \; z_2 = \frac{(1-c)(1-\delta)-c\delta^2}{1-\delta^2}$$

and gives Player-1 a pay-off

$$\frac{1-c-\delta+2c\delta}{1-\delta^2}.$$

One must note again in this context that in fact this is a quasi Nash equilibrium in pay-off (i.e infinitesimal tolerance needs to be invoked). Observe that for the largest penalty charge considered ($c = \frac{1}{2}(1-\delta)$) this now gives Player-1

$$\frac{1+2\delta}{2(1+\delta)}$$

i.e somewhere near 75% of the cake for δ nearing unity.

In the case of the play for BOTTOM the Nash equilibria are essentially:

$$z_1 = 1; \quad z_2 = 1,$$

with pay-off

$$\frac{1-c(1-\delta)}{1+\delta}, \frac{\delta + c(1-\delta)}{1+\delta}.$$

For the largest charge we get

$$\frac{1+2\delta-\delta^2}{2(1+\delta)}$$

near 50% for δ nearing unity, which gives Player-2 a not un-expected improvement over TOP. One can of course analyse the game in which one of the Players announces his claim prior to the announcement of the other but we leave this to the interested reader.

We recall that the analysis makes the simplifying assumption that $c < (1-\delta)/(2-\delta)$ according to the natural limitation described earlier and $z_i > c$ since the Players will not wish to agree a contract below the claim and so be left out of pocket; notwithstanding this the choice $z_i = 0$ is allowed. It is easiest to suppose that each Player evaluates commitment bids according to *maximin*.

SEE FIGURE 4 IN APPENDIX A

[Figure 4i. Equilibrium payoff to Player-1 in the commitment game vs. z_2]

[Figure 4ii Equilibrium pay-off in the commitment game to Player 1vs. z_1]

3. The Binmore approach

In order to interpret the various fixed points, we will need to refer to a sequential approach using Binmore's "staircase" argument which we briefly review in the standard Rubinstein game (without committments). Figure 5 depicts a series of parallel lines of the form $x + y = \delta^n$ drawn here with $\delta = 1/2$. These represent the cake in its various shrunk realizations: a point (x, y) on one of these line segments now represents a division of the cake giving x to Player-1 and $y = \delta^n - x$ to Player-2. We can play the standard bargaining game backwards from round N. Assume that it is Player-2 to propose a division. Take the view that any division point of the cake might be agreed upon; the interval $[a_N, b_N]$ on the x-axis represents possible cuts for Player-1 with $a_N = 0$ and $b_N = \delta^N$. Next observe that on the previous round those division points (x, y) which have the same utility for Player-2 (viz. y) as in round N, have a greater utility x for Player-1. **He** (Player 1) could therefore second guess **her** proposal (the proposal of Player-2) by making an identical y proposal using an x proposal lying in $[a_{N-1}, b_{N-1}]$ which leaves himself better off, thanks to the gain in cake value between the two rounds. Of course, she is able to make an earlier offer (x, y) in round $N-2$ with x in the same range $[a_{N-1}, b_{N-1}]$ but with a y greater than what she might have considered awarding herself in round N. This y range is represented by the interval $[A_{N-1}, B_{N-1}]$. The argument may be run backwards to round 0. In so doing we obtain a geometric counterpart traced out by a double "staircase" whose projections give the endpoints of the various intervals $[a_n, b_n]$ and $[A_n, B_n]$. The point to notice is that the staircase is of constant "width" $|b_N - a_N|$, whereas the cake size working backwards is growing to unity. We may repeat this argument for any starting value of N to obtain the corresponding interval $[a_0(N), b_0(N)]$ of values for Player-1. We have that

$$b_0(N) - a_0(N) = b_N(N) - a_N(N) = \delta^N \to 0$$

as $N \to \infty$, thus yielding the unique division point $(a_0(\infty), 1 - a_0(\infty))$ on the original cake of size 1. Thus ignorance of the optimum contract at round N leads to refined knowledge at round 0, which refinement improves to precision as $N \to \infty$. Our objective is to repeat this argument with the changed utility functions proposed in the amended game with prior claims, in order to interpret the multiple fixed points as a limiting value for $a_N(N)$ arising from different initial positions for $a_0(N)$ with a large enough N.

SEE FIGURE 5 IN APPENDIX A

[Figure 5 the staircase]

4. Staircase calculation with commitments

Step 1: Back-stepping compensations

Using a similar but more appropriate notation to that of the last section, we consider a y range of $[A_n, B_n]$ for a cake of size δ^{2n+1}. We need to compute an equivalent contract of the previous round (in which round the cake is worth δ^{2n}

and it is the turn of Player-1 to make a proposal). We must consider two main cases:

Main Case 1: $A_n < z_2 \cdot \delta^{2n+1}$.

The cut A_n at round $2n+1$ for Player 2 is worth

$$A'_n = A_n - c \cdot \delta^{2n+1}.$$

Hence Player 1 in the previous round must allow Player 2 a cut of the cake equal to $\delta^{2n} - a_n$ where:

$$A'_n + \text{compensation} = \delta^{2n} - a_n. \qquad (*)$$

The compensation term, which might be zero, has to deal with the fact that A'_n in round $2n$ may conceivably fall below Player-2's commitment level (viz. $z_2 \cdot \delta^{2n}$). If she sets aside her commitment she will have to be recompensed by an amount equal to the penalty, viz. $c\delta^{2n}$. Denoting the compensation term by k we need:

$$(A'_n + k) \text{ is equivalent in utility at round } 2n \text{ to } A'_n.$$

The compensation term is calculated by cases:

(0) $A'_n \geq z_2\delta^{2n}$

The compensation here is zero since the utility of A'_n at round $2n$ is exactly A'_n.

