Abstract

A graph is called superhamiltonian if any permutation of its vertex set can be approximated by a cyclic one, allowing a uniform error of at most one edge. We apply the theory of superhamiltonian graphs to approximate a measure preserving homeomorphism of a compact manifold X by one with a periodic orbit which enters all the elements of any given finite open cover of X. By repeated approximation we obtain transitive or chaotic homeomorphisms.

Superhamiltonian Graphs and Chaos

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The aim of this paper is to give a combinatorial construction, based on superhamiltonian graphs, of a chaotic homeomorphism. In an earlier paper [?], the author defined a graph $G = G(V, E)$ to be superhamiltonian if *any* permutation σ of its vertex set V can be approximated by a cyclic permutation τ , with a uniform error of at most one edge. That is, $d_G(\tau(v), \sigma(v)) \leq 1$, for all $v \in V$. Of course if a graph has this property for the *particular* permutation σ =identity, it is called Hamiltonian. An elementary result from that paper said that the 6'th power of any finite connected graph is superhamiltonian, or in terms more useful for this paper,

Theorem 1 *Let $G = G(V, E)$ be a finite connected graph, and let $\sigma : V \to V$ be any permutation. Then there is a cylic permutation $\tau : V \to V$ with $d_G(\tau(v), \sigma(v)) \leq 6$, for all vertices $v \in V$.*

For the applications in this paper, we think of V as the points in a finite family of periodic orbits of some homeomorphism h of a metric manifold (X, d) . We will use Theorem 1 to find a uniformly close homeomorphism $\hat{h}$ for which the set V represents a single periodic orbit. If the original set V is a fine $\varepsilon-$net for X, then the resulting homeomorphism $\hat{h}$ has properties which approximate the definition of transitivity and even chaos. Using a series of successive approximations, we obtain transitive and chaotic homeomorphisms. A homeomorphism $h : X \to X$ is called *transitive* if it has orbits entering both of any pair of nonempty open sets, and *chaotic* (in the sense of Devaney [5]) if it has periodic orbits with this property. The later property (see [4] and [9]) implies (for infinite sets) the notion of *sensitive dependence on initial conditions*: there is a $\delta > 0$ such that for any $x \in X$ and any neighborhood U of x, there is some y in U and some $k \geq 1$ such that $d\left(h^k(x), h^k(y)\right) > \delta$. A related but weaker topological property is called *recurrence*, which requires that given any nonempty open set, there is an orbit which starts in and returns to it.

If we look for combinatorial analogs of these topological concepts, we see that a map $\phi : V \to V$ of a finite set V is recurrent if it is a bijection, i.e., a permutation. Furthermore, it is transitive (and chaotic in the above sense) if it is a cyclic permutation. At the combinatorial level, we cannot distinguish between chaos and transitivity. Using this language, Theorem 1 asserts that recurrent maps of finite graphs can be uniformly approximated by transitive maps, with an error that is independent of the size (number of vertices) of the graph.

Our aim in this paper is to construct chaotic homeomorphisms on a metric manifold (X, d) of dimension $n \geq 2$. This was recently accomplished by J. Aarts and F. Daalderop [1], who began their construction with transitive homeomorphisms, using a result of J. Oxtoby (stated below) on their existence. Our construction does not use Oxtoby's result and in fact gives a new proof of it, probably the simplest. We base our construction instead on the combinatorial idea of Theorem 1. To ensure that the homeomorphisms we begin our construction with are at least recurrent, we assume that they preserve a finite measure μ (which we assume is a nonatomic Borel measure that is positive on open sets). The space of all $\mu-$preserving homeomorphisms of the manifold (X, d) is denoted by $\mathcal{M}$, which we endow with the uniform topology. If h belongs to $\mathcal{M}$, and U is any nonempty open subset of X, then the iterates $h^k(U)$, $k = 1, 2, \ldots$, cannot be disjoint because they all have the same positive measure $\mu(U)$, and hence eventually the sum of their measures would exceed the finite number $\mu(X)$. Consequently every homeomorphism h in $\mathcal{M}$ is recurrent. This argument goes all the way back to Poincaré. This fact is the reason we begin with measure preserving homeomorphisms (which are recurrent), when we want to produce transitivity or chaos. The next thing we have to do to reduce the analysis problem to a combinatorial problem accesible by Theorem 1, is to make it finite. Since the manifold X is assumed to be compact, the natural way to get a finite problem is to consider a finite open cover by small diameter sets. Finally, we have to transfer results obtained on finite subsets to similar results for homeomorphisms of the manifold. We do this using an easy 'extension lemma' ([7, Lemma 13], or [3, Lemma 3.3] for an elementary proof) which asserts that any injection $\phi : C \to X$ from a finite subset C of X with $\max_{c \in C} d(\phi(c), c) < \delta$ can be extended to a homeomorphism $f \in \mathcal{M}$ with $||f|| \equiv \max_{x \in X} d(f(x), x) < \delta$. We combine all these ideas to obtain the following.

