Convex Bodies, Graphs and Partial Orders

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Abstract

A *convex corner* is a compact convex down-set of full dimension in $\mathbb{R}^n_+$. Convex corners arise in graph theory, for instance as stable set polytopes of graphs. They are also natural objects of study in geometry, as they correspond to 1-unconditional norms in an obvious way. In this paper, we study a parameter of convex corners, which we call the *content*, that is related to the volume. This parameter has appeared before, either implicitly or in special cases, both in geometry and in combinatorics, and a major aim of the paper is to expose connections between work in these two areas.

We define the *content* $\mu(S)$ of a convex corner S in $\mathbb{R}^n_+$ recursively by

$$\mu(S) = \max_{\mathbf{x}\in S} \sum_{i=1}^{n} x_i \mu(S_i),$$

where S_i is the projection of S on to the i^{th} co-ordinate plane, and $\mu(\{0\}) = 1$ for the zero-dimensional convex corner $\{0\}$.

As part of his proof of Saint-Raymond's Inequality – $\text{Vol}(S)\text{Vol}(S^\circ) \geq 1/n!$ – Meyer [20] effectively showed that $\mu(S) \leq n!\text{Vol}(S)$ for any convex corner S. Sidorenko [25] showed that, if S is the stable set polytope of the comparability graph of a partial order P, then $\mu(S)$ is the number of linear extensions of P and thus, via a result of Stanley [26], the content $\mu(S)$ is equal to $n!\text{Vol}(S)$. We show that, if S is a 0-1 polytope, then $\mu(S) = n!\text{Vol}(S)$ if and only if S is the stable set polytope of a strongly perfect graph.

The other ingredient in Meyer's proof is that $\mu(S)\mu(S^\circ) \geq n!$ for any convex corner S and its polar S°. We extend this result by working with the *pointwise product* of convex corners. The pointwise product of two sets A and B is $A \circ B = \{\mathbf{x} \circ \mathbf{y} : \mathbf{x} \in A, \mathbf{y} \in B\}$, where $(\mathbf{x} \circ \mathbf{y})_i = x_i y_i$. We show that, if A, B and C are convex corners such that the pointwise product $A \circ B$ is contained in C, then we have $\mu(A) \leq \mu(B^\circ)\mu(C)$. This leads to the following generalization of Saint-Raymond's Inequality: if $A_1, \ldots, A_r$ are any bounded sets in $\mathbb{R}^n_+$, such that $A_1 \circ \cdots \circ A_r$ is contained in the ℓ_1 unit ball, then $\mu(A_1^\circ) \ldots \mu(A_r^\circ) \geq n!$ and so $\text{Vol}(A_1^\circ) \ldots \text{Vol}(A_r^\circ) \geq (n!)^{-r+1}$.

We prove various other results concerning content and volume of convex corners, and give counterexamples to two conjectures of Bollobás and Leader [5] about pointwise products. We conclude with a number of open problems.

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1 Introduction and Definitions

In a 1991 paper [25], Sidorenko introduces the function $\mu^{\mathrm{g}}(G)$, defined for a graph G recursively by setting $\mu^{\mathrm{g}}(G) = 1$ if G has one node, and

$$\mu^{\mathrm{g}}(G) = \max_{I \in \mathcal{I}} \sum_{v \in I} \mu^{\mathrm{g}}(G - v),$$

where $\mathcal{I}$ denotes the set of independent sets in G. (To be more accurate, our $\mu^{\mathrm{g}}(G)$ is Sidorenko's $\mu(\overline{G})$.) We shall call $\mu^{\mathrm{g}}(G)$ the *graph-content* of G. Sidorenko proves that, in the case where G is the comparability graph of a partial order P, $\mu^{\mathrm{g}}(G)$ is equal to the number of linear extensions of P.

A very similar concept had been used in 1986 by Meyer [20] in a completely different context, namely to provide a simple proof of Saint-Raymond's Inequality, which can be stated as follows.

Theorem 1.1 *If T is the unit ball of a 1-unconditional norm in $\mathbb{R}^n$, T^* is the unit ball of the dual norm, $S = T \cap \mathbb{R}^n_+$ and $S^\circ = T^* \cap \mathbb{R}^n_+$, then* $\mathrm{Vol}(S)\mathrm{Vol}(S^\circ) \geq 1/n!$.

The first proof of Theorem 1.1 was given by Saint-Raymond [23] in 1980. Besides Meyer's proof, other short proofs were found by Reisner [22] and by Bollobás, Leader and Radcliffe [6].

Meyer's approach amounts to considering a parameter $\mu(S)$, which we call the *content* of S, with a recursive definition very similar to that of μ^{g} above, showing that $\mu(S) \leq n!\mathrm{Vol}(S)$, $\mu(S^\circ) \leq n!\mathrm{Vol}(S^\circ)$, and $\mu(S)\mu(S^\circ) \geq n!$. Theorem 1.1 evidently follows from these three inequalities. Meyer does not define the content explicitly, and the three inequalities here are woven together in his proof.

It turns out that Sidorenko's parameter $\mu^{\mathrm{g}}(G)$ is equal to the content of the *stable set polytope* of the graph G, so Meyer's and Sidorenko's work reflect different aspects of the same theory. Another concept that turns out to be related is the *pointwise product* of bodies in $\mathbb{R}^n_+$, studied by Bollobás and Leader [4], [5].

Our purpose in this paper is to initiate study of the content for its own sake. We clarify the relationships among results of Meyer, Sidorenko and others, and, building on their work, we give a number of results (none of which is especially difficult to prove within its context), directed at bringing out the strong connections between the theories of volume products, perfect graphs, and partial orders. Our aim is to emphasize the interplay between these areas, and to answer the questions that seem to us to arise naturally.

Before going into more detail, we need a number of definitions. We then state the main results of the paper, along with further discussion.

1.1 Definitions

In this paper, we study concepts from several different areas of mathematics, and so there are a large number of basic terms, and a few results, that we feel obliged to introduce in detail. However, most of the definitions below are fairly standard, the principal exceptions being those of content and of pointwise product.

A *convex corner* in $\mathbb{R}^n_+$ is a compact convex subset S of $\mathbb{R}^n_+$, with non-empty interior, which is a down-set, i.e., if $x_i \leq y_i$ for $i = 1, \ldots, n$, and $\mathbf{y} \in S$, then $\mathbf{x} \in S$.

A norm $\|.\|$ on $\mathbb{R}^n$ is *1-unconditional* (with respect to the standard basis of $\mathbb{R}^n$) if it has the property that

$$\|(x_1, \ldots, x_n)\| = \|(|x_1|, \ldots, |x_n|)\|$$

for every vector $\mathbf{x} = (x_1, \ldots, x_n)$. The unit ball of a 1-unconditional norm is called a *1-unconditional body*. It is easy to see that the intersection of a 1-unconditional body in $\mathbb{R}^n$ with the closed positive quadrant is a convex corner. Conversely, if S is a convex corner, then $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : (|x_1|, \ldots, |x_n|) \in S\}$ is a 1-unconditional body. Thus convex corners correspond naturally to 1-unconditional norms. We work with convex corners throughout, but of course many of our results can be restated in terms of 1-unconditional bodies or norms. For more information about the geometry of finite-dimensional normed spaces, a good reference is the book by Lindenstrauss and Tzafriri [16]. The term "convex corner" was used by Csiszár, Körner, Lovász, Marton and Simonyi [10].

For S a subset of $\mathbb{R}^n_+$, we define the *polar* S° of S to be

$$\{\mathbf{y} \in \mathbb{R}^n_+ : \mathbf{x} \cdot \mathbf{y} \leq 1 \text{ for all } \mathbf{x} \in S\}.$$

It is easy to see that, if S is bounded, and not contained in any face $\{\mathbf{x} : x_i = 0\}$, then S° is a convex corner. Furthermore, if S is the convex corner associated with a 1-unconditional norm, then S° is the convex corner associated with the dual norm. It is clear that, if S is a convex corner, then $(S^\circ)^\circ = S$. In the literature of combinatorial optimization (when S will normally be a polytope), S° is often called the *antiblocker* or *blocking polytope*, and is typically denoted S^*. See for instance the book of Grötschel, Lovász and Schrijver [12].

A point $\mathbf{x}$ in a convex corner S is *maximal* if, whenever $\mathbf{y} \in S$, with $x_i \leq y_i$ for every i, we have $\mathbf{y} = \mathbf{x}$.

The *support* $\operatorname{supp}(\mathbf{z})$ of a vector $\mathbf{z} \in \mathbb{R}^n$ is the set of co-ordinates j such that $z_j \neq 0$. For S a convex corner in $\mathbb{R}^n_+$, and $i \in [n] \equiv \{1, \ldots, n\}$, we define S_i to be the set of points in S whose support does not include i. So S_i is an $(n-1)$-dimensional convex corner in the i^{th} co-ordinate plane. More generally, if $A \subseteq [n]$, then we define S_A to be the set of points in S whose support is contained in A; S_A is a convex corner in the lower-dimensional space defined by A. We shall use subscripts in this way throughout: a subscript i will indicate reduction by ignoring the i^{th} co-ordinate, a subscript A will indicate restriction to the subspace defined by the co-ordinates in set A.

An exception to this rule is that we use $\mathbf{e}_i$ to denote the i^{th} standard basis vector in $\mathbb{R}^n$. For every subset A of $[n]$, we denote by $\mathbf{e}_A$ the indicator vector of the set A.

We are now ready to define the *content* $\mu(S)$ of a convex corner S, recursively. First we define the content of $\{0\}$ (the unique zero-dimensional convex corner) to be 1. Now, for $n \geq 1$, and S a convex corner in $\mathbb{R}^n_+$, we set

$$\mu(S) = \max_{\mathbf{x} \in S} \sum_{i=1}^n x_i \mu(S_i).$$

We give some examples, starting with the convex corners B^n_p obtained by intersecting the unit balls of the various spaces ℓ^n_p with the positive quadrant:

$$B^n_p = \{\mathbf{x} \in \mathbb{R}^n_+ : x_1^p + \cdots + x_n^p \leq 1\},$$

for $1 \leq p < \infty$. Similarly, $B^n_\infty = [0,1]^n$, the intersection of the ℓ^n_∞ unit ball with the positive quadrant. We also define B^n_p as above for $0 < p < 1$, but this is not a convex set. For $1 \leq p \leq \infty$, the polar of B^n_p is B^n_q, where $\frac{1}{p} + \frac{1}{q} = 1$.

For $S = B^n_p$, $1 \leq p \leq \infty$, the maximization in the definition of the content $\mu(S)$ occurs at the point on the boundary of S where all co-ordinates are equal – indeed this is true whenever the symmetry group of S is transitive on the co-ordinate axes – namely $x_i = n^{-1/p}$ for all i, and S_i

is isomorphic to B_p^{n-1} for all i, so $\mu(S) = n^{1-1/p}\mu(S_i)$ and thus, by induction, $\mu(B_p^n) = (n!)^{1-1/p}$. For comparison, the volume of B_p^n, for $1 \le p < \infty$, is given by

$$\mathrm{Vol}(B_p^n) = \left(\frac{\Gamma(1/p)}{p}\right)^n \frac{1}{\Gamma(1+n/p)}.$$

Another important special case is where the convex corner S is a polytope, in which case the definition of μ involves maximizing a linear function over a convex polytope, so the maximum is attained at a vertex. If, in addition, all vertices of S are 0-1 vectors, we call S a *0-1 convex corner*; in this case the set of vertices of S corresponds to a collection $\mathcal{J}$ of subsets of $[n]$, and we have

$$\mu(S) = \max_{J \in \mathcal{J}} \sum_{j \in J} \mu(S_j).$$

For instance, consider the 0-1 convex corner

$$T_t^n = \{\mathbf{x} \in [0,1]^n : \sum x_i \le t\},$$

for $n, t \in \mathbb{N}$. For $n \le t$, $T_t^n = B_\infty^n$, so $\mu(T_t^n) = n!$. For $n \ge t$, the maximization in the definition of $\mu(T_t^n)$ is achieved at a vertex, and all vertices $\mathbf{x}$ have t co-ordinates equal to 1 and the rest 0, so $\mu(T_t^n) = t\mu((T_t^n)_1) = t\mu(T_t^{n-1})$, so $\mu(T_t^n) = t!t^{n-t}$.

Let G be a graph with node set $[n]$ (we use the term *node* rather than the more widely accepted *vertex* to avoid confusion with the vertices of polytopes). The *stable set polytope* $\mathrm{Stab}(G)$ of G is the 0-1 convex corner whose vertices correspond to the stable sets (independent sets) of G:

$$\mathrm{Stab}(G) = \mathrm{conv}\{\mathbf{e}_I : I \in \mathcal{I}(G)\}.$$

where $\mathcal{I}(G)$ is the family of stable sets of G. Observe that $\mu(\mathrm{Stab}(G))$ is exactly $\mu^{\mathrm{g}}(G)$, as defined in §1.

The polar of a stable set polytope is a well-studied concept in the context of combinatorial optimization. Clearly we have, for G an n-node graph, that

$$(\mathrm{Stab}(G))^\circ = \{\mathbf{x} \in \mathbb{R}_+^n : \sum_{i \in I} x_i \le 1 \text{ for all } I \in \mathcal{I}(G)\},$$

and indeed we may restrict attention to the *maximal* independent sets in the above description. In particular, if C is a clique in a graph G, then the vector $\mathbf{e}_C$ is in $(\mathrm{Stab}(G))^\circ$. In other words, all vertices of the stable set polytope of the complement $\overline{G}$ are in $(\mathrm{Stab}(G))^\circ$. Thus we have that

$$\mathrm{Stab}(\overline{G}) \subseteq (\mathrm{Stab}(G))^\circ.$$

We do not always have equality above: the easiest example is when $G = \overline{G} = C_5$, the cycle on 5 nodes, when the vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is in $(\mathrm{Stab}(C_5))^\circ$ but not in $\mathrm{Stab}(C_5)$.

A graph G is *perfect* if, for every induced subgraph H of G, the chromatic number of H is equal to the size of its largest clique. Lovász [17] proved the *Perfect Graph Theorem*, stating that, if a graph is perfect, then so is its complement; as part of that proof, he obtained the following alternative characterization of perfect graphs.

Theorem 1.2 $\mathrm{Stab}(\overline{G}) = (\mathrm{Stab}(G))^\circ$ *if and only if* G *is perfect.*

The polytope $\mathrm{Stab}(\overline{G})^\circ$ is called the *fractional stable set polytope* of G, and is sometimes denoted $\mathrm{FStab}(G)$. An equivalent formulation of Theorem 1.2 is to say that $\mathrm{Stab}(G) = \mathrm{FStab}(G)$ if and only if G is perfect. See Grötschel, Lovász and Schrijver [12] or Toft [27] for further information on perfect graphs and related topics.

A *strongly perfect graph* is a graph G with the property that, for every induced subgraph H of G, there is an independent set in H meeting every maximal clique of H. (Note that a graph is perfect if and only if, for every induced subgraph H, there is an independent set meeting every *maximum size* clique, so a strongly perfect graph is certainly perfect.) Strongly perfect graphs were introduced by Berge and Duchet [2] in 1984, and further investigated in papers of Chvátal [9], Hoàng, Maffray and Preissman [14], and Hoàng [13]. Attention has principally focussed on relations between the class of strongly perfect graphs and other classes of perfect graphs.

For a partial order $P = (X, <)$, the *comparability graph* $C(P)$ of P is the graph on X obtained by setting x and y adjacent if they are comparable in P, i.e., if either $x < y$ or $y < x$. The complement of $C(P)$ is the *incomparability graph* $I(P)$ of P. An independent set in $C(P)$ is a set of mutually incomparable elements of P – an *antichain*. Comparability graphs of partial orders are strongly perfect, since the set $M(Q)$ of maximal elements of any induced subposet $Q \subseteq P$ is an independent set meeting all maximal chains in Q (i.e., all maximal cliques in $C(Q)$. That incomparability graphs are perfect follows either from the Perfect Graph Theorem or from Dilworth's Theorem.

