Periodic, Locally Linear, and Chaotic Approximations to Volume Preserving Homeomorphisms

Steve Alpern
Department of Mathematics
The London School of Economics
London WC2A 2AE, U.K.
email: s.alpern@lse.ac.uk

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Abstract

We modify a combinatorial idea of Peter Lax, to uniformly approximate any volume preserving homeomorphism h of the closed unit cube I^n, $n \geq 2$, by another one $\hat{h}$ such that either: (i) $\hat{h}$ rigidly permutes an infinite family of dyadic cubes of total volume one, or (ii) $\hat{h}$ is chaotic in the sense of Devaney. The first result (i) generalizes a similar result of Oxtoby and Ulam, yielding an approximation which is periodic and locally linear almost everywhere. The second result (ii) is the first combinatorial proof the existence of chaotic manifold homeomorphisms, a result obtained independently, using different methods, by J. Aarts and F. Daalderop. Our constructions may also be applied on compact connected manifolds of dimensional $n \geq 2$.

1 Introduction

This paper applies a combinatorial idea of Peter Lax to simplify and extend results obtained by J. Oxtoby and S. Ulam on the uniform approximation of measure preserving manifold homeomorphisms. Most of these results apply to arbitrary compact connected manifolds, but for simplicity we concentrate on the space $\mathcal{M} = \mathcal{M}[I^n]$ consisting of all volume preserving homeomorphisms (henceforth called *automorphisms*) of the closed unit cube I^n, $n \geq 2$, and the subspace $\mathcal{M}_0$ which fixes the boundary. We endow $\mathcal{M}$ with the uniform topology. In 1937 Oxtoby [19] proved that the transitive automorphisms form a dense G_δ subset of $\mathcal{M}$, and in 1941 he and S. Ulam [21] established the same result for ergodic automorphisms. (An automorphism h is transitive if given nonempty open sets U, V, we have $h^i(U) \cap V \neq \emptyset$ for some $i > 1$; it is ergodic if all h-invariant sets have volume 0 or 1.) Similar ideas to those given here were used earlier by the author [3] to simplify the above result on ergodicity, and subsequently [4] to extend it to other measure theoretic properties. However in this paper we extend the results of Oxtoby and Ulam in a different direction, involving concepts of chaos not current at the time of their work, or even at the time of the author's earlier extensions of it.

In order to state the main result of this section, we need some definitions concerning dyadic decompositions of I^n. A dyadic cube of rank m is an $n-$fold product of dyadic intervals of the form $[k/2^m, (k+1)/2^m]$, $0 \leq k \leq 2^m - 1$. For any α, $0 \leq \alpha \leq 1$, an α-cube of rank m is a closed cube concentric to and with a side length α times that of some dyadic cube of rank m. (A 0-cube is a dyadic center, a center of a dyadic cube.) We call α a dyadic rational if it has the form $k/2^m$. (See [21].) We can now state the main result of the paper, from which the approximations with the properties stated in the title will follow.

Theorem 1 *Given any automorphism $h \in \mathcal{M}[I^n]$, $n \geq 2$, any dyadic rational α, $0 \leq \alpha < 1$ and any $\epsilon > 0$, there is a automorphism $f \in \mathcal{M}_0$ with $||f|| = \max_{x \in I^n} |f(x) - x| < \varepsilon$, and an infinite strictly increasing sequence of positive integers $m_1, m_2, \ldots$ with the following property: For each $k = 1, 2, \ldots$, the $\alpha-$cubes of rank m_k which are not contained in any α-cube of rank $m_1, m_2, \ldots, m_{k-1}$, are rigidly permuted in a single cycle by $\bar{h} = fh$.*

This theorem is essentially an unpublished result from the author's Ph.D. dissertation [2, Theorem 5]. The only difference is that it was originally stated only for $\alpha = 1/2$ (although the proof is the same for all α) and that it had the dimensional restrictions $n \neq 4, 5$ because the proof uses the Annulus Theorem (see below) in dimensions n and $n-1$ and at that time the Annulus Theorem was not proved for dimension 4. The proof given here is slightly improved on that given in [2, Theorem]. This theorem extends a similar result of Oxtoby and Ulam [21, Theorem 12] which had the further restrictions that $h \in \mathcal{M}_0$. They remark that

