Optimising the signal-to-noise ratio

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A long-term aim of current research into the frequency assignment problem is to identify features that would enable us to classify instances of the problem in terms of certain easily-computed parameters. Although there are good mathematical reasons why this may be impossible in full generality, experimental evidence is emerging which suggests that instances which occur in practice may be amenable to such classification [1].

One parameter that contributes to the specification of an instance of the problem is the signal-to-noise ratio (SNR). Recently some experimental work [2] has been done on the question of finding the minimum SNR for randomly-generated arrays of transmitters. The results suggest that, for a given configuration, the minima occur on the boundaries of what is known as the Voronoi partition. When the number of transmitters is small, and they are 'regularly-spaced', it is intuitively clear that a result of this kind should hold. But it is possible to conceive configurations in which one powerful transmitter causes complex variations in the SNR within the neighbourhood of a cluster of less powerful transmitters. The question is further complicated by the practical difficulty of specifying an exact propagation law for radio waves under operational conditions.

In this paper we shall state general conditions under which it can be proved that the minima of the signal-to-noise ratio must lie on the boundaries of the Voronoi partition. This result is significant for two reasons.

1. The complexity of calculating the minimum in any specific instance is reduced by an order of magnitude.

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2. Because the propagation law cannot be specified exactly, it is useful to have results that hold under general conditions, valid in a wide range of practical situations.

These facts imply that the classification of a specific problem on the basis of the SNR can be done effectively, but without unnecessary assumptions about the details.

We begin by giving a mathematical formulation of the SNR problem.

Let T be a set of transmitters located on the plane. Each transmitter $t \in T$ has power $p(t)$, and is located on the plane at the coordinates $\overline{x}(t)$. Calculating the strength of a signal from a given transmitter is another non-trivial practical problem. Here we assume that the strength depends only on the distance to the transmitter, and on the power of the transmitter;

$$S_t(\overline{x}_0) = p(t)N(|\overline{x}_0 - \overline{x}(t)|) \tag{1}$$

for some function $N(d)$ (we shall call it *propagation function*) tending to zero with $d \to \infty$. It is often assumed in practice that $N(d) = d^{-\alpha}$, where α is a constant. The value of α may vary; usually $2 < \alpha < 4$.

In real life situations, the strength of the signal may fall at different rates in different directions, and it may depend on the frequency of the signal as well. Also, the picture may be different for different locations of the transmitter.

Suppose that each transmitter is assigned a transmitting frequency (or *channel*) $c(t)$. The signal-to noise ratio (SNR) at a given point $\overline{x}_0$ is calculated in two steps. First, the tuning frequency is chosen as the frequency of that transmitter t_0 whose signal at the point is the strongest.

After choosing the tuning frequency $c_0 = c(t_0)$, one calculates the SNR by the formula

$$SNR(\overline{x}_0) = \frac{S_{t_0}(\overline{x}_0)}{\sum_{t \neq t_0} e^{-\beta|c_0 - c(t)|} S_t(\overline{x}_0)}; \tag{2}$$

where β is another experimentally determined constant.

Now we can formulate the problem. We are given: the set T of transmitters located at points $\overline{x}(t)$ and of powers $p(t)$; a region Ω of possible locations of receivers; the minimum allowed value R of the signal-to-noise ratio. We must find a channel assignment c for which the SNR at every point in Ω is at least R, and which minimizes the span of frequencies, that is

$$\max_{t\in T} c(t) - \min_{t\in T} c(t).$$

We shall denote by $V(t)$ the region in which the signal from t is the strongest; in our model (in which the propagation law does not depend on the frequency) $V(t)$ does not depend on the chosen channel assignment.

If $\overline{x}_0$ lies on the boundary between several regions $V(t)$ then it may tune to either of the corresponding transmitters; in this case we shall use the notation $SNR(\overline{x}_0, c_0)$ where c_0 is the chosen tuning frequency.