There are many graphs that are strongly perfect but are not comparability graphs; the simplest are the 6-node graph H obtained by attaching a pendant node to each node of a triangle, and the complement of H. The interesting problem of determining the class of minimal non-strongly perfect graphs seems to be open. This class certainly includes:

- odd cycles C_{2n+1} of length at least 5,
- the complements $\overline{C_n}$ of cycles of length $n \geq 5$,
- graphs obtained from a triangular prism ($\overline{C_6}$) by replacing each of the three edges not in triangles by odd-length paths.

The *intersection* $P \cap Q$ of partial orders $P = (X, <_P)$ and $Q = (X, <_Q)$ on the same set X is defined by setting $x < y$ in $P \cap Q$ if both $x <_P y$ and $x <_Q y$. A *linear extension* of a partial order $P = (X, <)$ is a linear (total) order $\prec$ on X consistent with $<$, i.e., such that $x \prec y$ whenever $x < y$. The number of linear extensions of P is denoted $e(P)$.

We next introduce an operation on convex corners, or more generally on subsets of $\mathbb{R}^n$, which has attracted some interest, and for which content seems to be a useful tool. For vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, define the *pointwise product* of $\mathbf{x}$ and $\mathbf{y}$ to be the vector $\mathbf{x} \circ \mathbf{y}$ with $(\mathbf{x} \circ \mathbf{y})_i = x_i y_i$. The pointwise product of three or more vectors is defined in a similar way, e.g., $\mathbf{x} \circ \mathbf{y} \circ \mathbf{z} = (\mathbf{x} \circ \mathbf{y}) \circ \mathbf{z}$, with the i^{th} co-ordinate being simply $x_i y_i z_i$. It is immediate that, for instance, $(\mathbf{x} \circ \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot (\mathbf{y} \circ \mathbf{z})$.

Now, for sets S and T in $\mathbb{R}^n_+$, we define the pointwise product of S and T to be the set

$$S \circ T = \{\mathbf{x} \circ \mathbf{y} : \mathbf{x} \in S, \mathbf{y} \in T\}.$$

Study of the pointwise product $S \circ T$ for its own sake was initiated by Bollobás and Leader [4]; various properties of the product are discussed in that paper and in its more extended version [5]. The pointwise product also appears in the paper by Csiszár, Körner, Lovász, Marton and Simonyi [10], who use the notation $S \circ T$, and doubtless explicitly or implicitly in many other works.

Note that, even if S and T are 0-1 convex corners, $S \circ T$ need not be convex; for instance, if $S = T = B_1^n$, then

$$S \circ T = B_{1/2}^n \equiv \{\mathbf{x} \in \mathbb{R}_+^n : \sum_{i=1}^n x_i^{1/2} \leq 1\},$$

a non-convex set.

For S a convex corner and $r \in \mathbb{N}$, let $S^{1/r}$ be the set of vectors $(x_1, \ldots, x_n)$ such that $(x_1^r, \ldots, x_n^r) \in S$. For instance $(B_p^n)^{1/r} = B_{rp}^n$. It is easy to verify that, for any convex corner S, the pointwise product of r copies of $S^{1/r}$ is S.

From convex corners $S \subseteq (\mathbb{R}_+^n)_A$ and $T \subseteq (\mathbb{R}_+^n)_B$, where A and B are disjoint sets whose union is $[n]$, we can form a convex corner in $\mathbb{R}_+^n$ in several ways. The ℓ_∞-*sum* of S and T is just the cartesian product of S and T, while the ℓ_1-*sum* is the convex hull $\mathrm{conv}(S \cup T)$.

For a convex corner S in $\mathbb{R}_+^n$, let

$$V(S) = \max_{\mathbf{x} \in S} \prod_{i=1}^n x_i.$$

Thus $V(S)$ is the volume of the largest n-dimensional cuboid contained in S. The *entropy* of a convex corner (with respect to the uniform distribution) is defined as $-\log_2 V(S)$. See, for instance, Csiszár, Körner, Lovász, Marton and Simonyi [10]. As part of their work on entropy of stable set polytopes of comparability graphs, Kahn and Kim [15] proved the following result, showing that $V(S)$ approximates $\mathrm{Vol}(S)$ to within an exponential factor – to be more precise, within a factor of e^n.

Theorem 1.3 *For any convex corner S,* $\mathrm{Vol}(S) \leq \dfrac{n^n}{n!} V(S)$.

The example of the unit simplex B_1^n shows that Theorem 1.3 is best possible.

1.2 Main Results

Here, we present the main results concerning content, both from this paper and from other sources. As we explained earlier, the following two results, giving perhaps the most important properties of the content, are essentially due to Meyer [20].

Theorem 1.4 *For any convex corner S in $\mathbb{R}_+^n$, $\mu(S) \leq n!\mathrm{Vol}(S)$.*

Theorem 1.5 *For any convex corner S in $\mathbb{R}_+^n$, $\mu(S)\mu(S^\circ) \geq n!$.*

In §2.1, we give the (quite simple) proof of Theorem 1.4, and extend it by defining a class of (not necessarily convex) polyhedra called *skew-cubes*, and showing that $\mu(S)/n!$ is equal to the volume of the largest skew-cube contained in S. We also show that $\mu(S) = n!\mathrm{Vol}(S)$ if and only if S is a skew-cube. In §2.2 we give various characterizations of skew-cubes, and in §2.3 we briefly discuss how far from equality we can get.

In §3, we investigate Theorem 1.5 in a similar way: we give a proof, discuss cases of equality, and investigate how far from equality we can get.

The next two inequalities follow immediately on combining Theorems 1.4 and 1.5.

Corollary 1.6 *For any convex corner S in $\mathbb{R}_+^n$, $\mu(S) \geq 1/\mathrm{Vol}(S^\circ)$.* □

We shall show (Theorem 3.3) that we only have equality in Corollary 1.6 for a very small class of convex corners; if $\mu(S)\mathrm{Vol}(S^\circ) = 1$ then, after scaling so that all the unit vectors $\mathbf{e}_i$ are on the boundary, S is a 0-1 polytope. We describe in §7.3 a class of 0-1 polytopes for which we do have equality in Corollary 1.6, and conjecture that this is the entire class of extremal examples.

Corollary 1.7 *For any convex corner S in $\mathbb{R}^n_+$, $\mathrm{Vol}(S)\mathrm{Vol}(S^\circ) \geq 1/n!$.* □

Of course, this is just a restatement of Saint-Raymond's Inequality, Theorem 1.1, for which the cases of equality were determined by Meyer [20] and Reisner [22]. Throughout the paper, we prove various strengthenings of Corollary 1.7, such as this result from §3.

Theorem 1.8 *For any convex corner S in $\mathbb{R}^n_+$, there are convex polytopes $R \subseteq S$ and $R' \subseteq S^\circ$, each with at most 2^n vertices, such that $\mathrm{Vol}(R)\mathrm{Vol}(R') \geq 1/n!$.*

In the case where S is the stable set polytope $\mathrm{Stab}(G)$ of a graph G with node set $[n]$, so that $\mu(S) = \mu^{\mathrm{g}}(G)$, Theorem 1.4 translates to the following.

Theorem 1.9 *For any graph G, $\mu^{\mathrm{g}}(G) \leq n!\mathrm{Vol}(\mathrm{Stab}(G))$.* □

In §4.2, we shall prove the following result, which includes a characterization of the extremal graphs in Theorem 1.9.

Theorem 1.10 *A 0-1 convex corner S satisfies $\mu(S) = n!\mathrm{Vol}(S)$ if and only if it is the stable set polytope of a strongly perfect graph.*

Theorem 1.5 can also be interpreted in the case when $S = \mathrm{Stab}(G)$, especially when G is perfect, so that $S^\circ = \mathrm{Stab}(\overline{G})$. These themes are explored in §4.

As we have seen, the comparability graph $C(P)$ of any partial order P is strongly perfect, so $\mu^{\mathrm{g}}(C(P)) = n!\mathrm{Vol}(C(P))$. In fact, as we now discuss, both sides of this identity have earlier been shown to be equal to $e(P)$, the number of linear extensions of P. One part of this is a 1986 result of Stanley [26].

Theorem 1.11 *For any partial order P, $e(P) = n!\mathrm{Vol}(\mathrm{Stab}(C(P)))$.*

To see the direct connection between $e(P)$ and $\mu^{\mathrm{g}}(C(P))$, observe that the set $M(P)$ of maximal elements of a partial order P is an independent set in $C(P)$, and that we can partition the set of linear extensions of P according to which element of $M(P)$ is top. Moreover, the number of linear extensions with a given maximal element m top is equal to $e(P-m)$, and so we have

$$e(P) = \sum_{m \in M(P)} e(P-m).$$

Comparing with the definition of μ^{g} gives that $e(P) \leq \mu^{\mathrm{g}}(C(P))$, for every partial order P. To obtain the reverse inequality, Sidorenko [25] proved the next result.

Theorem 1.12 *For any partial order P, and any antichain A in P, $e(P) \geq \sum_{x \in A} e(P-x)$.*

As noted by Sidorenko, Theorem 1.12 implies that

$$e(P) = \max_A \sum_{x \in A} e(P-x),$$

where the maximum is over all antichains A of P. Since the one-element partial order has just one linear extension, we have the next result, again due to Sidorenko.

Theorem 1.13 *For any partial order P, $e(P) = \mu^{\mathrm{g}}(C(P))$.*

Given our results, Sidorenko's Theorem 1.13 and Stanley's Theorem 1.11 are equivalent. However, their proofs are quite different. Sidorenko proves Theorem 1.12 using a very elegant argument about a certain flow through the elements of a partial order. Stanley proves Theorem 1.11 by giving a measure-preserving bijection between $\mathrm{Stab}(C(P))$ and the *order polytope* of P.

We prove almost no new results about partially ordered sets in this paper. However it was consideration of the results of Stanley and Sidorenko that led us to work on content, and the applications to partial orders remain a major source of motivation. Moreover, as we shall see, Stanley's theorem is useful to us in that it provides a relatively practical method for finding or approximating the volumes of bodies that happen to be stable set polytopes of comparability graphs: counting linear extensions is #P-complete in general (this was proved by Brightwell and Winkler [8]), but in practice it is usually much easier than calculating volumes directly.

In §5, we are concerned with the pointwise product of convex corners. In §5.1, we prove the following inequality.

Theorem 1.14 *Let A, B and C be convex corners in $\mathbb{R}^n_+$. If $A \circ B \subseteq C$, then $\mu(A) \leq \mu(B^\circ)\mu(C)$.*

Note that this is a generalization of Theorem 1.5, which is the special case with $A = B^n_\infty$ and $B = C$. Thus, in view of Theorem 1.4, it can be seen as a more general version of Saint-Raymond's Theorem. Theorem 1.14 also generalizes results of Sidorenko [25], who deals with the special case where the convex corners are stable set polytopes of graphs, and in particular B is the stable set polytope of the incomparability graph of a partial order P (so that $\mu(B^\circ)$ is the number of linear extensions of P). Although there is not much more to the proof of Theorem 1.14 than to that of Theorem 1.5, the extra generality enables us to work with pointwise products of more than two bodies. In particular, we use Theorem 1.14 to prove the following result, which is an extension of Saint-Raymond's Theorem to pointwise products of several bodies, and might well be considered as the most attractive result in this paper.

Theorem 1.15 *If $A_1, \ldots, A_r$ are convex corners in $\mathbb{R}^n_+$ such that $A_1 \circ \cdots \circ A_r \subseteq B^n_1$, then*

$$\mathrm{Vol}(A_1^\circ) \ldots \mathrm{Vol}(A_r^\circ) \geq \frac{1}{n!^{r-1}}.$$

We also determine the class of r-tuples of convex corners $(A_1, \ldots, A_r)$ for which equality holds in Theorem 1.15, generalizing the results of Meyer [20] and Reisner [22] for the case $r = 2$.

A major motivation for Bollobás and Leader [5] in their investigation of the pointwise product was the idea of generalizing the notion of the polar: to say that $A \circ B \subseteq B^n_1$ is equivalent to saying that $A \subseteq B^\circ$, so one might hope to replace theorems about a body and its polar by theorems with the condition that $A \circ B$ is contained in some other body C. For instance, they prove the following pair of inequalities.

Theorem 1.16 (a) *If A and B are down-sets in $\mathbb{R}^n_+$, then*

$$\mathrm{Vol}(A)\mathrm{Vol}(B) \geq \mathrm{Vol}(A \circ B).$$

(b) *If A, B and C are convex corners in $\mathbb{R}^n_+$ with $A \circ B \subseteq C$, then*

$$\mathrm{Vol}(A)\mathrm{Vol}(B) \leq (\mathrm{Vol}(C^{1/2}))^2.$$

Theorem 1.16(b) extends Santaló's Theorem [24] for unconditional bodies, which states that, for a convex corner S in $\mathbb{R}_+^n$, $\mathrm{Vol}(S)\mathrm{Vol}(S^\circ) \le (\mathrm{Vol}(B_2^n))^2$. (The full version of Santaló's Theorem is that $\mathrm{Vol}(T)\mathrm{Vol}(T^*) \le (\mathrm{Vol}(C_2^n))^2$, whenever T and T^* are the unit balls of dual norms on $\mathbb{R}^n$, and C_2^n is the ℓ_2 unit ball. For a short proof, see Meyer and Pajor [21].)

Theorem 1.16(a) is another generalization of Saint-Raymond's Inequality, since if $B = A^\circ$, then $A \circ B = B_1^n$, of volume $1/n!$. One possible drawback of this result is that $A \circ B$ is not in general convex, and its volume may not really be the object of interest.

In [5], Bollobás and Leader conjecture (Conjecture 7 of [5]) that, if A, B and C are convex corners such that $A \circ B \subseteq C$, with A and B maximal and C minimal subject to this condition, then $A \circ B = C$. They prove that this holds, even without the assumption of minimality of C, for a large family of bodies C, including all the B_p^n $(1 \le p \le \infty)$ – this generalizes Lozanovskii's Theorem [18], which is the result for $C = B_1^n$ – and also in the case where A satisfies a "smoothness" condition and C is "strictly unconditional". An equivalent version of the conjecture is that, if S, T and U are convex corners that are maximal subject to the condition that $S \circ T \circ U \subseteq B_1^n$, then $S \circ T \circ U = B_1^n$ – to see the equivalence, take $S = A$, $T = B$ and $U = C^\circ$, and use Lozanovskii's Theorem.

Bollobás and Leader also conjecture (Conjecture 17 of [5]) that, under the same conditions, $\mathrm{Vol}(A)\mathrm{Vol}(B) \ge \mathrm{Vol}(C)$ (compare with Theorem 1.16(a)). By Saint-Raymond's Inequality, this would imply that, if S, T and U are maximal convex corners subject to $S \circ T \circ U \subseteq B_1^n$, then $\mathrm{Vol}(S)\mathrm{Vol}(T)\mathrm{Vol}(U) \ge 1/n!$.

In §5.2, we prove the following result, providing a counterexample to all these conjectures, showing in particular that some smoothness condition is vital for Conjecture 7 of [5], even for $n = 3$.

Theorem 1.17 *There are convex corners S_1, S_2 and S_3 in $\mathbb{R}_+^3$ with the following properties.*

- $S_1 \circ S_2 \circ S_3 \subseteq B_1^3$, *and the S_i are each maximal subject to this condition.*
- $S_1 \circ S_2 \circ S_3 \ne B_1^3$.
- $\mathrm{Vol}(S_i) = \frac{1}{2}$ *for each i, so that* $\mathrm{Vol}(S_1)\mathrm{Vol}(S_2)\mathrm{Vol}(S_3) < \frac{1}{3!}$.

Indeed, we can take, for $i = 1, 2, 3$, $S_i = \mathrm{Stab}(G_i)$, where G_1, G_2 and G_3 are the graphs on $\{1, 2, 3\}$ with edge sets $\{12\}$, $\{13\}$ and $\{23\}$ respectively.

The maximality of the S_i is the most difficult part of the theorem to verify: we deduce it from a general result (Theorem 5.8) characterizing the r-tuples of graphs $(G_1, \dots, G_r)$ on $[n]$ such that each polytope $S_i = \mathrm{Stab}(G_i)$ is maximal subject to $S_1 \circ \cdots \circ S_r \subseteq B_1^n$.