We now notice that this case cannot arise, since $z_2\delta^{2n} \leq A'_n < A_n < z_2\delta^{2n+1} < z_2\delta^{2n}$.

If $A'_n < z_2\delta^{2n}$ then we must consider whether or not $A'_n + k$ is below the commitment level. In the former case she pays a penalty and is left with $A'_n + k - c\delta^{2n}$. We thus have $k = c\delta^{2n}$ provided $A'_n + c\delta^{2n} < z_2\delta^{2n}$. Thus

(1) if $A'_n < (z_2 - c)\delta^{2n}$, then $k = c\delta^{2n}$.

On the other hand,

(2) if $(z_2 - c)\delta^{2n} \leq A'_n \leq z_2\delta^{2n}$, then $k = z_2\delta^{2n} - A'_n$ gives Player 2 a cut at her commitment level (which is not penalised) at round $2n$ and which is better than the cut A_n in the next round (i.e. at round $2n+1$).

It is to be noted that the maximal compensation is $c\delta^{2n}$ is less than $\delta^{2n}(1-\delta) = \delta^{2n-1} - \delta^{2n}$ which represents the gain to Player-1 (see earlier section).

Main Case 2: $A_n \geq z_2\delta^{2n+1}$. This comes easy: we can just quote the same formulas (0) through (2) but with the simplification that $A'_n = A_n$ (i.e. no penalty was incurred at round $2n+1$). Note that this time Case(0) does occur (with $k = 0$).

Step 2: Computing a_n from A_n We do the calculation from the Main Case 1 above splitting this into the two subcases (1.1) and (1.2). A similar notation is used for the Main Case 2. We thus have:

$$a_n = \delta^{2n} - A_n + c\delta^{2n+1} - c\delta^{2n} = \delta^{2n}(1-c) - A_n + c\delta^{2n+1}, \tag{1.1}$$

$$a_n = \delta^{2n} - \delta^{2n} z_2 = \delta^{2n}(1-z_2), \tag{1.2}$$

since in this case the compensation takes A'_n up to the commitment level. Also

$$a_n = \delta^{2n} - A_n, \tag{2.0}$$

$$a_n = \delta^{2n}(1-c) - A_n, \tag{2.1}$$

$$a_n = \delta^{2n}(1-z_2). \tag{2.2}$$

There are thus five formulas which we note in the form of a lemma.

Lemma 1 *If $A_n < z_2\delta^{2n+1}$, then*
either:
1.1 $A_n < (z_2 - c)\delta^{2n} + c\delta^{2n+1}$ and

$$a_n = \delta^{2n}(1-c) - A_n + c\delta^{2n+1},$$

or:
1.2 $(z_2 - c)\delta^{2n} + c\delta^{2n+1} \leq A_n \leq z_2\delta^{2n} + c\delta^{2n+1}$ and

$$a_n = \delta^{2n}(1-z_2).$$

Similarly, if $A_n \geq z_2\delta^{2n+1}$, then
either:
2.0 $z_2\delta^{2n} \leq A_n$ and

$$a_n = \delta^{2n} - A_n,$$

or:
2.1 $A_n < (z_2 - c)\delta^{2n}$ and

$$a_n = \delta^{2n}(1-c) - A_n,$$

or:
2.2 $(z_2 - c)\delta^{2n} \leq A_n \leq z_2\delta^{2n}$ *and*

$$a_n = \delta^{2n}(1 - z_2).$$

Remarks to Lemma 1 Regarding condition 1.2 we note that $(z_2 + c\delta)\delta^{2n} > (z_2\delta)\delta^{2n}$ since $z_2(1-\delta) > -c\delta$, so the upper inequality need not be quoted.

Case 2.1 vanishes if $\delta > 1 - c$, for then $z_2 \leq 1 < \frac{c}{1-\delta}$ whence$(z_2 - c)\delta^{2n} < z_2\delta^{2n+1}$.

Proposition 1. *Lemma 1 defines a piecewise linear function on* $[0, 1]$ *of the form* $a_0 = f_1(A_0, z_2)$ *which is continuous in* A_0 *for* $z_2 \leq c$ *and otherwise is decreasing and continuous except at* $A_0 = \delta z_2$ *where it is discontinuous from the left. The slope of the function, whenever it is defined, is either zero or* -1. *Evidently,* $a_n/\delta^{2n} = f_1(A_n/\delta^{2n}, z_2)$.

Proof. It is enough to compare pairwise the formulas at the end-points of their range of validity. Breakdown in continuity occurs when $z_2 - c + c\delta > z_2\delta$, i.e. when $z_2 > c$. Note that A_0 represents the part of a cake of size δ offered to Player-2 at round 1 but is being considered in relation to a cake of size 1 at round 0. Thus the function is defined on $[0, 1]$ but we are concerned only with values below δ.

SEE FIGURE 6 IN APPENDIX A
[Figure 6. Graph of $f_1(A_0, z_2)$.]

Corollary 1. *If Case 1.2 of Lemma 1 above occurs then* $z_2 < c$.