'... the same methods can be extended to apply to $r-$dimensional polyhedra embedded in $r-$space, and to automorphisms that are merely isotopic to the identity. An attempt to extend them to general automorphisms and polyhedra seems to encounter more serious difficulties...'. Our proof does not require that h is isotopic to the identity and extends immediately to cover any boundary identification on I^n, and so applies to any compact connected manifold by Brown's result [12], if the notion of rigidity is suitably extended. Also, our proof is much simpler.

There are three reasons why the author has decided to publish this result now. First, the remaining case $n = 4$ of the Annulus Theorem has been proved by Quinn [17], so there are no longer any dimensional restrictions. Second, we wish this result to appear in the literature prior to its inclusion in the author's planned book with V. Prasad [9], which should come out in a few years. Finally, the author has just recently noticed its connection with chaos, to be discussed below, which should increase the interest in the techniques presented here.

We now discuss the consequences of Theorem 1, which are very different in the two cases $\alpha > 0$ and $\alpha = 0$. For $\alpha > 0$, the permuted cubes have total volume 1. This means that for almost every point x in I^n, x is periodic under $\bar{h}$, and belongs to a closed cube on which $\bar{h}$ acts rigidly and hence linearly. So we immediately obtain the following.

Corollary 1 *Every volume preserving homeomorphism h of the closed unit cube $I^n, n \geq 2$, can be uniformly approximated by another which is measure theoretically periodic and is locally linear almost everywhere.*

For the case $\alpha = 0$ (which we will establish first as a lemma for Theorem 1), observe that the α-cubes are simply the centers of their dyadic cubes, and so *all* these centers of cubes of rank m_k are permuted in a single cycle of $\bar{h}$. It follows that since any given pair U, V of nonempty open subsets of I^n contain dyadic cubes of some rank m_k, there is an orbit of $\bar{h}$ which enters both U and V. This condition clearly implies that $\bar{h}$ is (i) transitive, and (ii) has a dense set of periodic points. It has been shown by Banks *et. al.* [10] that these two conditions imply that $\bar{h}$ has(iii) sensitive dependence on initial conditions. (This last condition (iii) applies to $h : X \to X$ if there is some $\delta > 0$ such that for any $x \in X$ and any neighborhood U of x, there is some y in U and some $k \geq 1$ such that $d\left(h^k(x), h^k(y)\right) > \delta$, where d is a given metric on X.) Thus the approximation $\bar{h}$ given by Theorem 1 for $\alpha = 0$ satisfies the three conditions (i), (ii), and (ii), which constitute chaos, in the sense defined by Devaney [14]. (In fact the condition that $\bar{h}$ has periodic orbits entering both of any given pair of open sets is equivalent to chaos in the sense of Devaney [23].) Thus for the case $\alpha = 0$, Theorem 1 has the following consequence.

Corollary 2 *Every volume preserving homeomorphism of the closed unit cube*

$I^n, n \geq 2$, can be uniformly approximated by another which is chaotic in the sense of Devaney.

We should observe that our proof of this corollary (actually our proof of Lemma 2 on which it is based) gives a direct demonstration, without using the G_δ property, of Oxtoby's result cited earlier on the denseness of transitivity. As mentioned above, Brown's result, suitably modified to preserve measures, can be used to extend both corollaries to the context of homeomorphisms of a compact connected $n-$manifold, $n \geq 2$, which preserve a nonatomic Borel probability measure which is positive on open sets.