In §6, we go on to show that Bollobás and Leader's Conjecture 17 is not even approximately true: for larger values of n, there are convex corners S, T, U, maximal subject to $S \circ T \circ U$ being contained in B_1^n, with $\mathrm{Vol}(S)\mathrm{Vol}(T)\mathrm{Vol}(U)$ around $\frac{1}{n!} n^{-n/6}$.

In the light of the example in Theorem 1.17, we regard the condition that $A_1 \circ \cdots \circ A_r = B_1^n$ as the most appropriate extension of the condition that A_1 and A_2 are polar bodies. In §6, we prove the inequalities in the next result. For $r = 2$, the condition is equivalent to saying that A_1 and A_2 are polars; thus both lower bounds for $r = 2$ amount to Saint-Raymond's Inequality, while the upper bounds for $r = 2$ are exactly Santaló's Inequality for unconditional bodies.

Theorem 1.18 *Suppose that $A_1, \dots, A_r$ are convex corners in $\mathbb{R}_+^n$ such that $A_1 \circ \cdots \circ A_r = B_1^n$. Then*

$$\frac{1}{n!} \le \mathrm{Vol}(A_1) \dots \mathrm{Vol}(A_r) \le (\mathrm{Vol}(B_r^n))^r, \tag{1}$$

$$\frac{1}{n!^{r-1}} \leq \mathrm{Vol}(A_1^\circ)\ldots\mathrm{Vol}(A_r^\circ) \leq (\mathrm{Vol}(B^n_{r/(r-1)}))^r. \tag{2}$$

All these inequalities are best possible.

One of the inequalities above is just Theorem 1.15; the others are all fairly easy consequences of results of Bollobás and Leader from [5].

In §7, we remark on complexity issues, give an assortment of other inequalities, and conclude with some open problems.

2 Content and Volume

2.1 Skew-cubes

We begin with Meyer's proof of Theorem 1.4, recast to make explicit the involvement of the content. Aspects of this proof form the basis of everything we do in this subsection.

Proof of Theorem 1.4. We work by induction on n, noting that the result is trivial for $n = 0$. For a convex corner S in $\mathbb{R}^n_+$ with $n \geq 1$, take a point $\mathbf{x} \in S$ such that $\mu(S) = \sum_{i=1}^n x_i \mu(S_i)$. For each i, consider the cone $\mathrm{conv}(\mathbf{x}, S_i)$ of volume $\frac{1}{n} x_i \mathrm{Vol}(S_i)$. The interiors of these cones are disjoint, and all the cones lie inside S. Thus $\mathrm{Vol}(S) \geq \frac{1}{n}\sum_{i=1}^n x_i \mathrm{Vol}(S_i)$. By induction, we have that $\mathrm{Vol}(S_i) \geq \mu(S_i)/(n-1)!$, and the result follows. □

It is clear from inspection of the proof that we do not always have equality. What we shall now show is that $\mu(S)/n!$ is equal to the volume of the largest set of a certain type inside S. We need a little more notation before we can make the required definition. All of what follows can be understood by consideration of the proof of Theorem 1.4 above.

For each permutation σ of $[n]$, and each $i = 0, \ldots, n$, set $\mathbf{e}_{\sigma,i} = \mathbf{e}_{\{\sigma(1),\ldots,\sigma(i)\}}$. Thus, for any σ, $\mathbf{e}_{\sigma,0} = \mathbf{0}$, and $\mathbf{e}_{\sigma,n} = \mathbf{1}$. Now, for each σ, define a simplex X_σ to be the convex hull of the $n+1$ vectors $\mathbf{e}_{\sigma,i}$. It is easy to see that a vector $\mathbf{x} \in [0,1]^n$ is in X_σ if and only if $x_{\sigma(1)} \geq x_{\sigma(2)} \geq \cdots \geq x_{\sigma(n)}$. Thus the $n!$ simplices X_σ make up a simplicial decomposition of the unit cube. Any function f from the vertices $\mathbf{e}_A$ of $[0,1]^n$ to $\mathbb{R}^n$ can be extended to a simplicial map $\hat{f}$ from the whole of $[0,1]^n$ to $\mathbb{R}^n$, by extending f linearly on each simplex X_σ.

We say that a function f from $\{0,1\}^n$ to $\mathbb{R}^n_+$ *respects the co-ordinates* if, for every subset A of $[n]$, $\mathrm{supp}(f(\mathbf{e}_A)) \subseteq A$.

We define a *skew-cube* to be the image Q_f of $\hat{f}$, for some function $f : \{0,1\}^n \to \mathbb{R}^n_+$ respecting the co-ordinates. Note that, if T is any convex set, then Q_f is contained in T if and only if the image of f is. However, it is easy to construct examples of skew-cubes that are not themselves convex.

For any co-ordinate $i \in [n]$, and any permutation σ with $\sigma(n) = i$, let $(X_\sigma)^i$ be the convex hull of the n vectors $\mathbf{e}_{\sigma,0}, \ldots, \mathbf{e}_{\sigma,n-1}$. All of these vectors have i^{th} co-ordinate equal to 0, so if f respects co-ordinates then $\hat{f}((X_\sigma)^i)$ is contained in the i^{th} co-ordinate plane. Now set $(Q_f)^i$ to be the union of all the simplices $\hat{f}((X_\sigma)^i)$ with $\sigma(n) = i$. Evidently $(Q_f)^i$ is an $(n-1)$-dimensional skew-cube in the i^{th} co-ordinate plane; in fact it is exactly the skew-cube defined from f by restricting attention to the subcube of $\{0,1\}^n$ consisting of those $\mathbf{e}_A$ with $i \notin A$.

Note that $(Q_f)^i \subseteq (Q_f)_i = \{\mathbf{x} \in Q_f : x_i = 0\}$, but we need not have $(Q_f)^i = (Q_f)_i$, since there may be some sets B with $i \in B$ but $f(\mathbf{e}_B)_i = 0$. However, if $f(\mathbf{1})_i > 0$, then for any permutation σ with $\sigma(n) = j \neq i$, we have that $\hat{f}(X_\sigma) \cap \{\mathbf{x} : x_i = 0\} \subseteq \{\mathbf{x} : x_j = 0\}$. In other words, provided that $f(\mathbf{1})_i > 0$, the difference $(Q_f)_i \setminus (Q_f)^i$ is contained in $(n-2)$-dimensional subspaces, and thus has $(n-1)$-dimensional volume 0.

More generally, if A is any subset of $[n]$, and f is defined for all vectors $\mathbf{e}_B$ with $B \subseteq A$, and respects the co-ordinates, let $(Q_f)^A$ be the skew-cube defined by the points $f(\mathbf{e}_B)$ with $B \subseteq A$; this is a skew-cube defined on the subspace spanned by the basic unit vectors from the set A. Again, we have $(Q_f)^A \subseteq (Q_f)_A$, but not necessarily equality.

Lemma 2.1 *Let Q_f be the skew-cube associated with a function $f : \{0,1\}^n \to \mathbb{R}^n_+$. Then, up to a set of measure 0, Q_f is the disjoint union of the n cones*

$$C_i = \{\lambda f(\mathbf{1}) + (1-\lambda)\mathbf{z} : 0 \le \lambda \le 1, \mathbf{z} \in (Q_f)^i\}.$$

Thus $\mathrm{Vol}(Q_f) = \frac{1}{n}\sum_{i=1}^n f(\mathbf{1})_i \mathrm{Vol}((Q_f)^i) = \frac{1}{n}\sum_{i=1}^n f(\mathbf{1})_i \mathrm{Vol}((Q_f)_i)$.

Proof. Note that Q_f is the union of simplices $\hat{f}(X_\sigma)$ (some of which may have volume 0), each of which has $f(\mathbf{1})$ as a vertex, and whose other vertices are those of $\hat{f}((X_\sigma)^{\sigma(n)})$, which is one of the simplices making up $(Q_f)^i$. Thus each point $\mathbf{x}$ in Q_f has a representation as $\lambda f(\mathbf{1}) + (1-\lambda)\mathbf{z}$, where $0 \le \lambda \le 1$ and $\mathbf{z}$ is in some $(Q_f)^i$, so Q_f is contained in the union of the C_i.

Conversely, if $\mathbf{z}$ is a vector in $(Q_f)^i$, then $\mathbf{z}$ is in some $\hat{f}((X_\sigma)^i)$, with $\sigma(n) = i$. Therefore any point on the line between $\mathbf{z}$ and $f(\mathbf{1}) = f(\mathbf{e}_{\sigma,n})$ is in the convex hull of $\hat{f}((X_\sigma)^i)$ and $f(\mathbf{1})$, which is just $\hat{f}(X_\sigma)$. We conclude that all the C_i are contained in Q_f.

To show that the C_i are disjoint up to a set of measure 0, note that, if $\mathbf{x}$ is in the interior of some C_i, then the representation $\mathbf{x} = \lambda f(\mathbf{1}) + (1-\lambda)\mathbf{z}$, with $\mathbf{z} \in \cup (Q_f)^i$, is unique, since $\mathbf{z}$ must be the point obtained by projecting from $f(\mathbf{1})$ through $\mathbf{x}$ onto $\cup (Q_f)^i$. Evidently the set of points $\mathbf{x}$ in C_i for which this projection $\mathbf{z}$ has a zero co-ordinate other than z_j has measure 0, as desired.

The volume of the cone C_i is just $\frac{1}{n} f(\mathbf{1})_i \mathrm{Vol}((Q_f)^i)$, and this is equal to $\frac{1}{n} f(\mathbf{1})_i \mathrm{Vol}((Q_f)_i)$, since either $f(\mathbf{1})_i = 0$, or $(Q_f)_i$ and $(Q_f)^i$ differ only on a set of measure 0. Therefore the volume of Q_f is given by either of the stated sums. $\square$

We now give the promised extension of Theorem 1.4.

Theorem 2.2 *For any convex corner S in $\mathbb{R}^n_+$, the content of S is equal to $n!$ times the maximum volume of a skew-cube contained in S. Moreover, $\mu(S) = n!\mathrm{Vol}(S)$ if and only if S is a skew-cube.*

If Q_f is any skew-cube of maximum volume inside S, and A is a subset of $[n]$, then

$$\mu(S_A) = \frac{1}{|A|} \sum_{i \in A} f(\mathbf{e}_A)_i \mu((S_A)_i)$$

(in other words, the maximizations involved in the definition of $\mu(S)$ are achieved at the vertices $f(\mathbf{e}_A)$).

Proof. We claim that a skew-cube Q_f of maximum volume contained in S can be obtained by the following recursive procedure to choose f. First, perforce, define $f(\mathbf{e}_\emptyset) = \mathbf{0}$. Having defined $f(\mathbf{e}_B)$ for all proper subsets B of A, choose $f(\mathbf{e}_A)$ to be any point $\mathbf{x}$ in S_A maximizing $\sum_{i \in A} x_i \mathrm{Vol}(((Q_f)^A)^i)$.

Moreover, we claim that, for each A, the skew-cube $(Q_f)^A$ thus defined has maximum volume over all ($|A|$-dimensional) skew-cubes contained in S_A. Indeed, let A be a minimal subset of $[n]$

such that this is not the case, and let $(Q_g)^A$ be a larger skew-cube in S_A. Then we have

$$\begin{aligned}\mathrm{Vol}((Q_g)^A) &= \frac{1}{|A|}\sum_{i\in A} g(\mathbf{e}_A)_i \mathrm{Vol}(((Q_g)^A)^i)\\ &\le \frac{1}{|A|}\sum_{i\in A} g(\mathbf{e}_A)_i \mathrm{Vol}(((Q_f)^A)^i)\\ &\le \frac{1}{|A|}\sum_{i\in A} f(\mathbf{e}_A)_i \mathrm{Vol}(((Q_f)^A)^i)\\ &= \mathrm{Vol}((Q_f)^A),\end{aligned}$$

using Lemma 2.1, the optimality of Q_f for smaller subspaces, and the choice of $f(\mathbf{e}_A)$.

This proves our claims, and we have that, for any subset $A \subseteq [n]$,

$$\mathrm{Vol}((Q_f)^A) = \frac{1}{|A|}\max_{\mathbf{x}\in S_A}\sum_{i\in A} x_i \mathrm{Vol}(((Q_f)^A)^i).$$

Therefore $|A|!\mathrm{Vol}((Q_f)^A)$ obeys the same recursive law as $\mu(S_A)$, and has the same value for $A = \emptyset$, so the two are equal. This also shows that

$$\mu(S_A) = \frac{1}{|A|}\sum_{i\in A} f(\mathbf{e}_A)_i \mu((S_A)_i),$$

as in the statement of the theorem.

Finally, if S is a convex corner with $\mu(S) = n!\mathrm{Vol}(S)$, then there is a skew-cube Q_f inside S with volume equal to S. If $Q_f \neq S$, then take some point $\mathbf{x}$ in $S \setminus Q_f$. Since S has full dimension, there is a point $\mathbf{y}$ in S with all co-ordinates non-zero. Since S is convex and Q_f is closed, there is a point $\mathbf{z}$ of $S \setminus Q_f$ on the line-segment between $\mathbf{x}$ and $\mathbf{y}$ with all co-ordinates at least some $\delta > 0$. Let the ℓ_∞ distance between $\mathbf{z}$ and Q_f be $\varepsilon > 0$. Since S is a down-set in $\mathbb{R}^n_+$, all the points of the form $\mathbf{z} - (\varepsilon_1, \dots, \varepsilon_n)$, with $0 \le \varepsilon_i \le \min(\varepsilon, \delta)$ for all i, are in $S \setminus Q_f$. This set of points has positive volume, contradicting the assumption that Q_f has the same volume as S.

This completes the proof. □

2.2 Characterizations of skew-cubes

We now address the problem of determining whether a given convex corner is a skew-cube; we shall give two characterizations of skew-cubes, although unfortunately neither is especially attractive. The next theorem is a consequence of the results of the previous subsection. Surprisingly, however, the proof seems to require delicate handling; the difficulty is that it is not immediately apparent that S_A is a skew-cube whenever S is; this does follow as an immediate consequence of the theorem.

Theorem 2.3 *Let S be a convex corner in $\mathbb{R}^n_+$. Then S is a skew-cube if and only if, for every subset A of $[n]$, there is a point $\mathbf{x}_A \in S_A$ such that S_A is the union of the cones $\mathrm{conv}(\mathbf{x}_A, (S_A)_i)$, for $i \in A$.*

Proof. If the condition is satisfied, then we can define $f(\mathbf{e}_A) = \mathbf{x}_A$ for every $A \subseteq [n]$. Using Lemma 2.1, it follows immediately by induction that $S_A = (Q_f)^A$, for every A.

For the converse, suppose that $S = Q_f$ is both a convex corner and a skew-cube. Then certainly S itself is the union of cones $\mathrm{conv}(f(\mathbf{1}), S_i)$, i.e., the condition of the theorem is satisfied for $A = [n]$. Furthermore, the point $\mathbf{x} = f(\mathbf{1})$ satisfies $\mu(S) = \sum_{i=1}^n x_i \mu(S_i)$, by Theorem 2.2.

Suppose that the condition is not satisfied for some $A \subset [n]$; then we have that S_A is not a skew-cube. We now claim that no S_B, for $A \subseteq B \subseteq [n]$, is a skew-cube. For this we use induction on $|B|$; it is true when $|B| = |A|$.

Suppose then that $A \subset B$, with $(S_B)_i$ not a skew-cube for any $i \in B \setminus A$. We claim that S_B is not a skew-cube. If it is, then there is some point $\mathbf{x} \in S_B$ with S_B equal to the union of the cones $\mathrm{conv}(\mathbf{x}, (S_B)_i)$, and with $\mu(S_B) = \sum_{i \in B} x_i \mu((S_B)_i)$, as above.