Proof: We have:

$$(z_2 - c)\delta^{2n} + c\delta^{2n+1} \leq A_n < z_2\delta^{2n+1}.$$

Hence

$$(z_2 - c) + c\delta \leq A_n < z_2\delta,$$

yielding:

$$z_2(1-\delta) < c(1-\delta).$$

The dual result to Lemma 1 is obvious; we write it out for convenience:

Lemma 2 *If* $a_n < z_1\delta^{2n}$ *, then*
either:
1.1 $a_n < (z_1 - c)\delta^{2n-1} + c\delta^{2n}$ *and*

$$A_{n-1} = \delta^{2n-1}(1-c) - a_n + c\delta^{2n};$$

or:
1.2 $(z_1 - c)\delta^{2n-1} + c\delta^{2n} \le a_n \le z_1\delta^{2n-1} + c\delta^{2n}$ *and*

$$A_{n-1} = \delta^{2n-1}(1 - z_1).$$

Similarly, if $a_n \ge z_1\delta^{2n}$ *, then*
either:
2.0 $z_1\delta^{2n-1} \le a_n$ *and*

$$A_{n-1} = \delta^{2n-1} - a_n$$

or:
2.1 $a_n < (z_1 - c)\delta^{2n-1}$ *and*

$$A_{n-1} = \delta^{2n-1}(1-c) - a_n;$$

or:
2.2 $(z_1 - c)\delta^{2n-1} \le a_n \le z_1\delta^{2n-1}$ *and*

$$A_{n-1} = \delta^{2n-1}(1 - z_1).$$

Remarks to Lemma 2 Regarding condition 1.2 we note that again $z_1\delta^{2n} > z_1\delta^{2n-1} + c\delta^{2n}$, since $z_1(\delta - 1) < 0 < c\delta$, so the upper inequality need not be quoted.

Proposition 2. *Lemma 2 defines a piecewise linear function from* $[0, \delta^2]$ *to* $[0, \delta]$ *of the form* $A_0 = f_2(a_1, z_1)$ *which is continuous in* a_1 *for* $z_2 \le c$*; otherwise it is decreasing and continuous except at* $a_1 = z_1\delta^2$ *where it is discontinuous from the left.The slope of the function, whenever it is defined, is either zero or* -1.

Evidently, $A_{n-1}/\delta^{2n-2} = f_2(a_n/\delta^{2n-2}, z_2)$.

SEE FIGURE 7 IN APPENDIX A
[Figure 7 Graph of $f_2(a_1, z_1)$.]

Corollary 2. *If Case 1.2 of Lemma 2 above occurs then* $z_1 < c$.

The proof is identical to that of Corollary 1.

Theorem. *Lemmas 1 and 2 establish a relationship of the form* $a_0 = F(a_1, z_1, z_2)$ *which if* $z_1, z_2 > c$ *is increasing and continuous in* a_1*, unless*

$$a_1 \in \{\delta^2 z_1, \delta(1 - z_2), \delta(1 - z_2 - c), \delta(1 - z_2 - c + c\delta)\}.$$

Thus if $z_2 > 1 - c + c\delta$ *there will be only one discontinuity.*

Proof. Indeed $F(a_1, z_1, z_2) = f_1(f_2(a_1, z_1), z_2)$ and the rest is clear from an inspection of the two graphs. We note that $c - c\delta + \delta > c/(1 - \delta)$ since this is equivalent to our natural limitation $c < (1 - \delta)/(2 - \delta)$.

We now proceed to consider the relation of the Binmore approach to the pseudo-fixed points.

4.1. Equilibrium divisions as pseudo-fixed points

We shall look for the equilibrium cuts by seeking left- and right-pseudo-fixed points which solve the equation:

$$a = F(\delta^2 a\pm, z_1, z_2).$$

This is related to the Rubinstein approach, whereby one solves:

$$x = d_2(d_1(x))$$

provided we amend the definition of d_i consonant with our definitions of f_i whereupon $d_2(d_1(x)) = F(x\delta^2, z_1, z_2)$. Our argument is now based on the back-tracking method reviewed in Section 3 which requires us to study the sequence of equivalent contracts $x_0(N), ..., x_N(N)$ starting at round $2N$ and finishing at round 0. This sequence uniquely defines $x_0(N)$ as a function *of* N and the initila value $x(N) = u$, so we write $x_0(u, N)$. Setting $v_r = x_{N-r}/\delta^{2(N-r)}$ (a number in the interval$[0, 1]$) we obtain:

$$v_r = F(v_{r-1} \cdot \delta^2, z_1, z_2). \quad (r = 0, \dots, N).$$

This iteration establishes a unique sequence $\{v_r(u)\}$ defined by its initial value $u = v_0$ and is monotone increasing or decreasing as $r \to \infty$ towards respectively a right- or a left- **nearest pseudo-fixed point** $f(u)$ to u of $F^*(x) :=$

$F(x \cdot \delta^2, z_1, z_2)$. Although we are entitled only to a finite part of the sequence terminating at v_N (representing round 0), we may usefully re-interpret the formalism; for if say $x_N(N) = u$, is fixed, then, since

$$x_0(u, N) = v_N(u)$$

we conclude:

Theorem 2. *The sequence* $\{x_0(u, N) : N = 1, 2, ...\}$ *is monotonically convergent with limit a pseudo-fixed point of* F^* *and in fact :*

$$\lim_N x_0(u, N) = \lim_N v_N(u) = f(u)$$

which may therefore validly be interpreted as a solution point of the bargaining game.

Of course one could have argued intuitively on the grounds that if $a_1(N) \to a_1 = \delta^2 a$ on the cake of size δ^2 as $N \to \infty$, then $a_0(N) \to a$. By Theorem 1 the limit of $F(a_1(N), z_1, z_2)$ is the fixed point $a = F(\delta^2 a\pm, z_1, z_2) = F(\delta^2 a, z_1, z_2)$ except possibly when a falls under one of the cases listed in the theorem, but this would have been a weaker argument altogether.

The function F^* is piecewise linear with slope either zero or δ^2. Its pseudo-fixed points are thus (unusually so) all attractive.