After sending a preprint of an earlier version of this paper to Beverly Brechner, she informed us that the above corollary (for compact manifolds) had been obtained independently, using different methods, by Jan Aarts and Fons Daalderop [1]. Their method is similar to one we had discussed in our earlier version [5] of this paper (but for obvious reasons have now deleted here). She also mentioned that Phil Boyland has obtained the above result, for $n = 2$, by a method in dynamics which uses the Weierstrass P-function. A different notion of chaos (which implies sensitive dependence on initial conditions but not the other properties required by Devaney's definition) was shown by Kato [15] to be supported by all compact connected n-manifolds, $n \geq 2$.

As for chaos on noncompact manifolds, the author and V. Prasad [8] have shown that the homeomorphisms of the $n-$torus T^n, $n \geq 2$, which have chaotic lifts to R^n (in the sense of Devaney), form a uniform topology dense subset of the volume preserving homeomorphisms with mean rotation zero, extending our similar result [7] for transitivity. We note that in the space of all volume preserving homeomorphisms of R^n, with the compact-open topology, the fact that transitive automorphisms (and even ergodic ones) are dense G_δ was proved by Prasad [22], following the much earlier *example* of such a transitive planar homeomorphism by Besicovitch [11]. It was not until much later that such a specific example was given by Xu [24] of a transitive homeomorphism of the square.

If we consider both cases of Theorem 1, for α zero or positive, we see that an arbitrary automorphism h has both chaotic ($\alpha = 0$) and nonchaotic ($\alpha > 0$) uniform approximations. This means that if h is certain to be an automorphism, but otherwise known only approximately (*e.g.*, experimentally), the question as to whether or not it is chaotic can never be answered.

The rest of the paper is organized as follows. In section 2 we prove Theorem 1 for the case $\alpha = 0$, a result we call Lemma 2, because we use it in section three to prove the remaining case $\alpha > 0$.

2 Proof of Theorem 1 ($\alpha = 0$)

The homeomorphism f of Theorem 1 will be obtained as a limit, so we need to observe that the space $\mathcal{M}$ is complete with respect to the metric $\rho(f,g) = \max\left(\max_{x\in I^n}|f(x)-g(x)|, \max_{x\in I^n}|f^{-1}(x)-g^{-1}(x)|\right)$, which is topologically equivalent to the simpler 'uniform topology' metric $\max_{x\in I^n}|f(x)-g(x)|$. We shall need an 'extension theorem' in the space $\mathcal{M}$ which was proved by Oxtoby and Ulam [21, Lemma 13], so we state it now. Oxtoby and Ulam prove this result as a corollary of a deep result on equivalence of measures under homeomorphisms - however a simple constructive proof is given in [6, Lemma 3.3].

Lemma 1 *Let $p_1, p_2, \ldots, p_k$ and $q_1, q_2, \ldots, q_k$ be two sets of k distinct points in the interior of I^n, with $|p_i - q_i| < \varepsilon$ for $i = 1, \ldots, k$. Then there is a automorphism $h \in \mathcal{M}_0$ with $||h|| < \varepsilon$ and $h(p_i) = q_i$ for $i = 1, \ldots, k$.*

We now state and prove the special case $\alpha = 0$ of Theorem 1, which we call a 'lemma' because we use this result in the proof given in the next section for $\alpha > 0$.

Lemma 2 (Theorem 1 for $\alpha = 0$) *Given any automorphism $h \in \mathcal{M}$ and $\varepsilon > 0$, there is an automorphism $f \in \mathcal{M}_0$ with $||f|| < \varepsilon$, and an infinite strictly increasing sequence of positive integers $m_1, m_2, \ldots$, with the following property: For each $k = 1, 2, \ldots$, the centers of all the dyadic cubes of rank m_k are cyclically permuted by fh.*