Now we note that $\mu((S_B)_j) \le (|B|-1)! \,\mathrm{Vol}((S_B)_j)$ for every j, by Theorem 2.2, with equality only when $(S_B)_j$ is a skew-cube. If $x_i > 0$ for some $i \in B \setminus A$, then we have

$$x_i \mu((S_B)_i) < x_i (|B|-1)! \,\mathrm{Vol}((S_B)_i),$$

and so

$$\mu(S_B) < (|B|-1)! \sum_{j \in B} x_j \mathrm{Vol}((S_B)_j) \le |B|! \,\mathrm{Vol}(S_B),$$

as in the proof of Theorem 1.4. Hence in this case S_B is not a skew-cube, by Theorem 2.2 again.

If on the other hand $x_i = 0$ for every $i \in B \setminus A$, then $\mathbf{x} \in S_A$. Then there is a point $\mathbf{z}$ of S_A not in the union of the cones $\mathrm{conv}(\mathbf{x}, (S_A)_i)$. Clearly $\mathbf{z}$ is also a point of S_B not in the union of the cones $\mathrm{conv}(\mathbf{x}, (S_B)_i)$, which gives a contradiction. □

The previous characterization is perhaps unsurprising given the proof of Theorem 1.4. The other result we give characterizing skew-cubes is quite different in character. Our experience so far suggests that this is in many ways a more useful result than Theorem 2.2.

Theorem 2.4 *A convex corner S in $\mathbb{R}^n_+$ is a skew-cube if and only if, for every subset A of $[n]$, all the maximal points of S°_A lie on a hyperplane. Furthermore, if S is a skew-cube, then the vector $\mathbf{x}$ satisfies $\mu(S) = \sum_{i=1}^n x_i \mu(S_i)$ if and only if all the maximal points $\mathbf{y}$ of S° lie on the hyperplane $\mathbf{x} \cdot \mathbf{y} = 1$.*

Before moving to the proof of Theorem 2.4, we need a lemma characterizing maximal points in convex corners that are polytopes.

Lemma 2.5 *Let S be a convex corner that is a polytope, and let $\mathbf{x}$ be a point of S. Then the following are equivalent.*
(a) $\mathbf{x}$ *is maximal.*
(b) *For $i = 1, \dots, n$, there is a point $\mathbf{z} \in S^\circ$, with $z_i > 0$, such that $\mathbf{x} \cdot \mathbf{z} = 1$.*
(c) *There is a point $\mathbf{y} \in S^\circ$, with $y_i > 0$ for every i, such that $\mathbf{x} \cdot \mathbf{y} = 1$.*

Proof. (a) $\Rightarrow$ (b) Suppose that there is no point $\mathbf{z} \in S^\circ$ with $\mathbf{x} \cdot \mathbf{z} = 1$ and $z_1 > 0$. Then there is $\varepsilon > 0$ such that, for all of the finitely many *vertices* $\mathbf{z}$ of S°, $\mathbf{x} \cdot \mathbf{z} \le 1 - \varepsilon z_1$. Then we have $(\mathbf{x} + \varepsilon \mathbf{e}_1) \cdot \mathbf{z} \le 1$, for every vertex $\mathbf{z} \in S^\circ$, where $\mathbf{e}_1$ denotes the first basic unit vector. This implies that $\mathbf{x} + \varepsilon \mathbf{e}_1 \in S$, contradicting the maximality of $\mathbf{x}$.
(b) $\Rightarrow$ (c) If, for each i, there is a point $\mathbf{z}(i) \in S^\circ$ with $z(i)_i > 0$ and $\mathbf{x} \cdot \mathbf{z}(i) = 1$, then the point $\mathbf{y} = \frac{1}{n} \sum_{i=1}^n \mathbf{z}(i)$ has $\mathbf{x} \cdot \mathbf{y} = 1$ and $y_i > 0$ for every i.
(c) $\Rightarrow$ (a) Let the vector $\mathbf{y}$ be as given. For any vector $\mathbf{w}$ in S with $w_i \ge x_i$ for every i, we have $1 \ge \mathbf{w} \cdot \mathbf{y} \ge \mathbf{x} \cdot \mathbf{y} = 1$, and so $w_i y_i = x_i y_i$ for every i, which implies that $\mathbf{w} = \mathbf{x}$. Therefore $\mathbf{x}$ is maximal. □

Proof of Theorem 2.4. Note that, if S is a skew-cube, then it is a polytope, and hence so is S°. The condition of the theorem also implies that S° is a polytope, as it is defined by at most 2^n

hyperplanes, and hence so is S. Thus we may restrict attention to those convex corners that are polytopes.

Thus, by Theorem 2.3, it is enough to prove that, for convex corners S that are polytopes, the condition that all the maximal points $\mathbf{y}$ of S° lie on the hyperplane given by $\mathbf{x} \cdot \mathbf{y} = 1$ is equivalent to S being the union of the cones $\text{conv}(\mathbf{x}, S_i)$.

Suppose first that all the maximal points $\mathbf{y}$ of S° lie on the hyperplane $\mathbf{x} \cdot \mathbf{y} = 1$. Then $\mathbf{x} \cdot \mathbf{w} \leq 1$ for every point $\mathbf{w}$ of S°, and so (since S is a convex corner) $\mathbf{x} \in S$. By Lemma 2.5, $\mathbf{x}$ is maximal in S, since for every i there is a maximal point of S° with non-zero i^{th} co-ordinate.

Now take any point $\mathbf{z} \neq \mathbf{x}$ of S, and choose a co-ordinate i to minimize z_i/x_i. This minimum λ is strictly less than 1, since otherwise $z_i \geq x_i$ for each i, and so $\mathbf{z} = \mathbf{x}$ by maximality of $\mathbf{x}$. Now consider the point $\mathbf{v}$ satisfying $\mathbf{z} = \lambda \mathbf{x} + (1-\lambda)\mathbf{v}$. Note that $v_i = 0$, and $v_j \geq 0$ for all j, by choice of λ. If $\mathbf{v} \notin S$, then there is a point $\mathbf{y}$ of S° with $\mathbf{v} \cdot \mathbf{y} > 1$, and we may take $\mathbf{y}$ to be maximal. But then

$$\mathbf{z} \cdot \mathbf{y} = \lambda \mathbf{x} \cdot \mathbf{y} + (1-\lambda)\mathbf{v} \cdot \mathbf{y} > \lambda + (1-\lambda) = 1,$$

which is not possible since $\mathbf{z} \in S$ and $\mathbf{y} \in S^\circ$. Therefore $\mathbf{v} \in S$, and hence $\mathbf{v} \in S_i$, and so $\mathbf{z} \in \text{conv}(\mathbf{x}, S_i)$. But $\mathbf{z}$ was arbitrary, so S is equal to the union of the cones, as required.

Conversely, suppose that S is the union of the cones $\text{conv}(\mathbf{x}, S_i)$, but there is a maximal point $\mathbf{w}$ of S° with $\mathbf{x} \cdot \mathbf{w} < 1$. By Lemma 2.5, there is a point $\mathbf{z}$ of S, with all co-ordinates z_i positive, and $\mathbf{z} \cdot \mathbf{w} = 1$. We can take $\mathbf{z} = \lambda \mathbf{x} + (1-\lambda)\mathbf{v}$, for some $\mathbf{v}$ in some S_i, and some λ between 0 and 1. We have $\lambda > 0$, since $\mathbf{z}$ is not in any S_i. This gives

$$1 = \mathbf{z} \cdot \mathbf{w} = \lambda \mathbf{x} \cdot \mathbf{w} + (1-\lambda)\mathbf{v} \cdot \mathbf{w} < \lambda + (1-\lambda)\mathbf{v} \cdot \mathbf{w}.$$

This implies that $\mathbf{v} \cdot \mathbf{w} > 1$, which is not possible since $\mathbf{v}$ is in S and $\mathbf{w}$ in S°.

This completes the proof. □

2.3 Lower bounds on the content

We conclude this section by discussing how far from equality we can get in Theorem 1.4. It is perhaps not immediately obvious that there is any lower bound for $\mu(S)$ in terms of the volume of S, but notice that any cuboid inside S is certainly a skew-cube, so $\mu(S) \geq V(S)$. Now we conclude from Kahn and Kim's result, Theorem 1.3, that

$$\frac{1}{n!}\mu(S) \leq \text{Vol}(S) \leq \frac{n^n}{n!^2}\mu(S) \leq \frac{e^n}{n!}\mu(S),$$

i.e., that $\frac{1}{n!}\mu(S)$ also approximates $\text{Vol}(S)$ to within a factor of e^n.

Presumably the factor of e^n above can be improved upon substantially. To pose this problem precisely, let us define, for a convex corner S in $\mathbb{R}^n_+$,

$$\phi(S) = \left(\frac{n!\text{Vol}(S)}{\mu(S)}\right)^{1/n}.$$

We have that $1 \leq \phi(S) < e$ for all convex corners S. It would be of interest to find the value of the supremum ϕ of $\phi(S)$ over all convex corners S.

Two families of examples that seem to be reasonable candidates for large values of $\phi(S)$ are the families B^n_p and T^n_t defined in §1.1. It turns out that $\phi(B^n_p) \leq 1.2883$ for all p and n, while $\phi(T^n_{an}) > 1.2984$ when $a \simeq 0.3581$ and n is large. (Surprisingly, estimating the volume of T^n_t seems not to be easy – we intend to give details elsewhere.) Thus we have the bounds

$$1.2984 < \phi \leq e.$$

We strongly believe that the lower bound is closer to the true value of ϕ.

3 Inequalities Involving the Polar

We now turn our attention to Theorem 1.5, stating that $\mu(S)\mu(S^\circ) \geq n!$ for every n-dimensional convex corner S. Before proving this, we need one easy preliminary lemma.

Lemma 3.1 *For any convex corner S in $\mathbb{R}^n_+$, and any co-ordinate i, $(S^\circ)_i = (S_i)^\circ$.*

Proof. Note that S_i, and hence also, by definition, $(S_i)^\circ$, lie in the i^{th} co-ordinate plane.

If $\mathbf{x} \in (S^\circ)_i$, then $x_i = 0$, and $\sum_{j=1}^n x_j y_j \leq 1$ for every $\mathbf{y} \in S$. Hence certainly $\sum_{j\neq i} x_j y_j \leq 1$ for every $\mathbf{y} \in S_i$, so $\mathbf{x} \in (S_i)^\circ$.

Conversely if $\mathbf{x} \in (S_i)^\circ$, then $x_i = 0$, and $\sum_{j\neq i} x_j y_j \leq 1$ for every $\mathbf{y} \in S_i$. Since S is a down-set, for every point $\mathbf{z} \in S$, the projection of $\mathbf{z}$ onto the i^{th} co-ordinate plane is in S_i. Hence $\sum_{j=1}^n x_j z_j = \sum_{j\neq i} x_j z_j \leq 1$, so $\mathbf{x} \in S^\circ$, and therefore in $(S^\circ)_i$. □

It is worth observing that the assumption that S is a down-set (or, equivalently, that our underlying norm is 1-unconditional) is crucial to Lemma 3.1. From now on, this assumption will be used implicitly whenever we write S_i°, which we can do without ambiguity whenever S is a convex corner.

Proof of Theorem 1.5. We prove the result by induction on n. It is trivially true for $n = 0$. Suppose the result is true in dimension $n-1$, and let S be any convex corner in $\mathbb{R}^n_+$. We have that $\mu(S_i)\mu(S_i^\circ) \geq (n-1)!$ for each i.

Define the vector $\mathbf{y}$ by setting $y_i = \mu(S_i)/\mu(S)$. By the definition of $\mu(S)$, we have $\mathbf{x}\cdot\mathbf{y} \leq 1$ for every $\mathbf{x} \in S$, which means that $\mathbf{y} \in S^\circ$.

So

$$\begin{aligned}
\mu(S^\circ) &\geq \sum_{i=1}^n y_i \mu(S_i^\circ) \\
&= \frac{1}{\mu(S)} \sum_{i=1}^n \mu(S_i)\mu(S_i^\circ) \\
&\geq \frac{1}{\mu(S)} \sum_{i=1}^n (n-1)!,
\end{aligned}$$

and thus $\mu(S)\mu(S^\circ) \geq n!$. The result follows by induction. □

Let us stress once more that the only novelty here, compared with Meyer's proof of Saint-Raymond's Theorem, is in the explicit use of the content, and in the separation of Theorem 1.5 from issues concerning the volume.

We shall be interested in when we have equality in Theorem 1.5, i.e., for which S we have $\mu(S)\mu(S^\circ) = n!$; we say that S is *μ-minimal.* (The choice of terminology follows that used in connection with Saint-Raymond's Inequality, where a convex corner S in $\mathbb{R}^n_+$ is called *minimal* if $\mathrm{Vol}(S)\mathrm{Vol}(S^\circ) = 1/n!$.) Let us note here that, as an immediate consequence of the proof of Theorem 1.5, if S is μ-minimal, then so is every S_i. Also, if S is μ-minimal, then the vector $\mathbf{y}$ defined in the proof, with $y_i = \mu(S_i)/\mu(S)$, must maximize $\sum_{i=1}^n y_i \mu(S_i^\circ)$ over all vectors in S°.

We have already noted that Theorems 1.4 and 1.5 combine to give Saint-Raymond's Inequality, Corollary 1.7, and also Theorem 1.6. If we use Theorem 2.2 in place of Theorem 1.4, we get the following strengthening of Corollary 1.7.

Corollary 3.2 *For any convex corner S in $\mathbb{R}^n_+$, there are skew-cubes $Q \subseteq S$ and $Q' \subseteq S^\circ$ such that*

$$\mathrm{Vol}(Q)\mathrm{Vol}(Q') \geq \frac{1}{n!}.$$

Weaker versions of Corollary 3.2 might be more immediately appealing. For instance, we can conclude that there are convex polytopes $R \subseteq S$ and $R' \subseteq S^\circ$ with $\mathrm{Vol}(R)\mathrm{Vol}(R') \geq 1/n!$. Furthermore, R and R' each have at most 2^n vertices – this is Theorem 1.8. Even more, there are maps from the discrete cube $\{0,1\}^n$ to these sets of vertices that respect the co-ordinates.

Meyer [20] and Reisner [22] showed that there is only a very small class of convex corners S with $\mathrm{Vol}(S)\mathrm{Vol}(S^\circ) = 1/n!$, namely those S that can be built up from 1-dimensional convex corners by ℓ_1- and ℓ_∞-sums. However, it is not too hard to show that suitable symmetry conditions are enough to imply that S is μ-minimal.

For instance, suppose the symmetry group of S includes all the $n!$ isometries permuting the co-ordinate axes; the same is then true of S°. This also implies that all the S_i are isomorphic, as are all the S_i°. We may suppose by induction that all the S_i are μ-minimal, i.e., $\mu(S_i)\mu(S_i^\circ) = (n-1)!$. Let $\mathbf{y}$ and $\mathbf{z}$ be the points on the boundaries of S and S°, respectively, with all co-ordinates equal, say to y and z respectively. Then certainly $yz \leq 1/n$. As noted earlier, the maximizations defining $\mu(S)$ and $\mu(S^\circ)$ occur at $\mathbf{y}$ and $\mathbf{z}$ respectively, so

$$\mu(S)\mu(S^\circ) = \sum_{j=1}^{n} y\mu(S_j) \sum_{k=1}^{n} z\mu(S_k^\circ) = n^2 yz\mu(S_1)\mu(S_1^\circ) \leq n!,$$

and S is μ-minimal.

For example all of the convex corners in the families B_p^n and T_t^n are μ-minimal – for $S = B_p^n$, this can also easily be verified by direct calculation, since we have $\mu(S)\mu(S^\circ) = (n!)^{1-1/p}(n!)^{1-1/q} = n!$.

In fact, somewhat weaker symmetry conditions suffice for μ-minimality. On the other hand, it would be misleading to think that a μ-minimal S necessarily has any symmetry properties: when we consider the case of stable set polytopes of graphs in §4.1, we shall see examples of highly asymmetric bodies that are nevertheless μ-minimal.

We can say more about the situation where S is μ-minimal and S° is a skew cube, so that $\mu(S) = n!/\mu(S^\circ) = 1/\mathrm{Vol}(S^\circ)$, i.e., we have equality in Theorem 1.6. Note that there is no real loss of generality involved in imposing the condition that all the basic unit vectors are on the boundary of S, since any convex corner can be made to satisfy this condition by re-scaling, i.e., changing the scales on the axes.