Remark. For moderate values of N the iteration is for practical purposes at the limit already. More precisely, if the iteration begins in an interval where the slope is δ^2, then since:

$$v_{r+2} - v_{r+1} = \delta^2(v_{r+1} - v_r)$$

we have:

$$v_n - v_0 = (v_n - v_{n-1}) + (v_{n-1} - v_{n-2}) + ... + (v_1 - v_0)$$

so

$$v_N - v_0 = (v_1 - v_0)\{1 + \delta^2 + ... + \delta^{2(N-1)}\} = (v_1 - v_0) \cdot \frac{1 - \delta^{2N}}{1 - \delta^2}$$

and for large N the righthand side is numerically constant. For instance, if $\delta = 0.8$ we have $0.8^{40} = 0.0001...$;(and if that is insufficiently accurate $0.8^{100} = 2.0... \times$

10^{-10}.) It is for this reason that we consider the infinite game as terminating after some large but reasonable number of rounds.

The convergence is immediate when the slope is zero.

SEE FIGURE 8 IN APPENDIX A

[Figure 8. Convergence of iterates to a (pseudo) fixed point.]

4.2. Initialisation of the iteration

Our final quest in regard to the iteration is the starting value v_0. Evidently, if we suppose Player-1 is the last to choose a contract then he should claim almost all of the cake and so we should study the initialization $v_0 = 1$. This easily identifies $f(1)$ as the relevant equilibrium division.

If on the other hand we allow Player-2 to have the last word, she should take the whole cake. This means that in the previous round the equivalent contract for Player-1 yields $f_1(\delta, z_2)$ and so leaves us with the problem of selecting z_2.

We begin by considering the most obvious choice, viz. $z_2 = 1$. Since $c < 1 - \delta$ we have $\delta < 1 - c$ and so

$$f_1(\delta, 1) = 1 - c - \delta.$$

We enquire methodically and next consider when $c/(1-\delta) < z_2 < 1$. Since $\delta < z_2\delta$ is false we have the three possibilities:

$$z_2\delta \leq \delta \leq z_2 - c, z_2 - c \leq \delta \leq z_2, z_2 \leq \delta < 1$$

with corresponding values for $f_1(\delta, z_2)$ - giving rise to potential initialisations u - as follows:

$$(1 - c) - \delta, 1 - z_2, 1 - \delta.$$

Evidently Player-2 wants to choose the smallest of these (subject to $\frac{c}{1-\delta} \leq z_2$) in order to maximise her share of the original cake. We begin by noting that

$$\frac{c}{1-\delta} \leq c + \delta$$

holds since this is equivalent to

$$c \leq (1 - \delta)$$

and that

$$\frac{c}{1-\delta} < \delta$$

only if we assume $c < \delta(1-\delta)$ for the sake of genericity. Thus for consistencey we check that

(i) to choose $1-\delta$ she must take $z_2 \leq \delta$ and this case need not even occur unless we restrict c;

(ii) to choose $1-z_2$ she must take

$$\delta \leq z_2 \leq c+\delta$$

which allows a maximum value of $z_2 = c+\delta$ and in effect chooses $1-c-\delta$ doing better than case (i) and as it transpires as well as in in (iii);

(iii) to choose $1-c-\delta$ she must take

$$z_2 \geq c+\delta.$$

All these being possible options, it is optimal to choose z_2 arbitrarily with $1 \geq z_2 \geq c+\delta$.

Finally suppose that

$$\frac{c}{1-\delta} > z_2.$$

Assuming $\delta > \frac{c}{1-\delta}$ we must have

$$z_2 \leq \delta < 1$$

and so $f_1(\delta, z_2) = 1-\delta$ which is evidently sub-optimal. Otherwise, we have the two cases:

$$z_2\delta \leq \delta \leq z_2, z_2 \leq \delta < 1$$

the latter offering $1-\delta$ as before, and the former offering

$$1-z_2$$

with best value again $1-\delta$.

In conclusion:

Initialisation Theorem. *If Player-2 is the last to make a proposal in the terminated bargaining game, then it is optimal to select any z_2 such that*

$$c+\delta \leq z_2 \leq 1,$$

whereupon

$$f_1(\delta, z_2) = 1-c-\delta$$

and the corresponding equilibrium division is thus the least pseudo-fixed point of $F^*(z_1, z_2, .)$ on or to the right of $a = 1-c-\delta$.

We should therefore study an initialization with $v_0 = \Delta = 1 - \delta - c$; at its smallest relative to the limit for c the initialisation will be equal to

$$\Delta_0 = 1 - \delta - \frac{1-\delta}{2-\delta} = \frac{(1-\delta)^2}{2-\delta}$$

e.g. for $\delta = 0.9$ we have $\Delta_0 = 0.01/1.1 = 0.0090909...$ quite small, especially, if, as seems likely, Player-2 commits herself to a sizeable chunk of the cake. We thus identify the alternative pseudo-fixed point as being $f(0)$.

5. Formulas for $F(z_1, z_2, x)$ and associated fixed points

We begin with a statement of the results all of which may be computed routinely (details of he calculations are available for the interested reader directly from the author). Values are tabulated against side conditions to be placed on z_1, z_2. The latter arise by conjuction of the side conditions associated with the definitions of f_1, f_2.

5.1. Formulas for F

By Lemma 1 we have if $z_i > c$ that

$$f_1 = \begin{array}{ll} 1 - c + c\delta - x & x < \delta z_2 \\ 1 - c - x & \delta z_2 \leq x \leq z_2 - c \\ 1 - z_2 & z_2 - c \leq x \leq z_2 \\ 1 - x & z_2 \leq x \leq 1 \end{array}$$

and

$$f_2 = \begin{array}{ll} \delta(1 - c + c\delta) - x & x < \delta^2 z_1 \\ \delta(1-c) - x & \delta^2 z_1 \leq x \leq \delta(z_1 - c) \\ \delta(1 - z_1) & \delta(z_1 - c) \leq x \leq \delta z_1 \\ \delta - x & \delta z_1 \leq x \leq \delta \end{array}.$$

We derive the definition of

$$F = f_1(f_2(x))$$

which is evidently an increasing function since both f_1 and f_2 are decreasing functions of x. The first is discontinuous at $x = \delta z_2$ and the second at $x = \delta^2 z_1$.