Proof. We begin by showing that if h already satisfies the cyclic permutation condition for $k = 1, \ldots, K$, then for any $\delta > 0$ we can find a $g \in \mathcal{M}_0$ with $||g|| < \delta$ which is the identity on the dyadic centers of ranks $m_1, \ldots, m_K$ and such that gh satisfies the condition for $k = 1, \ldots, K+1$. (We intend this to include the case $K = 0$, which covers the original automorphism h.) We begin by choosing the next rank $r = m_{K+1}$ so that dyadic cubes of rank r have diameters $d = d_r$ less than some positive number to be determined later. Let $b_1, b_2, \ldots, b_R$ be an enumeration of all the $R = 2^{nr}$ dyadic cubes of rank r (which we call simply 'r-cubes') and number them so that consecutively numbered cubes have an $n-1$ dimensional face in common. Let c_i denote the center of the cube b_i. We say that the r-cube b_i 'knows' the r-cube b_j if $h(b_i) \cap b_j \neq \emptyset$. Observe that if B is the union of l distinct r-cubes, it has the volume of l r-cubes and therefore cannot be a subset of any set consisting of fewer than l distinct r-cubes. In other words, any set of l r-cubes 'knows' at least l r-cubes. According to the well known Marriage Theorem this condition implies that there is a permutation σ of the set $\{1, \ldots, R\}$ such that the r-cube b_i knows the r-cube $b_{\sigma(i)}$, or equivalently,

$$h(b_i) \cap b_{\sigma(i)} \neq \emptyset, \text{ for all } i = 1, \ldots, R. \tag{1}$$

We wish to approximate the permutation σ by another permutation τ (of the same set) which is cyclic and for which $b_{\sigma(i)}$ is close to $b_{\tau(i)}$ for all i. The simplest way is to apply the algorithm given by the author in [4] which approximates any permutation of $\{1,\ldots,R\}$ (for any integer R) by a cyclic one which differs uniformly by at most 2. In particular,

$$\text{there is a cyclic permutation } \tau \text{ with } |\sigma(i)-\tau(i)| \leq 2, i=1,\ldots,R. \tag{2}$$

The algorithm is easy: We modify σ by consecutively looking at each of the pairs $(1,2),(3,4),\ldots,(R-1,R),(2,3),(4,5),\ldots,(R-2,R-1)$, and in each case transposing the two numbers iff they lie in different cycle of the current permutation. Eventually there is only one cycle and in the process any number is moved at most twice, both times to an adjacent number. For example, if (in cycle notation) $\sigma=[(1),(2,7),(3),(4),(56),(8),(9)]$, then $\tau=[(1,3,5,6,4,2,9,8,7)]$.

It follows from (1) that for every center c_i of the r-cube b_i, we have $\left|h(c_i)-c_{\sigma(i)}\right| \leq \omega(d_r)+d_r$, where ω is the uniform modulus of continuity of h on I^n, since the diameter of $h(b_i)$ is bounded above by $\omega(d_r)$. It follows from (2) that $\left|c_{\sigma(i)}-c_{\tau(i)}\right| \leq 2d_r$. Consequently we have

$$\left|h(c_i)-c_{\tau(i)}\right| \leq \omega(d_r)+3d_r<\delta,$$

if we choose $m_{K+1}=r>m_K$ sufficiently large. For $i=1,\ldots,r$, set $p_i=h(c_i)$ and $q_i=c_{\tau(i)}$. For $i=R+j$, where $j=1,\ldots,J$, and $J=\sum_{k=1}^{K}2^{nm_k}$ is the number of the centers of ranks $m_1,\ldots,m_K$ permuted by h, define $p_i=q_i$ to be some enumeration of these centers. Then $|p_i-q_i|<\delta$ for $i=1,\ldots,R+J$, so Lemma 1 gives us an automorphism $g\in\mathcal{M}_0$ with $||g||<\delta$ with $g(p_i)=q_i$. Hence g is the identity on the centers already permuted by h and $gh(c_i)=c_{\tau(i)}$ for the centers c_i, $i=1,\ldots,R$, of the cubes of rank $r=m_{K+1}$. Thus g satisfies the requirements claimed in the first sentence of the proof.