Theorem 2 *Let $\mathcal{U}$ be a finite open cover of an n-manifold X, $n \geq 2$, by nonempty sets of $d-$diameter less than ε. Then given any $h \in \mathcal{M}$ with a finite invariant set F, there is an $f \in \mathcal{M}$ which fixes F, has $||f|| \equiv \max_{x \in X} d(f(x), x) < 13\varepsilon$, and such that fh has a periodic orbit which enters all the elements (open sets) of $\mathcal{U}$.*

Proof. Since h is recurrent, it follows that for every open set $U \in \mathcal{U}$ there is a point $y_U \in U - F$ and a least positive integer k_U with $h^{k_U}(y_U) \in U$. It follows from the extension lemma that there is an $f_1 \in \mathcal{M}$ with $||f_1|| < \varepsilon$, which fixes F and $B = B_{\mathcal{U}} = \bigcup_{U \in \mathcal{U}} \bigcup_{k=0}^{k_U - 1} h^k(y_U)$, and such that $f_1 h^{k_U - 1}(y_U) = y_U$, $U \in \mathcal{U}$. It follows that B is invariant under the homeomorphism $g = f_1 h$. Consider the finite graph G with nodes B and with an edge between two points in B if and only if the distance between these points is less than 2ε. Since X is a connected metric space and each U contains $y_U \in B$, it follows that the graph G is connected. According to Theorem 1, the permutation $g : B \to B$ can be approximated by a cyclic permutation $\tau : B \to B$ such that $d_G(g(b), \tau(b)) \leq 6$, or equivalently, $d(g(b), \tau(b)) < 6(2\varepsilon) = 12\varepsilon$, for all $b \in B$. It follows from extension lemma that there is a homeomorphism $f_2 \in \mathcal{M}$, which fixes F, has $f_2(g(b)) = \tau(b)$, $b \in B$, and $||f_2|| < 12\varepsilon$. The composition $f = f_2 f_1$ satisfies $||f_2|| < 13\varepsilon$, fixes the set F, and fh cyclically permutes the points of B, which is the required periodic orbit.

Corollary 1 *Given $h \in \mathcal{M}$, and $\delta > 0$, there is an $f \in \mathcal{M}$ such that fh is chaotic.*

