Real-valued frequency assignment

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Abstract

We consider the binary constraints formulation of the frequency assignment problem in its most general form: for an arbitrary metric space, with frequencies taking arbitrary real values, and with possibly infinitely many constraints. We obtain some necessary and sufficient conditions for the problem to have a solution with a finite span. When the metric space is the set of integers, we give an exact criterion.

Also we demonstrate a connection of this problem in one-dimensional case with one combinatorial question about finite permutations; and pose some unsolved problems.

The practical problem of assigning frequencies to transmitters in a cellular network has been approximated by various combinatorial and/or geometrical formulations. Most often it is treated as a problem of finding a colouring of a metric space (e.g. a graph). The required colouring should satisfy a certain set of restrictions; usually each restriction involves two vertices, and the restrictions therefore are called *binary constraints*.

In this note we consider a variant of the standard binary constraints problem in which frequencies (or *colours*) can take on arbitrary real values. Apart from its theoretical significance, this problem can also have some practical importance; especially for the question to what extent one can improve the performance by using more channels within the same frequency interval.

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Let (M, d) be an arbitrary metric space. By a (real-valued) *colouring* of M we mean any function $c : M \to \mathbf{R}$; the *span* of the colouring being

$$sp(c) = \sup_{m \in M} c(m) - \inf_{m \in M} c(m).$$

By a *restriction function* we mean any non-increasing function $f : \mathbf{R}^+ \to \mathbf{R}^{\geq 0}$. A colouring c of M *satisfies* the restriction f if for every two distinct elements x, y of M we have

$$|c(x) - c(y)| \geq f(d(x, y)).$$

The real-valued frequency assignment problem in its most general form can now be stated as follows:

given a metric space M and a restriction function f, find the value of $\inf\ sp(c)$ for all colourings c of M satisfying the restriction f.

We shall denote this number by $sp(M, f)$.

First we shall study this problem for $M = \mathbf{Z}$, the set of integers, with the natural metric $d(x, y) = |x - y|$. The restriction function in this case is determined by a non-increasing sequence $f = (f_n)_{n>0}$ of non-negative real numbers. The first question which arises in this situation is whether $sp(\mathbf{Z}, f)$ is finite or infinite. We shall give the complete answer to this question, and also we'll find upper bounds on the span when it is finite.

THEOREM 1 *For the restrictions sequence $f_n = 1/n$, the span $sp(\mathbf{Z}, f)$ is finite and does not exceed*

$$1 + \varphi = 2.618\ldots,$$

where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio. There exists a colouring c of span $1 + \varphi$ satisfying f.

THEOREM 2 *For an arbitrary restrictions sequence $f = (f_n)$, the span $sp(\mathbf{Z}, f)$ is finite if and only if $C = \sup\{nf_n \mid n > 0\}$ is finite; and the span does not exceed $C(1 + \varphi)$.*

THEOREM 3 *If the sum $s = \sum_{n>0} f_n$ is finite then the span $sp(\mathbf{Z}, f)$ is finite, and the span does not exceed $2s$.*

First we shall derive Theorem 2 from Theorem 1, and prove Theorem 3; both proofs are fairly easy. Then we'll give a proof of Theorem 1.

PROOF OF THEOREM 2. The "if" part immediately follows from Theorem 1. Indeed, if the number C is finite then for all n, $f_n \leq C/n$; and we can take the colouring c from Theorem 1 multiplied by C.

The "only if" part is just as easy. Given $L > 0$, choose n such that $nf_n > L$. As $f_i \geq f_n$ for $i < n$, all differences between colours of elements $0, 1, \ldots, n$ in a colouring satisfying f are at least f_n; and the span of these $n+1$ colours is at least $nf_n > L$. $\square$

PROOF OF THEOREM 3. We shall assign colours to elements of $\mathbf{Z}$ one by one, in the following order: $0, 1, -1, 2, -2, \ldots$, choosing on each step a value $c(k) \in [0, 2s]$ which satisfies the colouring conditions. This is always possible, since the measure of the set of values for $c(k)$ which are forbidden by any of the previously chosen values of $c(i)$, $i \leq |k|$, is at most $2\sum_{i=1}^{2|k|} f_i$ and is less than $2s$. $\square$

PROOF OF THEOREM 1.

Denote by a mod b with real a, b the number q such that $0 \leq q < b$, and $(a-q)/b \in \mathbf{Z}$. Let also $\mathbf{R}_r$ mean the factorgroup $\mathbf{R}/r\mathbf{Z}$ of the additive group of reals. The distance between elements of the group is naturally defined as $d_r(a, b) = \min((a-b) \bmod r, (b-a) \bmod r)$.

Let X be the half-open interval $[-(\varphi+1)/2, (\varphi+1)/2)$. Define a colouring $c : \mathbf{Z} \to X$ by the formula: $c(n) = ((\varphi+1)/2) + n\varphi) \bmod(\varphi+1) - (\varphi+1)/2$. If we consider elements of X as representatives of the cosets $\mathbf{R}/(\varphi+1)\mathbf{Z}$ then the colouring c becomes a group homomorphism $c : \mathbf{Z} \to \mathbf{R}_{\varphi+1}$.

Let us prove that this colouring satisfies the restriction f. Since c is a homomorphism, it is enough to show that for every $n > 0$ the distance $d_{\varphi+1}(c(0), c(n)) \geq 1/n$.

Our colouring has the following easy properties:

(i) $c(0) = 0$;

(ii) $c(n+1) - c(n)$ is equal to either φ or -1;

(iii) $d_{\varphi+1}(0, c(n)) = |c(n)|$.

