Abstract

We establish the existence of self homeomorphisms of R^n, $n \geq 2$, which are chaotic in the sense of Devaney, preserve volume and are spatially periodic. Moreover, we show that in the space of volume preserving homeomorphisms of the n-torus with mean rotation zero, those with chaotic lifts to R^n are dense, with respect to the uniform topology. An application is given for fixed points of 2-dimensional torus homeomorphisms (Conley-Zehnder-Franks Theorem).
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Chaotic Homeomorphisms of R^n, Lifted from Torus Homeomorphisms

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1 Introduction

In two previous papers, the authors showed that volume-preserving homeomorphisms of the annulus or the n-torus, with mean rotation zero, typically have lifts (to $[0,1] \times R$ or to R^n) which are respectively recurrent [5] or transitive [6]. The later result, which also covered the case of the annulus, clearly strengthened the earlier one. In this paper we further strengthen these results by showing that for such homeomorphisms of the $n-$torus T^n, $n \geq 2$, those with chaotic lifts to R^n are dense, with respect to the uniform topology. In particular, we establish the existence of homeomorphisms of R^n, $n \geq 2$, which are chaotic (in the sense of Devaney, defined below). This result continues a line of research which stems from Besicovitch's example [8] of a transitive homeomorphism of the plane, generalized by Prasad [16] to ergodic volume preserving ones.

A homeomorphism $h : T^n \to T^n$ will be called *rotationless* if it preserves volume and has a lift $\overline{h} : R^n \to R^n$ with mean translation zero. This last property means that $\int_\Omega \left(\overline{h}(x) - x\right) \, dx = 0$, where Ω denotes the closed unit $n-$cube $[0,1]^n$. (Such a lift is unique, and we always choose this one.) Let $\mathcal{H}_0 = \mathcal{H}_0[T^n]$ denote the space of all rotationless homeomorphisms of the n-torus T^n, and endow it with the uniform topology given by the complete metric $\rho(f,g) = \max(\max\limits_{x \in T^n} |f(x) - g(x)|, \max\limits_{x \in T^n} |f^{-1}(x) - g^{-1}(x)|)$, which is topologically equivalent to the simpler 'uniform topology' metric $\max\limits_{x \in I^n} |f(x) - g(x)|$. (See [15].) A self homeomorphism of a metric space (X,d) is said to be *chaotic, in the sense of Devaney* [10], if it (i) is transitive (iterates of any open set intersect any open set), (ii) its periodic points are dense, and (iii) *it has sensitive dependence on initial conditions.* (This last condition (iii) requires that there is a $\delta > 0$ such that for any $x \in X$ and any neighborhood U of x, there is some y in U and some $k \geq 1$ such that $d\left(h^k(x), h^k(y)\right) > \delta$.) We can now state our main result.

Theorem 1 *In the space $\mathcal{H}_0[T^n]$, $n \geq 2$, of all rotationless homeomorphisms of the n-torus T^n, those homeomorphisms with Devaney-chaotic lifts to R^n are dense, with respect to the uniform topology.*

In particular, Theorem 1 establishes the existence of Devaney-chaotic homeomorphisms of R^n. The existence of such homeomorphisms of *compact* manifolds has recently been established by Aarts and Daalderop [1], who used Oxtoby's 1937 result [14] that transitivity is typical for measure preserving homeomorphism of compact manifolds. A combinatorial construction of Devaney-chaotic homeomorphisms of compact manifolds has been given by one of the authors [2]. Our proof of Theorem 1 is similar to that in [1], except that we use instead our result [6] that transitive lifts are typical in $\mathcal{H}_0$. We should note that a chaotic homeomorphism of T^n does not necessarily have a chaotic lift to R^n, so that Theorem 1 does not follow from these cited results on compact manifolds.

2 Proof of Theorem 1

We now give a proof of Theorem 1 that uses our previous result [5] that those homeomorphisms in $\mathcal{H}_0$ with transitive lifts to R^n are dense. We will need the following result, adapted from Oxtoby and Ulam, where $|x-y|$ denotes Euclidean distance on T^n, and the norm of a homeomorphism is defined by $||h|| = \max_{x \in T^n} |h(x) - x|$.

Lemma 1 *Let $p_1, p_2, \ldots, p_k$ and $q_1, q_2, \ldots, q_k$ be two sets of k distinct points in T^n, $n \geq 2$, with $|p_i - q_i| < \varepsilon$ for $i = 1, \ldots, k$. Then there is a rotationless homeomorphism $f \in \mathcal{H}_0$ with $||f|| < \varepsilon$ and $f(p_i) = q_i$ for $i = 1, \ldots, k$.*

This result was proved by Oxtoby and Ulam [15, Lemma 13] for I^n and for homeomorphisms h which are merely volume-preserving. However the proof given in [4, Lemma 3.3] is constructive and shows that the homeomorphism h is the product of homeomorphisms supported on small disks, and consequently is rotationless. We shall use this lemma and our previous result for transitive lifts to prove the following lemma, which is similar to [1, Lemma 3] and [2, Lemma 2]. Let $\pi : R^n \to T^n$ denote the covering projection.