Theorem 3.3 *Let S be a convex corner in $\mathbb{R}^n_+$ with all the basic unit vectors $\mathbf{e}_i$ on the boundary of S. Suppose that S is μ-minimal, and that S° is a skew-cube. Then S is a 0-1 polytope.*

Proof. First observe that, since S° is a skew-cube, it is a polytope, and hence so is S.

Suppose that S has a vertex that is not a 0-1 vector, and take one, say $\mathbf{x}$, with minimal support, say A. By the remarks after the proof of Theorem 1.5, we have that S_A is μ-minimal, and furthermore that the vector $\mathbf{y}$ given by $y_i = \mu((S_A)_i)/\mu(S_A)$ maximizes $\sum_{i\in A} y_i\mu((S_A^\circ)_i$ over S_A°. Moreover, Theorem 2.4 tells us that all the maximal points $\mathbf{z}$ of S_A lie on the hyperplane $\mathbf{y}\cdot\mathbf{z} = 1$. In other words, all maximal points $\mathbf{z}$ of S_A satisfy $\mu(S_A) = \sum_{i\in A} z_i\mu((S_A)_i)$. This implies that, for each maximal $\mathbf{z}$, S_A is the union of the cones $\mathrm{conv}(\mathbf{z}, (S_A)_i)$.

Since $\mathbf{x}$ is non-integral, there is a co-ordinate i with $0 < x_i < 1$. Since $\mathbf{e}_i \in S_A$, there is a maximal point $\mathbf{z} \in S_A$ with $z_i = 1$, so in particular $\mathbf{z} \neq \mathbf{x}$. By the above, $\mathbf{x}$ is a convex combination of $\mathbf{z}$ and some point $\mathbf{w}$ in some $(S_A)_j$. This contradicts the assumption that $\mathbf{x}$ is a vertex, since $\mathbf{x}$ is equal to neither $\mathbf{z}$ nor (since $\mathbf{x}$ has support A) $\mathbf{w}$. Thus S is a 0-1 polytope. $\square$

Theorem 3.3 goes some way towards determining the convex corners S for which $\mu(S) = 1/\mathrm{Vol}(S^\circ)$, but we have not managed to completely describe the class of such convex corners. We comment further on this problem in §7.2.

We close this section by considering how far $\mu(S)\mu(S^\circ)$ can be from $n!$. An upper bound for the product follows from Theorem 1.4 and Santaló's Theorem, namely that

$$\frac{\mu(S)\mu(S^\circ)}{n!^2} \leq \mathrm{Vol}(S)\mathrm{Vol}(S^\circ) \leq (\mathrm{Vol}(B_2^n))^2 = \frac{(\pi/4)^n}{\Gamma(1+n/2)^2}.$$

Since $\Gamma(1+n/2)^2 \geq n!2^{-n}$ for all integers n, we have that

$$n! \leq \mu(S)\mu(S^\circ) \leq n!\left(\frac{\pi}{2}\right)^n.$$

As we have already seen, actually $\mu(B_2^n)^2 = n!$, so we would expect the upper bound above on $\mu(S)\mu(S^\circ)$ to be rather weak.

For S a convex corner in $\mathbb{R}_+^n$, set

$$\eta(S) = \left(\frac{\mu(S)\mu(S^\circ)}{n!}\right)^{1/n}.$$

The previous argument shows that $1 \leq \eta(S) \leq \pi/2 \simeq 1.5708$ for all convex corners S. As for the constant ϕ in the previous section, it would be interesting to find the value of the supremum η of $\eta(S)$ over all convex corners S.

The best lower bound for η we have been able to find is based on the 2-dimensional convex corner T maximizing $\mu(T)\mu(T^\circ)$.

Theorem 3.4 *For a convex corner S in $\mathbb{R}_+^2$, $2 \leq \mu(S)\mu(S^\circ) \leq \frac{9}{4}$. The upper bound is achieved when S is the convex hull T of $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1/2)$, in which case S° is the reflection of T in the diagonal $x = y$.*

We omit the straightforward proof. There is a notion of the "T-sum" of convex corners, for the 2-dimensional body T above: repeatedly taking T-sums yields a sequence of bodies T_j with

$$\eta(T_j) = \frac{\prod_{i=1}^{j}\left(2^{2^{i-1}-1} + \binom{2^i-1}{2^{i-1}}2^{-2^{i-1}}\right)^{2^{1-j}}}{(2^j!)^{1/2^j}} \to 1.31527\ldots$$

as $j \to \infty$. Hence we have

$$1.3152 \leq \eta \leq \frac{\pi}{2} \simeq 1.5708.$$

We think it is just about possible that the lower bound gives the correct value of η; however, we have not been able to show even that $\eta < \pi/2$.

4 Stable set polytopes of graphs

4.1 Graph-content

Recall that the graph-content $\mu^{\mathrm{g}}(G)$ of a graph G, introduced by Sidorenko [25], is $\mu(\mathrm{Stab}(G))$. Since the stable set polytope $\mathrm{Stab}(G)$ of G is a convex corner, the results of the previous sections apply; for instance, we obtain Theorem 1.9. We have also noted that the polar $\mathrm{FStab}(\overline{G})$ of $\mathrm{Stab}(G)$ is equal to $\mathrm{Stab}(\overline{G})$ if and only if G is perfect. Thus we have the following consequence of Theorem 1.5.

Theorem 4.1 (a) *For any graph G on n nodes, $\mu^{\mathrm{g}}(G)\mu(\mathrm{FStab}(\overline{G})) \geq n!$.*
(b) *For any perfect graph G on n nodes, $\mu^{\mathrm{g}}(G)\mu^{\mathrm{g}}(\overline{G}) \geq n!$.*

Theorem 4.1(b) is essentially proved by Sidorenko [25]; although he states the result only for comparability graphs of partial orders, his proof works in the general setting of perfect graphs. As noted by Sidorenko in [25], the result does not hold for all graphs: the smallest example is, not surprisingly, $G = \overline{G} = C_5$, when $\mu^{\mathrm{g}}(G)\mu^{\mathrm{g}}(\overline{G}) = 10^2 < 5!$.

Of course we can also combine Theorems 1.9 and 4.1, or apply our earlier results on volume directly, to obtain the following results.

Theorem 4.2 *Let G be a graph on n nodes.*
(a) $\mu^{\mathrm{g}}(G) \geq \dfrac{1}{\mathrm{Vol}(\mathrm{FStab}(\overline{G}))}$.
(b) $\mathrm{Vol}(\mathrm{Stab}(G))\mathrm{Vol}(\mathrm{FStab}(\overline{G})) \geq \dfrac{1}{n!}$.
(c) *If G is perfect, then* $\mathrm{Vol}(\mathrm{Stab}(G))\mathrm{Vol}(\mathrm{Stab}(\overline{G})) \geq \dfrac{1}{n!}$. □

Parts (b) and (c) of Theorem 4.2 were first stated in Bollobás, Brightwell and Sidorenko [3]. Again, some condition on G is required for the conclusion of Theorem 4.2(c) to hold. Rather than give an explicit example to illustrate this, let us consider what happens when we take G to be a *random graph* $G_{n,1/2}$.

Almost surely, there is no stable set in $G = G_{n,1/2}$ containing more than $2\log_2 n$ nodes. Therefore, all of $\mathrm{Stab}(G)$ lies inside $\{\mathbf{x} \in [0,1]^n : \sum x_i \leq 2\log_2 n\}$, a body of volume at most $(2\log_2 n)^n/n!$. The same is true for the complement, another copy of $G_{n,1/2}$ and so, for almost every G, we have

$$\mathrm{Vol}(\mathrm{Stab}(G))\mathrm{Vol}(\mathrm{Stab}(\overline{G})) \leq \frac{(2\log_2 n)^{2n}}{n!^2},$$

which is much less than $1/n!$ for large enough n.

This argument shows that, for $G = G_{n,1/2}$, we almost surely have $\mu^{\mathrm{g}}(G) \leq (2\log_2 n)^n$ – indeed it is trivial that, for any graph G with largest independent set of size k, $\mu^{\mathrm{g}}(G) \leq k^n$. In fact, one can show that this is a fairly good estimate of $\mu^{\mathrm{g}}(G)$ for a random graph G.

We next look at cases of equality in the various inequalities in Theorems 4.1 and 4.2. We start with the strongest inequality, namely Theorem 4.1(a); to say that we have equality here for a graph G amounts to saying that $\mathrm{Stab}(G)$ is μ-minimal.

Let P_4 denote the path on 4 nodes. It is easy to check that $\mu^{\mathrm{g}}(P_4) = 5$. Now P_4 is perfect, and its complement is again a copy of P_4, so we have

$$\mu^{\mathrm{g}}(P_4)\mu(\mathrm{FStab}(\overline{P_4})) = \mu^{\mathrm{g}}(P_4)\mu^{\mathrm{g}}(P_4) = 25 > 4!,$$

so $\mathrm{Stab}(P_4)$ is not μ-minimal.

Now, if $\mathrm{Stab}(G)$ is μ-minimal, then, as noted after the proof of Theorem 1.5, so is every $(\mathrm{Stab}(G))_A$, for A a subset of the node set of G. This means that $\mathrm{Stab}(H)$ is μ-minimal for every induced subgraph H of G. Thus we have that, if G is μ-minimal, then G contains no induced copy of P_4.

Graphs with no induced copy of P_4 are called *cographs*, and they are made up recursively from one-node graphs by the two operations of

- *disjoint union* $G_1 \cup G_2$: taking copies of graphs G_1 and G_2 on disjoint node sets, and putting no edges between the two,
- *disjoint sum* $G_1 + G_2$: taking copies of graphs G_1 and G_2 on disjoint node sets, and adding all the edges between the two sets.

Cographs are comparability graphs of partial orders. Thus we can use Theorem 1.13 and/or Theorem 1.11 to calculate $\mu^{\mathrm{g}}(G) = n!\mathrm{Vol}(\mathrm{Stab}(G))$ for G a cograph, or indeed we can work directly from the definition of graph-content. One obtains the recursive formulae below, as first noted by Sidorenko [25].

Lemma 4.3 *If G_1 and G_2 are graphs on n_1 and n_2 nodes respectively, then*

$$\mu^{\mathrm{g}}(G_1 \cup G_2) = \binom{n_1+n_2}{n_1}\mu^{\mathrm{g}}(G_1)\mu^{\mathrm{g}}(G_2),$$
$$\mu^{\mathrm{g}}(G_1 + G_2) = \mu^{\mathrm{g}}(G_1)\mu^{\mathrm{g}}(G_2).$$

□

Theorem 4.4 *For any of the inequalities in Theorems 4.1 and 4.2, the graph G satisfies the inequality with equality if and only if G is a cograph.*

Proof. We have already seen that, if G is not a cograph, then it does not achieve equality in Theorem 4.1(a), and so does not achieve equality in any of the inequalities.

It remains to be shown that cographs satisfy all the inequalities exactly. Since we know that $\mathrm{Vol}(\mathrm{FStab}(G)) = \mathrm{Vol}(\mathrm{Stab}(G)) = \mu^{\mathrm{g}}(G)/n!$ for G an n-node cograph, it is enough to show that $\mu^{\mathrm{g}}(G)\mu^{\mathrm{g}}(\overline{G}) = n!$ for all cographs G. As shown by Sidorenko in [25], this follows readily on using Lemma 4.3. □

4.2 0-1 convex corners

Combining Sidorenko's and Stanley's Theorems shows that the stable set polytopes of comparability graphs of partial orders are skew-cubes, i.e., $\mu^{\mathrm{g}}(G) = \mathrm{Vol}(\mathrm{Stab}(G))$ whenever G is the comparability graph of a partial order. Theorem 1.10, which we now prove, implies that this equality holds for the somewhat wider class of strongly perfect graphs, but not for any other graphs. Furthermore, there are no examples of 0-1 polytopes that are skew-cubes, but are not stable set polytopes of graphs. Our proof is a direct application of the characterization of skew-cubes given in Theorem 2.4.

Proof of Theorem 1.10. Let S be a 0-1 convex corner in $\mathbb{R}^n_+$.

Suppose first that S is not the stable set polytope of a graph. This implies that there is a subset A of $[n]$ such that all the 2-element subsets of A correspond to vertices of S, but A itself does not. If we further take A minimal with this property, then we have that all proper subsets of A correspond to vertices of S, but A does not. Now all the basic unit vectors $\mathbf{e}_i$ are maximal points of S_A°, and so is the vector $(\frac{1}{|A|-1}, \ldots, \frac{1}{|A|-1})$. These vectors do not lie on a common hyperplane so, by Theorem 2.4, S is not a skew-cube.

Now suppose that S is the stable set polytope of a graph G, and that S is a skew-cube. Take any induced subgraph H of G, say on node set A. Each maximal clique C in H corresponds to a maximal point $\mathbf{e}_C$ in S_A°. (Typically, there are other maximal points as well.) By Theorem 2.4, all the points of S_A° corresponding to maximal cliques lie on a hyperplane $\mathbf{x} \cdot \mathbf{y} = 1$, where $\mathbf{x} \in S_A$. Therefore there must be a *vertex* $\mathbf{z}$ of S_A such that $\mathbf{z} \cdot \mathbf{e}_C = 1$ whenever C is a maximal clique in H. This vertex $\mathbf{z}$ corresponds to a stable set in H, which therefore meets all the maximal cliques. We deduce that G is strongly perfect.

Now suppose that S is the stable set polytope of a strongly perfect graph G. Since G is perfect, Theorem 1.2 tells us that $S^\circ = \mathrm{Stab}(\overline{G})$, i.e., that the vertices of S° correspond to the cliques of G.

For any induced subgraph H of G, say on node set A, we have that there is a stable set $B \subseteq A$ meeting every maximal clique in H. The vector $\mathbf{e}_B$ then corresponds to a hyperplane through all the vectors $\mathbf{e}_C$, where C is a maximal clique in H, i.e., through all the maximal vertices of S_A°. This hyperplane therefore passes through all the maximal points of S_A°. We deduce from Theorem 2.4 that S is a skew-cube. □

5 Pointwise products of convex corners

Our aim in this section is to derive inequalities concerning the content and volume under the condition that the pointwise product of several convex corners is contained in another. Before we get properly underway, we need a few easy preliminary results.

As we have seen, the pointwise product of polytopes need not be a polytope. The next lemma shows that at least the convex hull of such a product is a polytope.

Lemma 5.1 *Let $A_1, \ldots, A_r$ be any sets in $\mathbb{R}_+^n$. Then*

$$\mathrm{conv}(\mathrm{conv}(A_1) \circ \cdots \circ \mathrm{conv}(A_r)) = \mathrm{conv}(A_1 \circ \cdots \circ A_r).$$

Proof. We give the proof for $r = 2$ only; the full result can be proved in a similar way, or can be deduced from the case $r = 2$ using induction.

The lemma amounts to saying that any point in $\mathrm{conv}(A_1) \circ \mathrm{conv}(A_2)$ can be written as a convex combination of points from $A_1 \circ A_2$.