The property (ii) enables us to give an alternative description of the colouring by an easy inductive process: $c(0) = 0$, and $c(n+1)$ is that one of the two numbers $c(n) + \varphi$, $c(n) - 1$ which belongs to X.

Properties (i) and (ii) imply that the number $c(n)$ is of the form $-a+b\varphi$ for some natural numbers a, b with $a+b=n$. Let $y=b+a\varphi$; we have $y \leq n\varphi$. Also,

$$c(n)y = ab(\varphi^2-1)+(b^2-a^2)\varphi = (ab+b^2-a^2)\varphi.$$

The number $ab+b^2-a^2$ is a non-zero integer; therefore $|c(n)| \geq \varphi/y \geq \varphi/(n\varphi) = 1/n$, and by (iii) we are done. $\square$

Note that the colouring described above is a cyclic channel colouring (in the sense of [2]) with the same constraints function; which is a stronger property.

Our next theorem gives a sufficient and a necessary condition for finiteness of the span for general discrete metric spaces. Though these conditions don't give a complete answer to the problem, they are rather close.

Let M be a metric space, and sequences (l_n), (u_n) be such that for every $n>0$, and for every $x \in M$, the following inequalities hold:

$$l_n \leq |\{y \in M \mid n-1 < d(y,x) \leq n\}| \leq u_n.$$

Let f be a restriction function; we extend its domain by setting $f(0) = \sup_{x>0} f(x)$, and assume that $f(0)$ is finite. This assumption is very natural. Indeed, if there is a positive lower bound ε on the distance between distinct elements of M then we can assume that $f(x) = f(\varepsilon)$ for all $0 < x < \varepsilon$; on the other hand, if one can have arbitrarily small distances then this assumption is necessary for the span to be finite.

THEOREM 4 *In the above notation, if* $\sup f(n) \sum_{i=1}^{n} l_i = \infty$ *then* $sp(M,f)$ *is infinite. If* $\sum_{i=1}^{\infty} f(i-1)u_i < \infty$ *then* $sp(M,f)$ *is finite.*

PROOF. The first assertion is proved exactly as the "only if" part of Theorem 2; the second assertion — as Theorem 3. $\square$

For the k-dimensional rectangular grid $\mathbf{Z}^k$ we can take $l_n = c_1 n^{k-1}$ and $u_n = c_2 n^{k-1}$ for suitable constants $0 < c_1 < c_2$. As an immediate corollary, we get that $sp(\mathbf{Z}^k, (n^{-\alpha}))$ is finite when $\alpha > k$, and infinite when $0 < \alpha < k$; and this holds for every metrics on $\mathbf{Z}^k$ equivalent to the Euclidean one — including the graph distance metrics.

The question whether $sp(\mathbf{Z}^k, (n^{-k}))$ is finite or infinite remains open for $k \geq 2$.

The result of Theorem 1 has an interesting application to monotone subsequences in permutations.

The classical theorem of Erdős and Szekeres [1] states that every permutation of more than n^2 numbers has a monotonic (increasing or decreasing) subsequence of length more than n.

Let $s \in S_n$ be a permutation of the set $\Omega = \{1, \ldots, n\}$, and $w = (i_0, \ldots, i_k)$ be an increasing sequence of elements of Ω such that the sequence $(s(i_0), \ldots, s(i_k))$ is monotonic (to avoid trivialities, we assume that $k \geq 2$). By the *density* of w we mean the number $d(w) = k^2/(i_k - i_0)$.

The above-mentioned theorem implies that for $n > 4$ there exists a lower bound $b(n) > 0$ such that every permutation in S_n has a monotonic subsequence of density at least $b(n)$; the bound $b(n)$ so obtained tends to 1 as n tends to infinity. A question arises: can one find such a lower bound which would tend to infinity with n? Theorem 1 gives a negative answer to this question.

Take the colouring c as in this theorem. For every n we define a permutation $f_n \in S_n$ as follows. The numbers $c(1), \ldots, c(n)$ are all distinct and linearly ordered; let $f_n(k)$ be the position of $c(k)$ in this ordering. We claim that every monotonic subsequence of any of the permutations f_n has density less than $1 + \varphi$. Indeed, let $i_0 < i_1 < \ldots < i_k$ be such that the sequence $(c(i_0), \ldots, c(i_k))$ is monotonic. We have:

$$\begin{aligned} i_k - i_0 &= (i_k - i_{k-1}) + \ldots + (i_1 - i_0) \geq \\ &\geq |c(i_k) - c(i_{k-1})|^{-1} + \ldots + |c(i_1) - c(i_0)|^{-1} \geq \\ &\geq k \cdot (|c(i_k) - c(i_0)|/k)^{-1} > \\ &> k^2/(1+\varphi); \end{aligned}$$

and the claim follows.

This leaves two related open questions: find the best possible value b_1 of the bound b from the above problem; and find the exact value of $b_0 = sp(\mathbf{Z}, (1/n))$. The above considerations show that

$$1 \leq b_1 \leq b_0 \leq 1 + \varphi = 2.618\ldots.$$

It is not difficult to show that $b_0 \geq 2$ by considering all possible orderings of the five values $c(0), \ldots, c(4)$.

References

[1] P. Erdős, G. Szekeres. A combinatorial problem in geometry. *Compositio Math.* 2(1935), 463–470.

[2] J. van den Heuvel, R.A. Leese, M.A. Shepherd. Graph Labelling and Radio Channel Assignment. *CDAM Research Report Series*, CDAM-96-23, 1996.