Returning to the proof of the full theorem, we use the claim already established to construct an automorphism $g=g_0\in\mathcal{M}_0$ with $||g_0||<\varepsilon/2$ so that g_0h permutes the dyadic centers of some fixed rank m_1. We then choose $\delta=\delta_1<\varepsilon/4$ so that any automorphism $\hat{g}\in\mathcal{M}_0$ with $||\hat{g}||<\delta_1$ satisfies $\rho(\hat{g}g_0,g_0)<\varepsilon/4$, and use the established claim to find a $g_1\in\mathcal{M}_0$ with $||g_1||<\delta_1$ which is the identity on the dyadic centers of rank m_1 and such that g_1g_0h cyclically permutes the dyadic centers of some rank $m_2>m_1$. We continue in this fashion and define $f=\lim_k g_k\circ g_{k-1}\circ\ldots\circ g_1$. This limit exists in $\mathcal{M}_0$ because the construction gives $\rho(g_k\circ g_{k-1}\circ\ldots\circ g_1,\, g_{k-1}\circ\ldots\circ g_1)<\varepsilon/2^{k+1}$, and has the required properties by construction. ■

This proof uses Peter Lax's idea [18] of applying the Marriage Theorem to approximate an automorphism by a dyadic permutation, and the author's method [4] of approximating a permutation by a cyclic one with error two. Since the

topological restrictions in Oxtoby and Ulam's weaker version of Theorem 1 (cited above) apply only to the portion of their proof which fattens up a periodic point into a period cube, one could say that this result is implicit in their paper, though the proof given here is much simpler.

3 Proof of Theorem 1 ($\alpha > 0$)

We now modify the approximation result of section 2 to obtain an approximation that has periodic cubes instead of periodic points. Furthermore, these cubes 'fill up' the unit cube in the sense that their total volume is 1. A final property of the approximation is that each of these closed cubes is permuted by a succession of rigid motions.

A dyadic subset of I^n is a finite union of dyadic cubes of the same rank such that any two of them may be joined by a sequence of these cubes in which consecutive cubes share a common ($n-1$ dimensional) face. Observe that if α is a dyadic fraction, that is $\alpha = a/2^b$ with $0 \leq a < 2^b$ (with a, b positive integers), then the removal of the interiors of all α-cubes from a dyadic set leaves a dyadic set.

Most of the proof of Theorem 1 ($\alpha > 0$) is the same as that of Lemma 2. The difference is that the period point produced in the proof of Lemma 2 must be 'fattened' into a periodic, rigidly permuted, α-cube. In order to prove this stronger result, we will need a deep result from topology called the Annulus Theorem, the final dimensional restrictions of which have been removed by Kirby [16] and Quinn [17]. In order to state it, we first introduce some definitions. Any space homeomorphic to $S^{n-1} \times [0,1]$ is called an annulus. An embedding $f : S^{n-1} \to R^n$ is called locally flat if for each $x \in S^{n-1}$, there is a neighborhood U of $f(x)$ and a homeomorphism h between $(U, U \cap f(S^{n-1}))$ and (R^n, R^{n-1}); i.e. h is a homeomorphism between U and R^n whose restriction to $U \cap f(S^{n-1})$ is a homeomorphism to R^{n-1}.

Theorem 2 (Annulus Theorem) *Let $f, g : S^{n-1} \to R^n$ be disjoint locally flat orientation preserving embeddings of the $n-1$ sphere S^{n-1}, with $f(S^{n-1})$ inside the bounded component of $R^n - g(S^{n-1})$. Then the closed region bounded by $g(S^{n-1})$ and $f(S^{n-1})$ is homeomorphic to the annulus $S^{n-1} \times [0,1]$. Furthermore, all orientation preserving homeomorphisms of $S^{n-1} \to S^{n-1}$ are isotopic.*

Remark 1 *Actually the 'Furthermore' part is not part of the Annulus Theorem, but was shown to be a consequence of it by Brown and Gluck [13].*

We are now in a position to state and prove the lemma which provides the additional step needed to establish Theorem 1. (This is Lemma 20 of [21], without the restriction that $h \in \mathcal{M}_0$.)