Accordingly, suppose that $\mathbf{x} = \sum_j \lambda_j \mathbf{x}^j$ is a convex combination of points $\mathbf{x}^j$ in A_1, and that $\mathbf{y} = \sum_k \mu_k \mathbf{y}^k$ is a convex combination of points $\mathbf{y}^k$ in A_2. Then we have

$$\begin{aligned}
\mathbf{x} \circ \mathbf{y} &= \Big(\sum_j \lambda_j \mathbf{x}^j\Big) \circ \Big(\sum_k \mu_k \mathbf{y}^k\Big) \\
&= \sum_{i=1}^n \Big(\sum_j \lambda_j (\mathbf{x}^j)_i\Big)\Big(\sum_k \mu_k (\mathbf{y}^k)_i\Big) \\
&= \sum_{i=1}^n \sum_{j,k} \lambda_j \mu_k (\mathbf{x}^j)_i (\mathbf{y}^k)_i \\
&= \sum_{j,k} \lambda_j \mu_k (\mathbf{x}^j \circ \mathbf{y}^k),
\end{aligned}$$

a convex combination of vectors from $A_1 \circ A_2$. □

Theorem 5.2 *If $C_1, \ldots, C_r$ are convex polytopes in $\mathbb{R}^n_+$, then so is $C = \mathrm{conv}(C_1 \circ \cdots \circ C_r)$, and C is the convex hull of the set of points of the form $\mathbf{x}^1 \circ \cdots \circ \mathbf{x}^r$, where each $\mathbf{x}^i$ is a vertex of the polytope C_i.*

Proof. This follows immediately on applying Lemma 5.1 to the sets A_i of vertices of C_i. $\square$

5.1 Further generalizations of Saint-Raymond's Theorem

Recall Theorem 1.5, stating that, for S a convex corner in $\mathbb{R}^n_+$, we have $\mu(S)\mu(S^\circ) \geq n!$. In terms of pointwise products, we can restate this as saying that, if S and T are any two convex corners in $\mathbb{R}^n_+$ with $S \circ T \subseteq B^n_1$, then $\mu(S^\circ)\mu(T^\circ) \geq n!$. Indeed, it is not hard to see that S and T can be taken to be any two bounded sets in $\mathbb{R}^n_+$ satisfying the condition: we discuss this carefully a little later.

We aim for generalizations of the inequality above. The main tool we need is Theorem 1.14, stating that, if A, B and C are convex corners in $\mathbb{R}^n_+$ with $A \circ B \subseteq C$, then $\mu(A) \leq \mu(B^\circ)\mu(C)$. The proof of this result is very similar to that of Theorem 1.5.

Proof of Theorem 1.14. As usual, we use induction on n and begin by observing that the result is trivial in one dimension.

Suppose the result is true in dimensions less than n, take any A, B and C in $\mathbb{R}^n_+$ satisfying the hypotheses, and find a point $\mathbf{x} \in A$ with $\mu(A) = \sum_{i=1}^n x_i \mu(A_i)$. Note that $A_i \circ B_i \subseteq C_i$ for each i so, by the induction hypothesis, we have

$$\mu(A) \leq \sum_{i=1}^n x_i \mu(B_i^\circ)\mu(C_i).$$

Consider the vector $\mathbf{y}$ defined by $y_i = x_i\mu(C_i)/\mu(C)$ for each i. We claim that $\mathbf{y} \in B^\circ$. To verify this, take any $\mathbf{z} \in B$; by our hypothesis we have $\mathbf{x} \circ \mathbf{z} \in C$. This means that $\mu(C) \geq \sum_{i=1}^n x_i z_i \mu(C_i)$, or that $\mathbf{z} \cdot \mathbf{y} \leq 1$. So $\mathbf{y} \in B^\circ$, as claimed.

Finally, observe that we have

$$\mu(A) \leq \mu(C) \sum_{i=1}^n y_i \mu(B_i^\circ) \leq \mu(C)\mu(B^\circ),$$

as required. The result follows by induction. $\square$

Since the condition that $A \circ B \subseteq C$ is equivalent to the condition that $A \circ B \circ C^\circ \subseteq B^n_1$, we can restate Theorem 1.14 in various forms, such as (a) and (b) below.

Theorem 5.3 *Let S, T and U be convex corners in $\mathbb{R}^n_+$, with $S \circ T \circ U \subseteq B^n_1$. Then:*
(a) $\mu(S) \leq \mu(T^\circ)\mu(U^\circ)$,
(b) $\mu((T \circ U)^\circ) \leq \mu(T^\circ)\mu(U^\circ)$,
(c) $\mu(S^\circ)\mu(T^\circ)\mu(U^\circ) \geq n!$.

Proof. As noted, (a) is equivalent to Theorem 1.14. Part (b) is the special case of (a) with $S = (T \circ U)^\circ$, and (c) follows on combining (a) with Theorem 1.5. $\square$

Theorem 5.3(c) is a particularly clear extension of Theorem 1.5. It is natural to ask whether it can be extended further to products of four or more sets. Indeed, as we now see, Theorem 1.14 forms the basis for a natural induction argument giving the desired results.

Theorem 5.4 *Let $A_1, \ldots, A_r, C$ be convex corners in $\mathbb{R}^n_+$ such that $A_1 \circ \cdots \circ A_r \subseteq C$. Then*

$$\mu(A_1) \leq \mu(A_2^\circ) \ldots \mu(A_r^\circ)\mu(C),$$

and so

$$n! \leq \mu(A_1^\circ) \ldots \mu(A_r^\circ)\mu(C).$$

In particular, if $A_1 \circ \cdots \circ A_r \subseteq B_1^n$, then

$$\mu(A_1^\circ) \ldots \mu(A_r^\circ) \geq n!.$$

Proof. The first assertion uses induction on r; the case $r = 2$ is just Theorem 1.14. Suppose we have the result for products of $r-1$ convex corners, and suppose that $A_1 \circ \cdots \circ A_r \subseteq C$. Now let $A = \mathrm{conv}(A_1 \circ \cdots \circ A_{r-1})$. By Lemma 5.1,

$$A \circ A_r \subseteq \mathrm{conv}((A_1 \circ \cdots \circ A_{r-1}) \circ A_r) \subseteq C.$$

Hence, by Theorem 1.14, we have

$$\mu(A) \leq \mu(A_r^\circ)\mu(C).$$

Also we have $A_1 \circ \cdots \circ A_{r-1} \subseteq A$ so, by the induction hypothesis,

$$\mu(A_1) \leq \mu(A_2^\circ) \ldots \mu(A_{r-1}^\circ)\mu(A).$$

Multiplying these two inequalities gives the desired result.

The second assertion of the theorem follows on multiplying through by $\mu(A_1^\circ)$ and using Theorem 1.5. The third is the special case with $C = B_1^n$. (Alternatively, the third assertion follows from the first with $A_1 = B_\infty^n$ and $C = B_1^n$; and the second follows from the third with $A_{r+1} = C^\circ$.) □

A special case of Theorem 5.4 was earlier given by Sidorenko [25]; see the notes to Theorem 1.14 above and Theorem 5.9 below. We think that Theorem 5.4 is interesting in its own right, but no doubt its consequences for volume are more immediately appealing. The second part of the next result is Theorem 1.15, extending Saint-Raymond's Theorem in a natural way.

Theorem 5.5 *Let $A_1, \ldots, A_r, C$ be convex corners in $\mathbb{R}^n_+$ such that $A_1 \circ \cdots \circ A_r \subseteq C$. Then*

$$\mathrm{Vol}(A_1^\circ) \ldots \mathrm{Vol}(A_r^\circ) \geq \frac{1}{n!^{r-1}\mu(C)} \geq \frac{1}{n!^r \mathrm{Vol}(C)}.$$

In particular, if $A_1, \ldots, A_r$ are convex corners in $\mathbb{R}^n_+$ such that $A_1 \circ \cdots \circ A_r \subseteq B_1^n$, then

$$\mathrm{Vol}(A_1^\circ) \ldots \mathrm{Vol}(A_r^\circ) \geq \frac{1}{n!^{r-1}}.$$

Proof. This is an immediate consequence of Theorem 5.4 and Theorem 1.4. □

Note also that one does not need the A_i in Theorem 5.5 to be convex corners. It is enough that the A_i are bounded sets in $\mathbb{R}^n_+$, none of them contained in any co-ordinate hyperplane. For then, if B_i is the smallest convex corner containing A_i, for each i, we have $A_i^\circ = B_i^\circ$, and Theorem 5.5 can be applied to the B_i. What is more, if all the A_i are bounded and one, say A_j, is contained in a co-ordinate hyperplane, then A_j° will have infinite volume, so the inequality holds in that case too. Thus it is enough to assume that the A_i are bounded sets in $\mathbb{R}^n_+$. Similar remarks apply to Theorem 5.4 and indeed to Theorem 1.14, where B can be any bounded set in $\mathbb{R}^n_+$.

Furthermore, the conclusion of Theorem 5.5 can be strengthened by noting that we can find skew-cubes inside $A_1^\circ, \ldots, A_r^\circ$ whose volumes satisfy the inequality. Let us couch Theorem 1.15 in this more general form.

Theorem 5.6 *Suppose that $A_1, \dots, A_r$ are bounded subsets of $\mathbb{R}^n_+$ with $A_1 \circ \dots \circ A_r \subseteq B^n_1$. Then there are skew-cubes $Q_1 \subseteq A_1^\circ, \dots, Q_r \subseteq A_r^\circ$ such that*

$$\mathrm{Vol}(Q_1) \dots \mathrm{Vol}(Q_r) \geq \frac{1}{n!^{r-1}}.$$

□

We discuss Theorem 1.15 further in §6.

5.2 Pointwise products of stable set polytopes

We wish to interpret some of the results from previous sections in the case where all the convex corners involved are stable set polytopes of graphs. Our results will naturally turn out a little more cleanly when some or all of the graphs are perfect.

In particular, we aim to prove Theorem 1.17, giving a counterexample to the two conjectures of Bollobás and Leader [5] mentioned in §1.2. Our example involves stable set polytopes, and accordingly we investigate when stable set polytopes of graphs G_1, G_2, G_3 satisfy the conditions of their conjectures, namely $\mathrm{Stab}(G_1) \circ \mathrm{Stab}(G_2) \circ \mathrm{Stab}(G_3) \subseteq B^n_1$, with all the $\mathrm{Stab}(G_i)$ maximal subject to this condition.

We begin with two lemmas explaining how the pointwise product operation acts on stable set polytopes.

Lemma 5.7 *Let $G_1, \dots, G_r$ be graphs on $[n]$, and let G be the edge-union of the G_i. Then*

$$\mathrm{conv}(\mathrm{Stab}(G_1) \circ \dots \circ \mathrm{Stab}(G_r)) = \mathrm{Stab}(G).$$

Proof. By Theorem 5.2, $S = \mathrm{conv}(\mathrm{Stab}(G_1) \circ \dots \circ \mathrm{Stab}(G_r))$ is the convex hull of the set of points of the form $\mathbf{e}_{X_1} \circ \dots \circ \mathbf{e}_{X_r} = \mathbf{e}_{\cap X_i}$, where each X_i is a stable set in the corresponding G_i. The sets that occur as $\cap X_i$ are exactly the stable sets in G, so the vertices of S are exactly the indicator vectors of stable sets in G, as required. □

We say that the r-tuple $(G_1, \dots, G_r)$ of graphs on $[n]$ is *exhaustive* if each $S_i = \mathrm{Stab}(G_i)$ is a maximal convex corner subject to the condition that $S_1 \circ \dots \circ S_r \subseteq \mathrm{Stab}(G)$, where G is the edge-union of the G_i.

Theorem 5.8 *Let $G_1, \dots, G_r$ be graphs on $[n]$, and let G be their edge-union.*
(a) *If $(G_1, \dots, G_r)$ is exhaustive, then the G_i are edge-disjoint.*
(b) *If the G_i are edge-disjoint and each G_i is perfect, then $(G_1, \dots, G_r)$ is exhaustive.*
(c) *If G is perfect, then $(G_1, \dots, G_r)$ is exhaustive if and only if the G_i are edge-disjoint and each G_i is perfect.*

Proof. (a) If the G_i are not edge-disjoint, then we can delete a duplicated edge from some G_j, increasing $\mathrm{Stab}(G_j)$, and preserving the union G of the G_i. So by Lemma 5.7, the original $\mathrm{Stab}(G_j)$ was not maximal, and the original r-tuple of graphs was not exhaustive.
(b) Suppose that the G_i are edge-disjoint, and that the G_i are all perfect. Suppose further that $\mathrm{Stab}(G_1)$ is not maximal, so that there is a point $\mathbf{x}$ not in $\mathrm{Stab}(G_1)$ with $\mathbf{x} \circ \mathrm{Stab}(G_2) \circ \dots \circ \mathrm{Stab}(G_r) \subseteq \mathrm{Stab}(G)$. Since G_1 is perfect, we have that $\mathbf{x} \notin \mathrm{Stab}(\overline{G_1})^\circ$, by Theorem 1.2. Thus there is a clique Y in G_1 with $\mathbf{x} \cdot \mathbf{e}_Y > 1$. Since the G_i are edge-disjoint, Y is a stable set in each other G_i, so $\mathbf{e}_Y \in \mathrm{Stab}(G_2) \circ \dots \circ \mathrm{Stab}(G_r)$. Hence $\mathbf{x} \circ \mathbf{e}_Y \in \mathrm{Stab}(G)$. Now we have

$$\mathbf{x} \cdot \mathbf{e}_Y = \sum_{i \in Y} (\mathbf{x} \circ \mathbf{e}_Y)_i \leq 1,$$

since Y is a clique in G. This is a contradiction, completing the proof of this part.
(c) One direction here follows from (b). For the other, suppose that G is perfect, and that G_1 is not perfect, so $\mathrm{Stab}(G_1) \subset \mathrm{Stab}(\overline{G_1})^\circ$. Now we claim that $\mathrm{Stab}(\overline{G}_1)^\circ \circ \mathrm{Stab}(G_2) \circ \cdots \circ \mathrm{Stab}(G_r) \subseteq \mathrm{Stab}(\overline{G})^\circ$, which will complete the proof since G is perfect and thus $\mathrm{Stab}(\overline{G})^\circ = \mathrm{Stab}(G)$. We know from Lemma 5.7 that $\mathrm{Stab}(G_2) \circ \cdots \circ \mathrm{Stab}(G_r) \subseteq \mathrm{Stab}(\cup_{i=2}^r G_i)$, which means that we may assume that $r = 2$.

Take any vectors $\mathbf{x} \in \mathrm{Stab}(\overline{G}_1)^\circ$ and $\mathbf{y} \in \mathrm{Stab}(G_2)$; we need to show that $\mathbf{x} \circ \mathbf{y} \in \mathrm{Stab}(\overline{G}_1 \cup G_2)^\circ$. We may assume that $\mathbf{y} = \mathbf{e}_Y$ for some stable set Y in G_2, since $\mathbf{x} \circ \mathbf{y}$ is a convex combination of vectors of the form $\mathbf{x} \circ \mathbf{e}_Y$. Now take any clique Z in $G_1 \cup G_2$, and note that

$$(\mathbf{x} \circ \mathbf{e}_Y) \cdot \mathbf{e}_Z = \mathbf{x} \cdot (\mathbf{e}_Y \circ \mathbf{e}_Z) = \sum_{i \in Y \cap Z} x_i \leq 1,$$

since $Y \cap Z$ is a clique in G_1. Hence $\mathbf{x} \circ \mathbf{e}_Y \in \mathrm{Stab}(\overline{G}_1 \cup G_2)^\circ$, as desired. □

What parts (b) and (c) of Theorem 5.8 actually prove is that the polytopes $S_i = \mathrm{Stab}(G_i)$ are maximal subject to the weaker condition that $S_1 \circ \cdots \circ S_r \subseteq \mathrm{Stab}(\overline{G})^\circ$, where G is the edge-union of the G_i, if and only if the G_i are disjoint and all perfect. This is in fact equivalent to the special case of Theorem 5.8(c) with $G = K_n$, since our condition is exactly that $S_1 \circ \cdots \circ S_r \circ \mathrm{Stab}(\overline{G}) \subseteq B_1^n$.

We can now prove Theorem 1.17, giving the counterexample to the conjectures of Bollobás and Leader [5]. Recall that G_1, G_2 and G_3 are the graphs with node set $\{1, 2, 3\}$ and edge-sets $\{12\}$, $\{13\}$ and $\{23\}$ respectively, and that the S_i are the stable set polytopes of the corresponding G_i.

Proof of Theorem 1.17. The first assertion amounts to saying that (G_1, G_2, G_3) is exhaustive, since the edge union of the graphs is K_3, and $\mathrm{Stab}(K_3) = B_1^3$. This follows from Theorem 5.8(b).