Lemma 2 *Let F be a finite subset of T^n and let U and V be nonempty open subsets of R^n. Let $\delta > 0$ and let $h \in \mathcal{H}_0$ leave F invariant. Then there is a homeomorphism $f \in \mathcal{H}_0$ which equals the identity on F, satisfies $||f|| < \delta$, and such that fh has a lift $\overline{fh} : R^n \to R^n$ with a periodic orbit entering U and V.*

Proof. According to [6] there is a homeomorphism $h_1 \in \mathcal{H}_0$ with $\rho(h_1, h) < \delta/2$ which has a transitive lift $g = \bar{h}_1 : R^n \to R^n$. Consequently there is an $x \in U$ whose orbit under g enters V and then returns to within a distance $\delta/2$ of x, while avoiding the discrete set $\pi^{-1}(F \cup gF)$. We may also assume that π is injective on this orbit. So for some positive integers $l < m$ we have $g^l(x) \in V$ and $|g^m(x) - x| < \delta/2$. For $i = 1, \ldots, m-1$, set $p_i = q_i = \pi(g^i(x))$. Set $p_m = \pi(g^m(x))$ and $q_m = \pi(x)$. Next, label the finite set F as $q_{m+1}, \ldots, q_k$ for some k, and set $p_j = h_1 h^{-1}(q_j)$, for these $j = 1, \ldots, k$. Now apply Lemma 1 to obtain an $f' \in \mathcal{H}_0$ with $||f'|| < \delta/2$, such that $f'(p_j) = q_j$ for $j = 1, \ldots, k$. It follows that $f'h_1$ has a lift under which the point x is periodic and its orbit enters V. Consequently the homeomorphism $f \in \mathcal{H}_0$ defined by $f'h_1 = fh$, or $f = f'h_1h^{-1}$ has the required properties. ■

Proof. (of Theorem 1): Let $\{U_i, V_i\}_{i=1}^{\infty}$ be an enumeration of all open dyadic cubes of R^n of all ranks. Setting $F = F_0 = \emptyset$ and choosing a $\delta > 0$ so that $||g|| < \delta$ implies that $\rho(h, gh) < \varepsilon/2$, use Lemma 2 to find $f_1 \in \mathcal{H}_0$, $||f|| < \delta$, so that there is a periodic $\overline{f_1 h}$ orbit O_1 which enters both U_1 and V_1. Set $F_1 = F_0 \cup \pi O_1$. We continue recursively finding, for each i, an automorphism $f_i \in \mathcal{H}_0$ which is the identity on $F_{i-1} = F_{i-2} \cup \pi O_{i-1}$, such that the lift $\overline{f_i \ldots f_1 h}$ has a periodic orbit entering both U_i and V_i, and satisfies $\rho(f_i f_{i-1} \ldots f_1 h, f_{i-1} \ldots f_1 h) < \varepsilon/2^i$. It follows from the completeness of ρ that the limit $f = \lim_i f_i f_{i-1} \ldots f_1$ exists, and $||f|| < \varepsilon$. We now show that fh is chaotic in the sense of Devaney. Given any two nonempty open sets U and V, there is some i with $U_i \subset U$ and $V_i \subset V$. Consequently the lift $\overline{f_i f_{i-1} \ldots f_1 h}$ has a periodic orbit O_i entering both U_i and V_i and hence also U and V. Since for $j > i$ the maps f_j fixes πO_i, it follows that $\overline{fh}$ agrees with $\overline{f_i f_{i-1} \ldots f_1 h}$ on the orbit periodic O_i. From this observation, we see that $\overline{fh}$ has the first two properties (i) and (ii) required for Devaney chaos. However, it has been shown by Banks *et. al.* [7] that (i) and (ii) imply (iii), completing the proof. ■

As with our earlier papers ([5],[6]) on this topic, there are particular applications to fixed point theory when $n = 2$. A special case of the Brouwer Plane Translation Theorem [9] says that if an orientation preserving homeomorphism of the plane R^2 has a periodic point, then it also has a fixed point. It follows that if $h \in \mathcal{H}_0[T^2]$ is orientation preserving and has a chaotic lift $\overline{h} : R^2 \to R^2$ (also orientation preserving), then $\overline{h}$ has a fixed point, and consequently so does h. So in the closed subspace $\mathcal{H}_0^{+}[T^2]$ of $\mathcal{H}_0[T^2]$ consisting of orientation preserving homeomorphisms, a dense subset possesses fixed points, and consequently (since the domain T^n is compact) *every* homeomorphism in $\mathcal{H}_0^{+}[T^2]$ has a fixed point. This result is of course the Conley-Zehnder-Franks Theorem for a single fixed point. See [12], [11], and [13] for more results, and more fixed points - the reference [13] gives three.

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