Lemma 3 *Fix any $\alpha \in (0,1)$. Suppose that the homeomorphism $h \in \mathcal{M}$ cyclically permutes the centers of some rank m dyadic cubes $b_1, b_2, \ldots, b_N$, and the union of these cubes is $h-$invariant. Then there exist a volume preserving homeomorphism $f \in \mathcal{M}_0$ and a homeomorphism $g : I^n \to I^n$ which are both equal to the identity outside the cubes b_i and on their boundaries, with the following property: The transformation $gfhg^{-1}$ belongs to $\mathcal{M}$ and permutes the rank m α-cubes concentric to $b_1, b_2, \ldots, b_N$ rigidly in a single cycle.*

Proof. Let p_i denote the center of b_i. We may suppose the cubes are numbered to that $h(p_i) = p_{i+1}$, for $i < N$, and $h(p_N) = p_1$. Let a_i, $i = 1, \ldots, N$, be cubes concentric to the cubes b_i, all chosen with the same size, and small enough so that $h(a_i)$ is interior to b_{i+1}, $i < N$, and $h(a_N)$ to b_1. For each i we seek a map $f_i \in \mathcal{M}_0$ which equals the identity off of b_i and on its boundary, such that $f_i h$ is a rigid motion of a_{i-1} onto a_i. (Arithmetic on the subscripts will henceforth be done mod N.) We begin by observing that the Annulus Theorem implies that both $b_i - h(a_{i-1})$ and $b_i - a_i$ are both homeomorphic to the annulus $S^{n-1} \times [0,1]$ and hence to each other. Hence there is a homeomorphism $\hat{f}_i : b_i - h(a_{i-1}) \to b_i - a_i$. Furthermore we may use the isotopy ('furthermore') part of the theorem to assume that $\hat{f}_i$ equals the identity on the boundary of b_i and $\hat{f}_i h$ is a rigid motion on the boundary of a_{i-1}. Finally, since the two sets $b_i - h(a_{i-1})$ and $b_i - a_i$ have the same volume, an important result of Oxtoby and Ulam [21, Corollary 3] says that we can take $\hat{f}_i$ to be volume preserving. We extend $\hat{f}_i$ to the required $f_i \in \mathcal{M}_0$ by taking $f_i h$ to be the same rigid motion on a_{i-1} as found above, and setting it equal to the identity off of b_i.

Define $f \in \mathcal{M}_0$ by $f = f_1 \circ f_2 \circ \ldots \circ f_N$ so that fh cyclically and rigidly permutes the cubes a_i. In general, the side length of a_i compared to b_i will be less than the required fraction α, that is, the cubes a_i are too small. To remedy this, let $g : I^n \to I^n$ be a homeomorphism which equals the identity outside the cubes b_i and on their boundaries, with the following property on each b_i : it radially maps each cube a_i onto the α-cube e_i concentric to b_i, and has constant measure magnification on all the a_i and on all the $b_i - a_i$. It follows that $gfhg^{-1}$ has the required property. ■

This lemma is what is needed to extend the proof of Lemma 2 to a proof of Theorem 1. When an approximation is found which permutes the dyadic centers of rank m_k not contained in any $\alpha-$cube of rank $m_i, i < k$, the lemma is used to find an approximation which permutes the $\alpha-$cubes of rank m_k containing those centers. If $m = m_k$ in the lemma, we write the approximation $gfhg^{-1}$ as $v_k h$ by setting $v_k = gfhg^{-1}h^{-1} \in \mathcal{M}_0$, and observing that we can make $||v_k||$ as small as we like by choosing m_k sufficiently large. If we do this, the only change in the proof of Lemma 2 to give a full proof of Theorem 1 (for $\alpha > 0$) is to define (in the last line of that proof) $f = \lim_k (v_k g_k) \circ (v_{k-1} g_{k-1}) \circ \ldots \circ (v_1 g_1)$ instead of $f = \lim_k g_k \circ g_{k-1} \circ \ldots \circ g_1$.

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