To see that $S_1 \circ S_2 \circ S_3 \neq B_1^3$, consider the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, which is in B_1^3. However, for any point $(a, b, c) \in \mathrm{Stab}(G_1)$, $a + b \leq 1$ and $c \leq 1$, so $abc \leq \frac{1}{4}$. Similarly, for $(d, e, f) \in \mathrm{Stab}(G_2)$ and $(g, h, i) \in \mathrm{Stab}(G_3)$, we have $def \leq \frac{1}{4}$ and $ghi \leq \frac{1}{4}$. Thus $abcdefghi \leq \frac{1}{64}$, so

$$(a, b, c) \circ (d, e, f) \circ (g, h, i) = (adg, beh, cfi) \neq \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Finally, we need to verify that $\mathrm{Vol}(\mathrm{Stab}(G_1)) = \frac{1}{2}$. This can be seen geometrically, analytically, or by observing that G_1 is the comparability graph of a poset with $3 = \frac{1}{2}3!$ linear extensions. □

We look briefly at applications of results from §5.1 to the case of stable set polytopes. By Theorem 1.14 and Lemma 5.7, we see that, for any graphs G_1 and G_2,

$$\mu^{\mathrm{g}}(G_1) \leq \mu(\mathrm{Stab}(G_2)^\circ)\mu^{\mathrm{g}}(G_1 \cup G_2).$$

More generally, using Theorem 5.4, we have that if $H, G_1, \ldots, G_r$ are graphs with edge-union H_0, then

$$\mu(\mathrm{Stab}(H)) \leq \mu(\mathrm{Stab}(G_1)^\circ) \ldots \mu(\mathrm{Stab}(G_r)^\circ)\mu(\mathrm{Stab}(H_0)).$$

If $G_1, \ldots, G_r$ are all perfect, with complements $H_1, \ldots, H_r$, then we obtain the following result.

Theorem 5.9 *Suppose that H, H_0 are graphs, and $H_1, \ldots, H_r$ are perfect graphs on $[n]$ with $H_0 \cap H_1 \cap \cdots \cap H_r \subseteq H$. Then*

$$\mu^{\mathrm{g}}(H) \leq \mu^{\mathrm{g}}(H_0)\mu^{\mathrm{g}}(H_1) \ldots \mu^{\mathrm{g}}(H_r).$$

□

Theorem 5.9 generalizes a result of Sidorenko, Theorem 14 of [25], which is the case of the theorem where $H_1, \ldots, H_r$ are comparability graphs. In this case, as we have seen, $\mu^{\mathrm{g}}(H_i)$ has an interpretation as the number of linear extensions of the corresponding partial order. Theorem 5.9 is of course also a generalization of Theorem 4.1(b), and we saw even in that case that the result will fail without some assumption on some of the graphs involved being perfect.

As usual, it would be straightforward to use Theorem 1.4 to derive inequalities involving volumes from Theorem 5.9 and other results above.

5.3 Cases of equality

The cases of equality in Theorem 1.1 were found by Meyer [20] and Reisner [22]: they showed that $\mathrm{Vol}(S)\mathrm{Vol}(S^\circ) = 1/n!$ if and only if S is, after rescaling, the stable set polytope of a cograph; in other words, if S can be built up from one-dimensional polytopes by ℓ_1- and ℓ_∞-sums. Our aim here is to see when we have equality in Theorem 1.15, i.e., for which r-tuples of convex corners $A_1, \ldots, A_r$ with $A_1 \circ \cdots \circ A_r \subseteq B_1^n$ we have $\mathrm{Vol}(A_1^\circ) \ldots \mathrm{Vol}(A_r^\circ) = n!^{1-r}$. Our proof does not rely on the result of Meyer and Reisner, so we obtain another proof of that result which, however, is not essentially different from Meyer's.

To describe all cases of equality in Theorem 1.15, it is easiest to regard the various bodies concerned as stable set polytopes of graphs. For each r, we define the family of *complementary* r-tuples of graphs recursively as follows. Firstly, the family $(G, \ldots, G)$, where each G is a one-node graph, is complementary. Then, if $V = V_1 \cup V_2$ is any partition of the node set into two parts, $(H_1, \ldots, H_r)$ is a complementary r-tuple on V_1, $(H_1', \ldots, H_r')$ is a complementary r-tuple on V_2, and $j \in [r]$ is any index, then the r-tuple $(G_1, \ldots, G_r)$ is a complementary r-tuple on V, where $G_i = H_i \cup H_i'$ for every index $i \neq j$, and $G_j = H_j + H_j'$.

If $(G_1, \ldots, G_r)$ is a complementary r-tuple of graphs on node set $[n]$, then clearly the G_i are edge-disjoint with union K_n, and furthermore each G_i individually is built up from one-node graphs by the operations of disjoint union and disjoint sum, so is a cograph. However, note that the three one-edge subgraphs of K_3 are all cographs, and their union is K_3, but the triple of graphs is not complementary.

Now suppose that $(G_1, \ldots, G_r)$ is a complementary r-tuple of graphs on $[n]$, and consider the stable set polytopes $S_i = \mathrm{Stab}(G_i)$. By Lemma 5.6, we have that $\mathrm{conv}(S_1 \circ \cdots \circ S_r) = \mathrm{Stab}(K_n) = B_1^n$.

Theorem 5.10 *Suppose that $(G_1, \ldots, G_r)$ is a complementary r-tuple of graphs on a node set of size n, and let $S_i = \mathrm{Stab}(G_i)$ for each i. Then*

$$\mathrm{Vol}(S_1^\circ) \ldots \mathrm{Vol}(S_r^\circ) = n!^{1-r}.$$

Proof. We work by induction on n, with the result being trivial for $n = 1$.

For the induction step, suppose that the complementary r-tuple $(G_1, \ldots, G_r)$ of graphs on V, of size n, was obtained from the two complementary r-tuples $(H_1, \ldots, H_r)$ and $(H_1', \ldots, H_r')$ on node sets V_1 and V_2 of sizes n_1 and n_2 respectively, with j being the distinguished index. Suppose also that $(H_1, \ldots, H_r)$ and $(H_1', \ldots, H_r')$ satisfy the conclusion of the theorem, i.e.,

$$\mathrm{Vol}(T_1^\circ) \ldots \mathrm{Vol}(T_r^\circ) = n_1!^{1-r}$$

and

$$\mathrm{Vol}(U_1^\circ) \ldots \mathrm{Vol}(U_r^\circ) = n_2!^{1-r},$$

where $T_i = \mathrm{Stab}(H_i)$ and $U_i = \mathrm{Stab}(H_i')$ for each i.

Note that all the graphs involved are strongly perfect, so that for instance $S_i^\circ = \mathrm{Stab}(\overline{G_i})$, and $\mathrm{Vol}(S_i^\circ) = \frac{1}{n!}\mu^{\mathrm{g}}(\overline{G_i})$ for each i. Now, for $i \neq j$, we have $S_i^\circ = \mathrm{Stab}(\overline{G_i}) = \mathrm{Stab}(\overline{H_i} + \overline{H_i'})$ so, by Lemma 4.3,

$$\mathrm{Vol}(S_i^\circ) = \frac{1}{n!}\mu^{\mathrm{g}}(\overline{G}_i) = \frac{1}{n!}\mu^{\mathrm{g}}(\overline{H}_i)\mu^{\mathrm{g}}(\overline{H}_i') = \frac{n_1!n_2!}{n!}\mathrm{Vol}(T_i^\circ)\mathrm{Vol}(U_i^\circ).$$

Similarly, $S_j^\circ = \mathrm{Stab}(\overline{H_j} \cup \overline{H_j'})$, and S_j° has volume $\mathrm{Vol}(T_j^\circ)\mathrm{Vol}(U_j^\circ)$.

Combining the various identities, we have that

$$\begin{aligned}\mathrm{Vol}(S_1^\circ)\dots\mathrm{Vol}(S_r^\circ) &= \left(\frac{n_1!n_2!}{n!}\right)^{r-1}\mathrm{Vol}(T_1^\circ)\dots\mathrm{Vol}(T_r^\circ)\mathrm{Vol}(U_1^\circ)\dots\mathrm{Vol}(U_r^\circ)\\ &= \left(\frac{n!}{n_1!n_2!}\right)^{1-r} n_1!^{1-r}n_2!^{1-r} = n!^{1-r},\end{aligned}$$

as desired. □

The converse of Theorem 5.10 is the content of the next theorem. Before that, we need one graph-theoretic lemma.

Lemma 5.11 *Suppose that $G_1, \dots, G_r$ are edge-disjoint cographs on node set $[n]$ with union K_n, such that no three-element subset of $[n]$ contains edges from three different G_i. Then $(G_1, \dots, G_r)$ is a complementary r-tuple.*

Proof. Suppose the result is false, and take a counterexample with minimal n.

Suppose that none of the G_i is connected, and let X be a largest component in any G_i, with u some node outside X and v some node inside. Then the edge uv is in some G_j, and the condition gives that uw is an edge of G_j for every $w \in X$. This means that $X \cup \{u\}$ is connected in G_j, contradicting the maximality of X.

Thus one cograph G_j is connected, which means it must contain all the edges between some set X and its complement Y, with X and Y non-empty. The graph G_j is then the disjoint sum of separate graphs on X and on Y, while all the other graphs G_i are the disjoint union of graphs on X and on Y. By minimality of n, the r-tuples $(G_1|_X, \dots, G_r|_X)$ and $(G_1|_Y, \dots, G_r|_Y)$ are complementary. Hence so is $(G_1, \dots, G_r)$, a contradiction. □

Theorem 5.12 *Suppose that $A_1, \dots, A_r$ are convex corners in $\mathbb{R}_+^n$ with $A_1 \circ \dots \circ A_r \subseteq B_1^n$, so that*

$$\mathrm{Vol}(A_1^\circ)\dots\mathrm{Vol}(A_r^\circ) \geq n!^{1-r}.$$

Suppose that each basic unit vector $\mathbf{e}_j$ is on the boundary of each A_i. Then we have equality above if and only if each A_i is the stable set polytope of a graph G_i on $[n]$, and the r-tuple $(G_1, \dots, G_r)$ is complementary.

Proof. By Theorem 5.4, we have that, if $A_1 \circ \dots \circ A_r \subseteq B_1^n$, then

$$\mu(A_1) \leq \mu(A_2^\circ)\dots\mu(A_r^\circ).$$

We also have that $n! \leq \mu(A_1)\mu(A_1^\circ)$, and that $\mu(A_i^\circ) \leq n!\mathrm{Vol}(A_i^\circ)$ for each i. Multiplying all these together gives that $n!^{1-r} \leq \mathrm{Vol}(A_1^\circ)\dots\mathrm{Vol}(A_r^\circ)$, just as in earlier proofs. Thus we must have equality in all of the above inequalities, i.e., A_1 is μ-minimal, and all the A_i° are skew-cubes. Also, A_1° has minimal volume subject to the condition that $A_1 \circ (A_2 \circ \dots \circ A_r) \subseteq B_1^n$, i.e., $A_1^\circ = \mathrm{conv}(A_2 \circ \dots \circ A_r)$.

Note that these conditions all pass to co-ordinate subspaces, so we may use induction on the dimension n of the space. Thus it is sufficient to show that all the A_i are stable set polytopes of graphs G_i, and that the r-tuple $(G_1, \ldots, G_r)$ can be broken down as in the definition of a complementary r-tuple.

By Theorem 3.3, we see that A_1 is a 0-1 polytope. By symmetry, all the other A_i are also 0-1 polytopes. Thus, by Theorem 5.2, $A_1^\circ = \mathrm{conv}(A_2 \circ \cdots \circ A_r)$ is a polytope, all of whose vertices are the products of 0-1 vectors, so are themselves 0-1 vectors. In other words, A_1° is a 0-1 convex corner. As it is also a skew-cube, Theorem 1.10 tells us that A_1° is the stable set polytope of a strongly perfect graph H_1. Furthermore, since A_1° is μ-minimal, we have by Theorem 4.4 (or, rather, the calculation preceding it concerning P_4) that H_1 is a cograph. By symmetry, each A_i° is the stable set polytope of a cograph H_i.

Each A_i is the stable set polytope of the complement G_i of H_i. We need to show that the r-tuple $(G_1, \ldots, G_r)$ is complementary.

By Lemma 5.7,

$$\mathrm{Stab}(G_1) = A_1^\circ = \mathrm{conv}(A_2 \circ \cdots \circ A_r) = \mathrm{Stab}(H_2 \cup \cdots \cup H_r).$$

This implies that $G_1 = H_2 \cup \cdots \cup H_r$, i.e., that the union of all the H_i is the complete graph K_n, and that H_1 as no edge in common with any other H_i. By symmetry, we deduce that the H_i are edge-disjoint.

Next we note from the proof of Theorem 5.4 that we have $\mu(A_1) = \mu(A_2^\circ)\mu(C)$, where $C = \mathrm{conv}(A_1 \circ A_2)$, which here is the stable set polytope of $G_1 \cup G_2$. Also, from the proof of Theorem 1.14, we see that this implies that $\mu((A_1)_i) = \mu((A_2^\circ)_i)\mu(C_i)$, for any i such that there is a vector $\mathbf{x} \in A_1$ with $x_i > 0$ and $\mu(A_1) = \sum_{i=1}^n x_i \mu((A_1)_i)$. But we know that all maximal vectors $\mathbf{x}$ of A_1 have this property, so we have $\mu((A_1)_i) = \mu((A_2^\circ)_i)\mu(C_i)$ for every i, and hence $\mu((A_1)_X) = \mu((A_2^\circ)_X)\mu(C_X)$ for every subset X of $[n]$.

Now consider the case $n = 3$, with G_1 consisting of one edge, and G_2 a different edge. Then $\mu(A_1) = 3$, $\mu(A_2^\circ) = 2$, and $\mu(C) = 2$. We conclude that no two of the G_i contain this configuration; i.e., if $uv \in G_i$ and $vw \in G_j$, then the edge uw is in either G_i or G_j.

The cographs $G_1, \ldots, G_r$ thus satisfy the conditions of Lemma 5.11, and hence $(G_1, \ldots, G_r)$ is complementary, as required. □

6 Products of volumes

Our main aim here is to prove the various assertions in Theorem 1.18, giving best possible upper and lower bounds on both $\prod \mathrm{Vol}(A_i)$ and $\prod \mathrm{Vol}(A_i^\circ)$, when $A_1, \ldots, A_r$ are convex corners with $A_1 \circ \cdots \circ A_r \subseteq B_1^n$.

For the upper bound on $\prod \mathrm{Vol}(A_i)$, we use Theorem 12 of Bollobás and Leader [5], which is as follows.

Theorem 6.1 *If $A_1, \ldots, A_r$ are convex corners, then*

$$\mathrm{Vol}\left((A_1 \circ \cdots \circ A_r)^{1/r}\right) \geq (\mathrm{Vol}(A_1) \ldots \mathrm{Vol}(A_r))^{1/r}.$$

Note that the case $r = 2$ is the result we quoted earlier as Theorem 1.16(b). An instant consequence of Theorem 6.1 is that, if $A_1, \ldots, A_r$ are convex corners (or, for that matter, bounded subsets of $\mathbb{R}_+^n$) with $A_1 \circ \cdots \circ A_r \subseteq B_1^n$, then

$$\mathrm{Vol}(A_1) \ldots \mathrm{Vol}(A_r) \leq (\mathrm{Vol}(B_r^n))^r.$$

Of course, the case $A_1 = A_2 = \cdots = A_r = B_r^n$ shows this to be best possible. The case $r = 2$ is just Santaló's Theorem for 1-unconditional bodies.

For the lower bound, we can use Theorem 1.16(a), also due to Bollobás and Leader, stating that, if A and B are down-sets in $\mathbb{R}_+^n$, then $\mathrm{Vol}(A \circ B) \le \mathrm{Vol}(A)\mathrm{Vol}(B)$. It follows by induction that, for any down-sets $A_1, \ldots, A_r$ in $\mathbb{R}_+^n$,

$$\mathrm{Vol}(A_1 \circ \cdots \circ A_r) \le \mathrm{Vol}(A_1) \ldots \mathrm{Vol}(A_r).$$

If $A_1 \circ \cdots \circ A_r = B_1^n$, the left-hand-side is $1/n!$. To see that this is best possible, let $(G_1, \ldots, G_r)$ be a complementary r-tuple of graphs and set $A_i = \mathrm{Stab}(G_i)$ for each i. Now $\mathrm{Vol}(A_i)\mathrm{Vol}(A_i^\circ) = 1/n!$ for each i, and $\mathrm{Vol}(A_1^\circ) \ldots \mathrm{Vol}(A_r^\circ) = 1/n!^{r-1}$, by Theorem 5.10, so $\mathrm{Vol}(A_1) \ldots \mathrm{Vol}(A_r) = 1/n!$. Thus we have proved part (1) of Theorem 1.18.