For the second clause for f_1 to be non-vacuous we need to assume that

$$\frac{c}{1-\delta} < z_2,$$

likewise for the second clause for f_2 to be non-vacuous we need to assume that

$$\frac{c}{1-\delta} < z_1.$$

We compute that:

$$F = \begin{array}{ll} \text{Case } A & \text{for } x < z_1\delta^2 : \\ x + 1 - \delta(1 - c + c\delta) & x \leq \delta(1 - c + c\delta) - z_2 \\ 1 - z_2 & \delta(1 - c + c\delta) - z_2 \leq x \leq \delta(1 - c + c\delta) - z_2 + c \\ x + (1 - c)(1 - \delta) - c\delta^2 & \delta(1 - c + c\delta) - z_2 + c \leq x \leq \delta(1 - c + c\delta - z_2) \\ x + (1 - \delta)(1 - c + c\delta) & \delta(1 - c + c\delta - z_2) < x \end{array}$$

$$F = \begin{array}{ll} \text{Case } X & z_1\delta^2 \leq x \leq (z_1 - c)\delta \\ x + (1 - c - \delta + 2c\delta) & \delta(1 - c - z_2) < x \\ x + (1 - c)(1 - \delta) & \delta(1 - c) - z_2 + c \leq x \leq \delta(1 - c - z_2) \\ 1 - z_2 & \delta(1 - c) - z_2 \leq x \leq \delta(1 - c) - z_2 + c \\ x + 1 - \delta(1 - c) & 0 \leq x \leq \delta(1 - c) - z_2 \end{array}$$

$$F = \begin{array}{ll} \text{Case } B & (z_1 - c)\delta \leq x \leq \delta z_1 \\ 1 - \delta(1 - z_1) & z_2 \leq \delta(1 - z_1) \\ 1 - z_2 & z_2 - c \leq \delta(1 - z_1) \leq z_2 \\ \delta z_1 + (1 - \delta - c) & z_1 + z_2 \leq 1 \text{ and } \delta(1 - z_1) \leq z_2 - c \\ (1 - c + c\delta) - \delta(1 - z_1) & 1 < z_1 + z_2 \end{array}$$

$$F = \begin{array}{ll} \text{Case } C & \delta z_1 \leq x \leq \delta \\ 1 - \delta + x & x \leq \delta - z_2 \\ 1 - z_2 & \delta - z_2 \leq x \leq \delta - z_2 + c \\ x + 1 - c - \delta & c + \delta - z_2 \leq x \leq \delta(1 - z_2) \\ (1 - c)(1 - \delta) + x & \delta(1 - z_2) < x \end{array}$$

5.2. Discontinuities in F

We list the discontinuities and corresponding cases:

$$\begin{array}{ll} x = \delta(1 - z_2) & \mathrm{C2/C3} \\ x = \delta(1 - z_2 - c) & \mathrm{X1/X2} \\ x = \delta(1 - z_2 - c + c\delta) & \mathrm{A3/A4} \end{array}$$

which need not all occur at once; in addition we have as well:

$$x \in \{z_1\delta^2, (z_1 - c)\delta, z_1\delta\}.$$

5.3. Evaluation of fixed points

We solve $a = F(z_1, z_2, a\delta^2)$. One argues routinely by cases following the definition of F to arrive at the following:

$$\begin{array}{ccc} & \text{Case } A & \\ \frac{1+c\delta}{1+\delta} & \frac{1+c\delta}{1+\delta} < z_1 & z_2 \leq \frac{1-c}{1+\delta}\delta \\ 1 - z_2 & 1 - z_2 < z_1 & \frac{(1-c)\delta}{1+\delta} \leq z_2 \leq \frac{\delta(1-\delta)(1-c)+c}{1-\delta^2} \\ \frac{(1-c)(1-\delta)-c\delta^2}{1-\delta^2} & \frac{(1-c)(1-\delta)-c\delta^2}{1-\delta^2} < z_1 & \frac{\delta(1-\delta)(1-c)+c}{1-\delta^2} \leq z_2 \leq \frac{(1-c-\delta+2c\delta)}{1-\delta^2} \\ \frac{1-c+c\delta}{1+\delta} & \frac{1-c+c\delta}{(1+\delta)} < z_1 & \frac{1-c+c\delta}{(1+\delta)} < z_2 \end{array}$$

$$\begin{array}{ccc} & \text{Case } X & \\ \frac{(1-c-\delta+2c\delta)}{1-\delta^2} & \frac{(1-c)\delta}{1+\delta} + \frac{c}{1-\delta^2} \leq z_1 \leq \frac{(1-c-\delta+2c\delta)}{1-\delta^2} & \frac{(1-\delta)(1-c)-c\delta^2}{(1-\delta^2)} < z_2 \\ \frac{(1-c)}{1+\delta} & \frac{\delta+c}{1+\delta} \leq z_1 \leq \frac{(1-c)}{1+\delta} & \frac{\delta+c}{1+\delta} \leq z_2 \leq \frac{(1-c)}{1+\delta} \\ 1 - z_2 & z_1 + z_2 \leq 1 \text{ and } (1 - z_2)\delta \leq (z_1 - c) & \frac{\delta(1-c)-\delta^2}{1-\delta^2} \leq z_2 \leq \frac{c+\delta}{1+\delta} \\ \frac{1-\delta(1-c)}{1-\delta^2} & \frac{\delta(1-\delta)+c}{1-\delta^2} \leq z_1 \leq \frac{1-\delta(1-c)}{1-\delta^2} & z_2 \leq \frac{\delta(1-c)-\delta^2}{1-\delta^2} \end{array}$$