Proof. The homeomorphism f will be obtained as a limit, so we need to observe that the space M is complete with respect to the metric $\rho(f,g) = \max\left(\max_{x\in X}|f(x)-g(x)|, \max_{x\in X}|f^{-1}(x)-g^{-1}(x)|\right)$, which is topologically equivalent to the simpler 'uniform topology' metric $\max_{x\in X}|f(x)-g(x)|$. We begin by choosing an open cover $\mathcal{U}_1$ of X with diameters ε_1 sufficiently small so that $13\varepsilon_1 < \delta/2$, and we apply Theorem 1 (with $F = F_1 = \emptyset$) to obtain an $f_1 \in \mathcal{M}$ with $||f_1|| < \delta/2$ and such that $f_1 h$ has a periodic orbit $B = B_1$ entering all the elements of $\mathcal{U}_1$. Next we choose ε_2 sufficiently small so that $13\varepsilon_2 < \delta/2^2$ and $||f_2|| < 13\varepsilon_2$ implies that $\rho(f_2 f_1, f_1) < \delta/2^2$. Let $\mathcal{U}_2$ be an open cover of X with diameters less than ε_2 and apply Theorem 1 to $f_1 h$ with $F = F_2 = F_1 \cup B_1$ to obtain an $f_2 \in \mathcal{M}$ with $||f_2|| < 13\varepsilon_2$ and such that $f_2 f_1 h$ has a periodic orbit B_2 which enters each element of $\mathcal{U}_2$. We continue recursively, finding for each k a homeomorphism f_k with $\rho(f_k f_{k-1} \dots f_1, f_{k-1} \dots f_1) < \delta/2^k$, which fixes the finite set $F_k = F_{k-1} \cup B_{k-1}$ and such that $f_k f_{k-1} \dots f_1 h$ has a periodic orbit B_k which enters every element of the cover $\mathcal{U}_k$ (which has diameter less than $\delta/2^k$). The completeness of $\mathcal{M}$ ensures that $f = \lim_k f_k f_{k-1} \dots f_1$ exists. To see that fh is chaotic, let W and U be any nonempty open subsets of X. For some k, both W and U contain elements of $\mathcal{U}_k$, and consequently $f_k f_{k-1} \dots f_1 h$ has a periodic orbit B_k which enters W and U. But for $j > k$, f_j fixes B_k, so that B_k is also a periodic orbit for fh. ■

Corollary 2 *The transitive homeomorphisms constitute a dense G_δ subset of $\mathcal{M}$.*

Proof. The previous result gives a direct proof, without the Baire Category Theorem, that the transitive homeomorphisms are dense in $\mathcal{M}$. As Oxtoby [6] showed, they may be represented as a G_δ subset of $\mathcal{M}$ by enumerating all pairs (W_i, U_i) of basic open sets and considering the sets $S_{i,j} = \{f \in \mathcal{M} : f^j(W_i) \cap U_i \neq \emptyset\}$. Each of these is open, and their intersection consists of the transitive homeomorphisms. ■

Corollary 2 was proved by J. Oxtoby [6] in 1937, using the Baire Category Theorem. Corollary 1 was recently proved by J. Aarts and F. Daalderop [1], using Oxtoby's result. The method used here (with Corollary 2 as a consequence of Corollary 1) gives a direct proof of both, without the Baire Category Theorem. Corollary 2 could also be proved as a consequence of Theorem 1, using the Baire Category Theorem, since Theorem 1 shows that each set $S_{i,j}$ is dense. Other constructions of transitive homeomorphisms can be found in [8] and [10].

References

[1] Aarts, J., and Daalderop, F., Chaotic homeomorphisms on manifolds. Preprint, 1997.

[2] Alpern, S., Superhamiltonian graphs. *J. Comb. Theory, B* **25** (1978), No. 1, 62-73.

[3] Alpern, S., and Edwards, R. D., Lusin's theorem for measure preserving homeomorphisms. *Mathematika* **26** (1979), 33-43.

[4] Banks, J., Brooks, J., Cairns, G., Davis., G., and Stacey, P., On Devaney's definition of chaos. *Amer. Math. Monthly* **99** (1992), no. 4, 332-334.

[5] Devaney, R., *An introduction to chaotic dynamical systems*, second edition. Addison-Wesley, 1989.

[6] Oxtoby, J. C., Note on transitive transformation. *Proc. Nat. Acad. Sci. USA* **23** (1937), 443-446.

[7] Oxtoby, J. C., and Ulam, S. M., Measure preserving homeomorphisms and metrical transitivity. *Ann of Math.* **42** (1941), 874-920.

[8] Prasad, V. S., Ergodic measure preserving homeomorphisms of R^n. *Indiana U. Math. J.* **28** (1979), 859-867.

[9] Touhay, P., Yet another definition of chaos. *American Mathematical Monthly* **104** (1997), 411–414.

[10] Xu, Xiao Quan, Explicit transitive automorphisms of the closed unit square. *Proc. Amer. Math. Soc.* **109** (1990), 571-572.