As we saw in §5.2, the lower bound $\prod \mathrm{Vol}(A_i) \ge 1/n!$ ceases to be valid if we demand only that the A_i are maximal convex corners subject to $A_1 \circ \cdots \circ A_r \subseteq B_1^n$. Indeed, we know of no non-trivial lower bound on $\prod_{i=1}^r \mathrm{Vol}(A_i)$ under these conditions. We give an example showing that the product can be substantially smaller than $1/n!$.

Let $r = 2^{d-1}$, and take d independent random linear orderings $L_1, L_2, \ldots, L_d$ on $[n]$. For a linear order L, let L^* denote its reverse. Now, for each of the 2^{d-1} possible choices $(M_2, \ldots, M_d)$, where each M_i is either equal to L_i or L_i^*, construct the partial order $L_1 \cap M_2 \cap \cdots \cap M_d$. Note that each pair of elements from $[n]$ is comparable in exactly one of these $r = 2^{d-1}$ partial orders $P_1, \ldots, P_r$, so K_n is the edge disjoint union of the comparability graphs $C(P_i)$. By Theorem 5.8(b), the r-tuple $(C(P_1), \ldots, C(P_r))$ is exhaustive, so the convex corners $\mathrm{Stab}(C(P_i))$ are maximal subject to the condition that their pointwise product is contained in $\mathrm{Stab}(K_n) = B_1^n$.

Stanley's Theorem, Theorem 1.11, tells us that $n!\mathrm{Vol}(\mathrm{Stab}(C(P_i))) = e(P_i)$, the number of linear extensions. For each i, the number of linear extensions is distributed as the number of linear extensions of a *random d-dimensional partial order $P_d(n)$*. Brightwell [7] showed that, for each fixed d, with probability tending to 1 as $n \to \infty$,

$$\left(e^{-2} n^{1-1/d}\right)^n \le e(P_d(n)) \le \left(2dn^{1-1/d}\right)^n.$$

Bollobás, Brightwell and Sidorenko [3] used Saint-Raymond's Theorem and Santaló's Theorem to improve the constants in both inequalities above in the case $d = 2$. For $d > 2$, the application is the other way around.

Indeed, with probability tending to 1 as $n \to \infty$ with $r = 2^{d-1}$ fixed, we have that

$$\prod_{i=1}^r \mathrm{Vol}(\mathrm{Stab}(C(P_i))) \le \left(\frac{(2dn^{1-1/d})^n}{n!}\right)^r \le \left(2den^{-1/d}\right)^{nr} = n^{-n2^{d-1}/d(1+o(1))}.$$

For $d \ge 3$, this is much smaller than $1/n!$.

One can obtain similar examples when r is not a power of 2. For instance, if $r = 3$, take three linear orders L_1, L_2, L_3 as above, and consider the three partial orders $L_1 \cap L_2$, $L_1 \cap L_2^* \cap L_3$ and $L_1 \cap L_2^* \cap L_3^*$, and the stable set polytopes of their comparability graphs. The same method shows that these polytopes satisfy the hypotheses, and gives an upper bound on the volume product around $n^{-7n/6}$.

Turning to the proof of part (2) of Theorem 1.18, the lower bound $\prod_{i=1}^r \mathrm{Vol}(A_i^\circ) \ge n!^{1-r}$ is just Theorem 1.15. The remaining assertion that we have to prove is that the product is largest when the A_i are all equal to B_r^n, in which case the A_i° are equal to $B_{r/(r-1)}^n$. We restate this as a theorem.

Theorem 6.2 *Suppose that $A_1, \ldots, A_r$ are convex corners in $\mathbb{R}^n_+$ such that $A_1 \circ \cdots \circ A_r = B^n_1$. Then*
$$\prod_{i=1}^{r} \mathrm{Vol}(A_i^\circ) \leq \left(\mathrm{Vol}(B^n_{r/(r-1)})\right)^r.$$

Proof. We apply Theorem 6.1 to the A_i°. Note that each A_i° is the pointwise product of all the A_j with $j \neq i$. Thus
$$\prod_{i=1}^{r} A_i^\circ = \prod_{j=1}^{r} A_j^{r-1} = (B^n_1)^{r-1},$$
so that $(\prod A_i^\circ)^{1/r} = (B^n_1)^{(r-1)/r} = B^n_{r/(r-1)}$. The result now follows from Theorem 6.1. □

The case $r = 2$, needless to say, is the unconditional form of Santaló's Theorem.

Again, the condition that the A_i are maximal subject to $A_1 \circ \cdots \circ A_r = B^n_1$ is not sufficient for the conclusion of Theorem 6.2. Our three-node example from Theorem 1.17 does not quite work this time, but $E(K_4)$ can be partitioned into three copies of the perfect graph H consisting of two independent edges, and $\mathrm{Vol}(\mathrm{Stab}(H)^\circ) = \frac{1}{6}$, while $\mathrm{Vol}(B^4_{3/2}) = 0.1655\cdots < \frac{1}{6}$.

We end this section with an inequality relating $\prod \mathrm{Vol}(A_i)$ and $\prod \mathrm{Vol}(A_i^\circ)$.

Theorem 6.3 *If $A_1, \ldots, A_r$ are convex corners with $A_1 \circ \cdots \circ A_r = B^n_1$, then*
$$\prod_{i=1}^{r} \mathrm{Vol}(A_i^\circ) \leq \left(\prod_{i=1}^{r} \mathrm{Vol}(A_i)\right)^{r-1}.$$

Proof. We note again that $A_i^\circ = \prod_{j\neq i} A_j$. By Bollobás' and Leader's Theorem 14, the volume of A_i° is thus at most $\prod_{j\neq i} \mathrm{Vol}(A_j)$, and the result follows. □

We always have equality when $r = 2$, and also when $A_i = \mathrm{Stab}(G_i)$, for $(G_1, \ldots, G_r)$ a complementary r-tuple of graphs.

7 Other topics

7.1 Complexity Issues

It is natural to ask whether the content of a convex corner, or at least of a suitably presented polytope, can be efficiently computed. The answer, as we explain below, is that exact computation is #P-hard, and even approximation may be difficult.

In the very special case where S is the stable set polytope of the comparability graph of a partial order P, we know from Theorem 1.12 that $\mu(S)$ is the number of linear extensions of P. A result of Brightwell and Winkler [8] states that it is #P-complete to calculate the number of linear extensions of an n-element partial order, and it follows that it is #P-hard to calculate $\mu(S)$ exactly, even when S has this special structure. This also proves that computing the volume of a convex corner is #P-hard.

It seems to be unknown whether it is NP-hard to decide whether a graph is strongly perfect, although Hoáng [13] has proved that it is NP-hard to decide whether a graph has a stable set meeting every maximal clique. In our terms, this means that we do not know if it is NP-hard to determine whether a 0-1 convex corner is a skew-cube, i.e., whether its content is equal to its volume.

One very weak positive result is that it is possible to approximate the content of a suitably presented convex corner in deterministic polynomial time, up to an exponential factor. This follows from the fact, noted by Kahn and Kim [15] that $V(S)$, the volume of the largest cuboid inside S, can be determined to within a factor of $(1+\varepsilon)$ in deterministic polynomial time, using the ellipsoid algorithm. They concluded from this that the volume of a convex corner can be approximated within a factor e^n in deterministic polynomial time (this is known *not* to be true for a general convex body – see Bárány and Füredi [1]), and the same evidently holds for the content.

One open problem in this direction is to decide whether or not $\mu(S)$ can be approximated to within any factor $(1+\varepsilon)$ in *randomized* polynomial time. Dyer, Frieze and Kannan [11] showed that the volume can be so approximated, even in more general circumstances.

7.2 Other inequalities

Let S be a convex corner in $\mathbb{R}^n_+$. We know from the proof of Theorem 1.4 that

$$\mathrm{Vol}(S) \geq \sum_{i=1}^{n} y_i \mathrm{Vol}(S_i)$$

for every $\mathbf{y} \in S$. This means that the vector $\mathbf{z}$ defined by $z_i = \mathrm{Vol}(S_i)/n\mathrm{Vol}(S)$ is in S°. So we have

$$\sum_{i=1}^{n} z_i \frac{\mathrm{Vol}(S)}{\mathrm{Vol}(S_i)} = 1.$$

A consequence of this is that there is an extreme point $\mathbf{w}$ of S° such that

$$\sum_{i=1}^{n} w_i \frac{\mathrm{Vol}(S)}{\mathrm{Vol}(S_i)} \geq 1.$$

If S° is a polytope, whose vertices we know, then this simple result will have an interpretation. In particular we have the following.

Theorem 7.1 *Let G be a perfect graph. Then there is a clique C in G such that*

$$\sum_{x \in C} \frac{\mathrm{Vol}(\mathrm{Stab}(G))}{\mathrm{Vol}(\mathrm{Stab}(G-x))} \geq 1.$$

Proof. Apply the preceding inequality to $\mathrm{Stab}(G)$. □

Theorem 7.2 *For every partial order P, there is a chain C in P with*

$$\sum_{x \in C} \frac{e(P)}{e(P-x)} \geq |P|.$$

Proof. Apply Theorem 7.1 to the comparability graph of P. □

Theorem 7.2 follows (in much the same way) from results of Sidorenko [25], and seems to us to be an attractive "dual" to Theorem 1.12.

Changing tack somewhat, let us give one more generalization of Saint-Raymond's Inequality. Suppose that S is a convex corner in $\mathbb{R}^n_+$, and that the natural number r is such that S^r, the pointwise product of r copies of S, is convex. Thus, by Theorem 5.5, with $A_1 = \cdots = A_r = S$ and $C = S^r$, we have

$$(\mathrm{Vol}(S^\circ))^r \geq \frac{1}{n!^r \mathrm{Vol}(S^r)}.$$

Rearranging gives the following result.

Theorem 7.3 *Suppose that S is a convex corner in $\mathbb{R}^n_+$, and that r is a natural number such that S^r is convex. Then*

$$\mathrm{Vol}(S)\mathrm{Vol}(S^\circ) \geq \frac{\mathrm{Vol}(S)}{(\mathrm{Vol}(S^r))^{1/r}} \frac{1}{n!}.$$

□

A consequence of Corollary 13 of Bollobás and Leader [5] is that $\mathrm{Vol}(S) \geq (\mathrm{Vol}(S^r))^{1/r}$, so this is (at least in some cases) a stronger form of Saint-Raymond's Inequality.

7.3 Open problems

We expect that there is much more to be discovered by following the themes of this paper. Here are just some of the problems that we have encountered that seem to us to be worth further study.

Problem 7.1 *Find better bounds for the constant $\phi = \sup_S \left(\frac{n!\mathrm{Vol}(S)}{\mu(S)}\right)^{1/n}$ defined in §2.3.*

A related problem is the following.

Problem 7.2 *Is there a positive ε such that, for every n and every convex corner S in $\mathbb{R}^n_+$, there is a point $\mathbf{x}$ in S such that*

$$\mathrm{Vol}(S) \leq \frac{1}{n\varepsilon} \sum_i x_i \mathrm{Vol}(S_i)?$$

In other words, is there always a point $\mathbf{x} \in S$ such that the union of the cones $\mathrm{conv}(\mathbf{x}, S_i)$ contains a positive proportion of the volume?

The example of $T^n_{n/2}$ shows that we cannot have $\varepsilon > 1/2$ in the above. If one could find such an ε, then it would follow immediately that $\mu(S) \geq \varepsilon^n n! \mathrm{Vol}(S)$ for every convex corner S in $\mathbb{R}^n_+$, i.e., that $\phi < 1/\varepsilon$.

Problem 7.3 *Find better bounds for the constant*

$$\eta = \sup_S \left(\frac{\mu(S)\mu(S^\circ)}{n!}\right)^{1/n}$$

defined in §3. In particular, is it true that $\eta < \pi/2$?

One possible approach to this would be to show that polytopes with a limited number of vertices are not close to being extremal examples in Santaló's Inequality. More precisely, the answer to the next problem is surely yes.

Problem 7.4 *Is it true that, for every constant c, there is a positive ε such that, whenever S is a polytope in $\mathbb{R}^n_+$ with at most c^n vertices,*

$$\mathrm{Vol}(S)\mathrm{Vol}(S^\circ) \leq (1-\varepsilon)^n (\mathrm{Vol}(B^n_2))^2?$$

The next problem is probably rather less innocent than it first appears. The result of Meyer [20] and Reisner [22] characterizing the cases of equality in Saint-Raymond's Theorem can be viewed as saying that, after scaling, the convex corners S such that both S and S° are skew-cubes, and S is μ-minimal, are exactly the stable-set polytopes of cographs, i.e., those polytopes that can be formed from 1-dimensional convex corners by ℓ_1- and ℓ_∞-sums. It seems that we may not get a much larger class of extremal examples if we demand only that S is μ-minimal and *one* of S and S° (say S°) is a skew-cube. This is, by the definitions, what the next problem amounts to.

Problem 7.5 *Determine the class of convex corners S such that*

$$\mu(S) = 1/\mathrm{Vol}(S^\circ). \qquad (*)$$

As we noted earlier, Theorem 3.3 implies that any S satisfying $(*)$ is, after re-scaling, a 0-1 convex corner. There are also a number of other conclusions that can be drawn about S from various results in this paper. We describe all the convex corners we know that satisfy $(*)$. Notice that all the convex corners T^n_t are μ-minimal, and their polars are skew-cubes by Theorem 2.4. The examples below include the T^n_t as a special case.

Let $U(t,m,r)$ be the set system on ground-set M, the disjoint union of r sets $M_1,\dots,M_r$, each of size m, with $X \in U(t,m,r)$ unless X contains t of the M_i. Now define the convex corner $S(t,m,r)$ to be the convex hull of the vectors $\mathbf{e}_X$, for $X \in U(t,m,r)$. One can check that $S(t,m,r)$ satisfies $(*)$ for all t, m and r.

It is easy enough to check that, if S and T both satisfy $(*)$, then so do both the ℓ_1- and ℓ_∞-sums of S and T. Indeed, we suspect that all convex corners satisfying $(*)$ can be constructed from convex corners of the form $S(t,m,r)$ by repeated ℓ_1- and ℓ_∞-sums.

Some other problems concerning pointwise products occurred to us, motivated at least partly by the special cases where the convex corners concerned are stable set polytopes of (perfect) graphs. We state two of these, although we have not investigated them at all thoroughly.

Problem 7.6 *Is it true that, for all convex corners A, B and C in $\mathbb{R}^n_+$ with $A \circ B \subseteq C$, we have* $\mathrm{Vol}(A) \le n!\mathrm{Vol}(B^\circ)\mathrm{Vol}(C)$*? Equivalently, if $S \circ T \circ U \subseteq B^n_1$, does it follow that* $\mathrm{Vol}(S) \le n!\mathrm{Vol}(T^\circ)\mathrm{Vol}(U^\circ)$*?*

This would be the analogue of Theorem 1.14 (and Theorem 5.3(a)), with content replaced by $n!$ times volume everywhere. It would of course be yet another extension of Saint-Raymond's Inequality. It can be shown that this would imply the perhaps even more attractive inequality below.

Problem 7.7 *Suppose that S, T and U are convex corners in $\mathbb{R}^n_+$ satisfying $S \circ T \supset U \circ B^n_1$. Does it follow that*

$$\mathrm{Vol}(S)\mathrm{Vol}(T) \ge \mathrm{Vol}(U)\mathrm{Vol}(B^n_1) = \frac{1}{n!}\mathrm{Vol}(U)?$$

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