$$\begin{array}{ccc} & \text{Case } B & \\ (1 - c + c\delta) - \delta(1 - z_1) & \frac{\delta(1-c)}{1+\delta} \leq z_1 \leq \frac{\delta(1-\delta)(1-c)+c}{1-\delta^2} & 1 < z_1 + z_2 \\ 1 - z_2 & \delta < z_1 + \delta z_2 \leq c + \delta & \delta \leq z_1\delta + z_2 \leq c + \delta \\ 1 - \delta(1 - z_1) & \frac{\delta}{1+\delta} \leq z_1 \leq \frac{c+\delta-\delta^2}{1-\delta^2} & z_1\delta + z_2 \leq \delta \\ \delta z_1 + (1 - \delta - c) & \frac{\delta(1-\delta-c)}{1-\delta^2} \leq z_1 \leq \frac{c+\delta(1-\delta-c)}{1-\delta^2} & z_1 + z_2 \leq 1,\ c + \delta \leq \delta z_1 + z_2 \end{array}$$

Case C

$\frac{1}{1+\delta}$	$z_1 \le \frac{\delta}{1+\delta}$	$z_2 \le \frac{\delta}{1+\delta}$
$1 - z_2$	$z_1 + \delta z_2 \le \delta$	$\frac{\delta}{1+\delta} \le z_2 \le \frac{\delta-\delta^2+c}{1-\delta^2}$
$\frac{1-c-\delta}{1-\delta^2}$	$z_1 \le \frac{(1-c-\delta)\delta}{1-\delta^2}$	$\frac{c+\delta-\delta^2}{1-\delta^2} \le z_2 \le \frac{1-(1-c)\delta}{1-\delta^2}$
$\frac{1-c}{1+\delta}$	$z_1 \le \frac{(1-c)\delta}{1+\delta}$	$z_2 > \frac{1+c\delta}{1+\delta}$

The pseudo-fixed points may be obtained by relaxing strong inequalities to weak ones in the side conditions.

6. Nash equilibria

Proposition 1. By playing $z_1 = 1/(1+\delta)$, Player-1 can ensure a payoff of $1/(1+\delta)$.

Proof. Note the potential overlap between cases C1 and X1 possible, since

$$\frac{1}{1+\delta} > \frac{(1-\delta)(1-c) - c\delta^2}{1-\delta^2}$$

(see Lemma 1). Thus if

$$\frac{\delta(1-\delta)(1-c)+c}{1-\delta^2} < \frac{1}{1+\delta} = z_1,$$

the minimum pay-off is at least $1/(1+\delta)$ or $(1-\delta-c+2c\delta)/(1-\delta^2)$. If however

$$\frac{\delta(1-\delta)(1-c)+c}{1-\delta^2} > \frac{1}{1+\delta},$$

then, since

$$\frac{1}{1+\delta} > \frac{\delta(1-c)}{1+\delta},$$

we invoke case B1 when

$$z_2 \ge \frac{1}{1+\delta}.$$

Then

$$z_1 + z_2 \ge \frac{2}{1+\delta} > 1$$

and the pay-off is

$$(1-c)(1-\delta) + \frac{\delta}{1+\delta}$$

which is greater than $1/(1+\delta)$, provided $1 < 2\delta + \delta^2$ and this certainly holds when $\delta \geq 0.5$.

Proposition 2. Player-2 can hold Player-1's pay-off down to at most $1/(1+\delta)$ if and only if

$$c < \frac{(1-\delta)^2}{1-\delta+\delta^2},$$

and otherwise the security level of Player-2 is

$$\frac{1-c-\delta+2c\delta}{1-\delta^2}.$$

Remark. Central to our argument is the condition

$$\frac{1}{1+\delta} < \frac{\delta(1-c)}{1+\delta} + \frac{c}{1-\delta^2}. \tag{6.1}$$

This is equivalent to

$$(1-\delta)^2 < c(1-\delta+\delta^2).$$

Clearly this fails for small values of c, but works for c large enough. Indeed we note that

$$\frac{(1-\delta)^2}{1-\delta+\delta^2} < \frac{1-\delta}{2}$$

holds in the range $(\sqrt{5}-1)/2 < \delta < 1$. Thus if the penalty charge c satisfies

$$\frac{(1-\delta)^2}{1-\delta+\delta^2} < c \leq \frac{1-\delta}{2},$$

the condition (6.1) holds.

Proof of Proposition 2.

6.1. When $z_2 = \delta/(1+\delta)$ secures $1/(1+\delta)$.

Suppose that

$$\frac{1}{1+\delta} > \frac{\delta(1-c)}{1+\delta} + \frac{c}{1-\delta^2}. \tag{6.2}$$

Thus case B1 fails since, if

$$z_1 \geq 1 - z_2 = \frac{1}{1+\delta},$$

then by our assumption z_1 is outside the range for the other condition in case B1.

We verify which other cases can occur:

Case A:

A1 fails since $z_2 > \frac{\delta(1-c)}{1+\delta}$; A2 can occur for z_1 large enough; A3 is ruled out by our assumed inequality and finallyA4 fails since $\delta < 1 - c(1-\delta)$.

Case X:

X1 fails since

$$\frac{(1-\delta)(1-c) - c\delta^2}{1-\delta^2} < \frac{\delta}{1+\delta}$$

is equivalent to

$$1 - 2\delta + \delta^2 < c(1 - \delta + \delta^2).$$

X2 fails since $c + \delta > \delta$; X4 fails since $1 > 1 - c - \delta$.