In what follows we assume that all transmitters are of equal power; $(\forall t \in T)\ p(t) = p$; and also that the region Ω is the whole of the plane.

Then the regions $V(t)$ form the Voronoi partition of the plane with respect to the set T: each $V(t)$ consists of those points which are closer to t than to any other $t' \in T$, because $S_t(x_0)$ depends only on $|x_0 - \overline{x}(t)|$. Therefore each $V(t)$ is a convex polygon the edges of which are straight line intervals lying on perpendicular bisectors of certain pairs of transmitter locations.

The first problem we encounter is, given a channel assignment c, find the point with the worst signal-to-noise ratio.

THEOREM 1 *Suppose that the propagation function $N(d)$ satisfies the following condition:*

$(*)$ *The derivative $(\log N(d))' = N'(d)/N(d)$ increases monotonically.*

Then for any set T of transmitters of equal power, and for any channel assignment $c : T \to \mathbf{R}$, the signal-to-noise ratio can achieve its minimum only on the boundary of some region $V(t)$.

PROOF. Instead of minimizing signal-to-noise ratio, we shall consider the equivalent problem of maximizing noise-to-signal ratio (NSR).

Let $V = V(t_0)$, and $\overline{x} \in V$ be an arbitrary interior point. By equations (2) and (1), NSR at the point $\overline{x}$ is given by the formula

$$NSR(\overline{x}) = \sum_{t\neq t_0} \lambda(t) \frac{N(|\overline{x} - \overline{x}(t)|)}{N(|\overline{x} - \overline{x}(t_0)|)} \tag{3}$$

where the factors $\lambda(t) = e^{-\beta|c(t_0)-c(t)|} > 0$ do not depend on the position of $\overline{x}$ within $V(t_0)$.

We shall show that NSR increases when $\overline{x}$ is moved radially from t_0 towards the boundary of V; moreover, that every summand in the formula (3) increases.

Let $r = |\overline{x} - \overline{x}(t)|$, $r_0 = |\overline{x} - \overline{x}(t_0)|$. Since $\overline{x} \in V$, we have $r > r_0$. We need to determine the sign of the derivative of

$$\frac{N(|\overline{x} - \overline{x}(t)|)}{N(|\overline{x} - \overline{x}(t_0)|)}$$

in the direction of the vector $\overline{x} - \overline{x}(t_0)$. Let φ be the angle between the vectors $\overline{x} - \overline{x}(t_0)$ and $\overline{x} - \overline{x}(t)$. Then the derivative is equal to

$$\frac{N'(r)N(r_0)\cos\varphi - N(r)N'(r_0)}{N(r_0)^2} =$$
$$= \frac{(\log N(r))'\cos\varphi - (\log N(r_0))'}{N(r_0)/N(r)}.$$

Both derivatives in the numerator are negative; so, if $\cos\varphi \leq 0$ then the expression is obviously positive. On the other hand, if $\cos\varphi > 0$ then the expression is positive because of $(*)$. The theorem is proved. $\square$

In particular, condition $(*)$ of this Theorem is satisfied for any function $N(d) = d^{-\alpha}$, $\alpha > 0$. Also, $(*)$ implies that N is strictly convex on $(0, \infty)$, i.e. that its first derivative increases monotonically.

In the case when, as it is in practical problems, the region Ω is bounded, the theorem can still be applied; neither its formulation nor the proof need to be changed. Indeed, the boundary of Ω is partitioned into parts of the boundaries of adjacent Voronoi regions; and the theorem implies that the minimum of SNR is achieved either on the edges of Voronoi diagram or on the boundary of Ω.

References

[1] S. Hurley, D.H. Smith, S.U. Thiel. Fasoft: A system for discrete channel frequency assignment. *Radio Science*, 32(5), 1997, 1921–1939.

[2] J. Bater, P. Jeavons, D. Cohen, N. Dunkin. Are there Effective Frequency Separation Constraints for FAPs with Irregular Tx Placements?. *Royal Holloway Technical Report*, 1998.