Case B:

B1 was dealt with above; B2 gives a pay-off of $1/(1+\delta)$;

B3 if this occurs we have $z_1 \leq 1 - \frac{1}{1+\delta} = \frac{\delta}{1+\delta}$ and also $z_1 \geq \frac{\delta}{1+\delta}$, thus the pay-off is $1/(1+\delta)$;

B4 cannot occur since $z_1 \leq 1 - z_2 = \frac{1}{1+\delta}$ and

$$\delta z_1 \geq c + \delta - \frac{\delta}{1+\delta} = c + \frac{\delta^2}{1+\delta},$$

i.e.

$$z_1 \geq \frac{c}{\delta} + \frac{1}{1+\delta};$$

so these two conditions are mutually contradictory.

Case C:

C3 can occur since $\delta + \delta^2 > c + \delta - \delta^2$ and $-1 + 2\delta - \delta^2 < c\delta$, but the pay-off is below $1/(1+\delta)$;

C4 fails since $\delta < 1 + c\delta$.

Thus in all cases which can hold the pay-off is no higher than $1/(1+\delta)$.

6.2. When $z_2 = \delta/(1+\delta)$ doesn't secure $1/(1+\delta)$.

This time assume

$$\frac{1}{1+\delta} < \frac{\delta(1-c)}{1+\delta} + \frac{c}{1-\delta^2}.$$

Now Case B1 can be achieved if $z_1 \geq 1 - z_2 = 1/(1+\delta)$ since we have

$$\frac{\delta(1-c)}{1+\delta} \leq \frac{1}{1+\delta}.$$

The pay-off here is at least

$$\frac{\delta}{1+\delta} + (1-c)(1-\delta)$$

and can be achieved with $z_1 = 1/(1+\delta)$; it is easy to verify that

$$\frac{\delta}{1+\delta} + (1-c)(1-\delta) > \frac{1}{1+\delta},$$

since

$$(1-c) > \frac{1}{1+\delta} \Leftrightarrow c < \frac{\delta}{1+\delta}.$$

But $c < 0.25$ and $1 + \delta < 4\delta$ for $\delta > 1/3$.

The only other pay-offs which do not depend on z_1 are those in A1, X1 and X4. In fact A1 cannot occur since

$$\frac{\delta(1-c)}{1+\delta} < \frac{\delta}{1+\delta},$$

and also X4 cannot occur since

$$\frac{\delta(1-c) - \delta^2}{1-\delta^2} < \frac{\delta}{1+\delta}.$$

A review of the earlier arguments shows that the strategy is secure against other pay-off values independent of $z_1$except case X1, which corresponds to the the next higher available pay-off value independent of z_1 (see Lemma 1 in the next section). Moreover

$$(1-c)(1-\delta) + \delta\frac{c + (1-c)(\delta - \delta^2)}{1-\delta^2} = \frac{1 - c - \delta + 2c\delta}{1-\delta^2},$$

since

$$(1-\delta^2)(1-c)(1-\delta) + c\delta + \delta^2(1-c)(1-\delta) = 1 - c - \delta + 2c\delta,$$

so the case B1 pay-off is dominated.

Proposition 3. When $z_1 = \delta/(1+\delta)$ does not yield a flat pay-off then it is conservative for Player-1 to play $z_1 = (1-c-\delta+2c\delta)/(1-\delta^2)$. Moreover the best response

$$z_2 = \frac{1-c-\delta+c\delta-c\delta^2}{1-\delta^2}$$

yields a Nash equilibrium.

Proof. For $z_2 \geq ((1-c)(1-\delta)-c\delta^2)/(1-\delta^2)$ we have by X1 a pay-off of

$$\frac{(1-c-\delta+2c\delta)}{1-\delta^2}.$$

By case A1 for $z_2 < \frac{1-c}{1+\delta}\delta$ we have a pay-off of

$$\frac{1+c\delta}{1+\delta},$$

which is greater than $1/(1+\delta)$. Passing to case A2 note that

$$\frac{\delta(1-\delta)(1-c)+c}{1-\delta^2} > \frac{1-c-\delta+c\delta-c\delta^2}{1-\delta^2}$$

since this is equivalent to $2c(1-\delta+\delta^2) > (1-\delta)^2$ and we already have $c(1-\delta+\delta^2) \geq (1-\delta)^2$. Thus case X1 takes over before case A2 fails. Now the minimum of $1-z_2$ before Case X1 takes over satisfies

$$1-\frac{\delta(1-\delta)(1-c)+c}{1-\delta^2} > \frac{1}{1+\delta}$$

and we note that

$$1-\frac{(1-c-\delta+2c\delta)}{1-\delta^2} \leq \frac{(1-c)\delta}{1+\delta}$$

for $\delta > 0.618..$.

If $z_2 = (1-c-\delta+c\delta-c\delta^2)/1-\delta^2$, then notice that X1 gives Player-1

$$\frac{(1-c-\delta+2c\delta)}{1-\delta^2},$$

provided we allow a weak inequality (as per pseudo-fixed point). This is the maximum under case B1. Case B4 fails since

$$z_1 + z_2 > 1$$

(equivalent to $c < (1-\delta)/(1-\delta)$). This shows also that A1 fails (A2 holds). Finally X4 fails since

$$\frac{1-c-\delta+c\delta-c\delta^2}{1-\delta^2} > \frac{\delta(1-c-\delta)}{1-\delta^2}$$

(equivalent to $c < 1$).

6.3. Some inequalities

In this section we make the **blanket assumptions** that $c < \frac{1}{2}(1-\delta)$ so that $c < (1-\delta)/(2-\delta)$, and that $\delta > 0.5$ so that $c < 0.25$. The following result allows us to place in order the various pay-offs.

Lemma 1:

$$\begin{aligned}\frac{1-\delta-c}{1-\delta^2} &< \frac{(1-c)(1-\delta)-c\delta^2}{1-\delta^2} < \frac{1-c}{1+\delta} < \\ &< \frac{1-c+c\delta}{1+\delta} < \frac{1}{1+\delta} < \left\{\frac{1-c-\delta+2c\delta}{1-\delta^2}, \frac{1+c\delta}{1+\delta}\right\} < \frac{1-\delta+c\delta}{1-\delta^2}.\end{aligned}$$

where $a < \{b, c\}$ means $a < b$ and $a < c$ etc. Furthermore

$$\frac{1-c-\delta+2c\delta}{1-\delta^2} < \frac{1+c\delta}{1+\delta}$$

if and only if

$$\delta^2+\delta-1 < 0,$$

which is equivalent to

$$\delta < \frac{1}{2}(-1+\sqrt{5}) \sim 0.61.. \; .$$

Remarks. Note that

$$\frac{1-\delta-c}{1-\delta^2} < \frac{1}{2}$$

if and only if

$$\frac{1}{2}(1-\delta)^2 < c$$

i.e. provided c is large enough and this is consitent with our blanket assumptions. Also

$$\frac{1}{2} < \frac{(1-c)(1-\delta)-c\delta^2}{1-\delta^2}$$

is equivalent to

$$c < \frac{(1-\delta)^2}{2(1-\delta+\delta^2)}$$

and this is true for $c = 0$, but false for $c = \frac{1}{2}(1-\delta)$, so the inequality is sensitive to the size of c. On the other hand

$$\frac{1}{2} < \frac{1-c}{1+\delta},$$

since this is equivalent to $c < \frac{1}{2}(1-\delta)$. Thus it is possible for an equilibrium position to be below 50%.

Next we consider the end-points defining various case. Here too we have on occasion inequalities sensitive to the size of c.

Lemma 2a

$$\frac{c+\delta}{1+\delta} < \frac{c+\delta-\delta c-\delta^2+c\delta^2}{1-\delta^2}.$$

Indeed $(c+\delta)(1-\delta) = c+\delta-c\delta-\delta^2$. But

Lemma 2b

$$\frac{c+\delta-\delta c-\delta^2+c\delta^2}{1-\delta^2} < \frac{1-c-\delta+\delta c-c\delta^2}{1-\delta^2}$$

is size sensitive since this requires

$$1-2\delta+\delta^2 > 2c[1-\delta+\delta^2]$$

i.e. small c. Also considering numerators only

$$c+\delta-c\delta-\delta^2 < 1-c-\delta+\delta c-c\delta^2$$

is equivalent to

$$c(2-2\delta+\delta^2) < 1-2\delta+\delta^2$$

and will be true only for small c, but for larger c the inequality reverses. Continuing with our numerator review

$$c+\delta-\delta^2 < 1-c-\delta+c\delta$$

is the same as $c(2-\delta) < 1-2\delta+\delta^2$ or

$$c < (1-\delta)\frac{(1-\delta)}{(2-\delta)}.$$

Substituting for c the largest value it is allowed, we have that

$$\frac{1}{2}(1-\delta) > (1-\delta)\frac{(1-\delta)}{(2-\delta)}.$$

Lemma 2c. The inequality

$$\frac{c+\delta-\delta^2}{1-\delta^2} < \frac{1-c}{1+\delta}$$

is true for small c only; however we do evidently have

Lemma 2d.

$$\frac{1-c-\delta+\delta c-c\delta^2}{1-\delta^2} = \frac{(1-c)(1-\delta)-c\delta^2}{1-\delta^2} < \frac{1-c}{1+\delta}.$$

Lemma 2. For $\delta > 0.618..$ we have

$$\begin{aligned}
\frac{\delta(1-c-\delta)}{1-\delta^2} &< \frac{\delta(1-c)(1-\delta)}{1-\delta^2} = \frac{\delta(1-c)}{1+\delta} < \frac{\delta}{1+\delta} < \frac{c+\delta}{1+\delta} < \\
&< \frac{1-c}{1+\delta} < \frac{1-c+c\delta}{1+\delta} < \frac{1}{1+\delta} < \\
&< \frac{1+c\delta}{1+\delta} < \frac{1-c-\delta+2c\delta}{1-\delta^2} < \frac{1-\delta+c\delta}{1-\delta^2} < \frac{(1-c)\delta}{1-\delta^2}.
\end{aligned}$$

Proof. The first seven are immediate, it is enough to show the following holds

$$\frac{1+c\delta}{1+\delta} < \frac{1-c-\delta+2c\delta}{1-\delta^2} < \frac{1-\delta+c\delta}{1-\delta^2}.$$

We need, since $(1-\delta)(1+c\delta) = 1-\delta+c\delta-c\delta^2$, that $0 < c(-1+\delta+\delta^2)$ which holds for $\delta > (-1+\sqrt{5})/2 \sim 0.618..$

Evidently

$$\frac{1-\delta+c\delta}{1-\delta^2} < \frac{(1-c)\delta}{1-\delta^2},$$

since $2c\delta < 2\delta - 1$, and indeed at the largest permitted value for c, we have

$$0 < \delta^2 + \delta - 1.$$

Remark. Note that

$$\frac{1+c\delta}{1+\delta} < \delta$$

since $c\delta < \delta^2+\delta-1$ when c is at its largest $\delta(1-\delta) < 2(\delta^2+\delta-1)$ as $3\delta^2+\delta-2 = (\delta+1)(3\delta-2) > 0$ for $\delta > 2/3 = 0.6666$.

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7. Appendix A: Figures for main text

8. Appendix B: Additional diagrams

We show the behaviour of the equilibrium payoffs versus z_1 and versus z_2 to Player-1 in the two generic cases with $\delta = 0.7$ of the low charge $c = 0.1$ and the high charge $c = 0.15$ regimes. The graphs are conveniently labelled Rez1l, Rez1h, Rez2l, Rez2h.