A Combinatorial Construction of Chaotic Homeomorphisms on Manifolds

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Abstract. We use a combinatorial idea of Peter Lax to construct, on any compact manifold, a self homeomorphism which is chaotic in the sense of Devaney. Moreover we show that such transformations are dense, with respect to the uniform topology, in the space of volume preserving homeomorphisms of the closed unit cube $I^n, n \geq 2$. This work extends that of John Oxtoby, who in 1937 proved similar results for transitive transformations.

1. Introduction and Statement of Results

This paper extends the work of John Oxtoby on the existence and prevalence of transitive homeomorphisms. For our purposes it is useful to define a map $f : X \to X$ to be transitive if given any nonempty open subsets U and V of X, there is an orbit of f which enters U and then V. Oxtoby proved [13] that the transitive homeomorphisms constitute a dense G_δ subset, with respect to the uniform topology, of the space $\mathcal{M}$ consisting of all volume preserving homeomorphisms of the closed unit cube $I^n = [0,1]^n$, $n \geq 2$. In fact, that result was one of the first to use the Baire Category Theorem to demonstrate the existence of a new object. (Besicovitch [8] in the same year gave an example of a transitive homeomorphism of the plane, but it was not until 1990 that Xu [17] gave a similar example for the square.) Using similar ideas to those of Oxtoby, but without the Baire Category Theorem, we obtain similar results to his for homeomorphisms which are chaotic in the sense of Devaney.

Our results use are based on an approximation (Theorem 1) which uses a combinatorial idea of Peter Lax. In order to state this result, we recall that a dyadic cube of rank of rank m is defined to be an $n-$fold product of closed intervals of the form $[k/2^m, (k+1)/2^m]$. We define $\mathcal{M}_0$ to be the closed subspace of $\mathcal{M}$ consisting of homeomorphisms which equal the identity on the boundary of I^n. For $h \in \mathcal{M}$, we define the norm of h by $||h|| = \max_{x \in I^n} |h(x) - x|$, where $|x - y|$ denote Euclidean distance. For simplicity, we will call a volume preserving homeomorphism an *automorphism.*

Theorem 1. *Given any automorphism $h \in \mathcal{M}$ and any $\epsilon > 0$, there is a automorphism $f \in \mathcal{M}_0$ with $||f|| < \epsilon$, and an infinite strictly increasing sequence of positive integers $m_1, m_2, \ldots,$ with the following property: For each $k = 1, 2, \ldots,$ the centers of all the dyadic cubes of rank m_k are cyclically permuted by fh.*

It turns out that the approximation fh given by Theorem 1 has an interesting property. Suppose that U and V are any nonempty open subsets of I^n. For some k, they must both contain a dyadic cube of rank m_k. Consequently, there is a *periodic* orbit of fh (namely the centers of all the dyadic cubes of rank m_k) which enters both U and V, and hence the approximation fh is *periodically transitive* as defined below.

Definition 1. *A map is called periodically transitive if given any pair of nonempty open sets, it has a periodic orbit entering both of them. In other words, $g : X \to X$ is periodically transitive if given nonempty open sets U and V, there is a $p \in U$ and positive integers $l < m$ such that $g^l(p) \in V$ and $g^m(p) = p$.*

Theorem 1 and the definition of periodic transitivity immediately give the following result.

Theorem 2. *The periodically transitive automorphisms are dense in $\mathcal{M}$, with respect to the uniform topology. Furthermore, given any automorphism h in $\mathcal{M}$ and any $\epsilon > 0$, there is a automorphism $f \in \mathcal{M}_0$ with $||f|| < \epsilon$ such that fh is periodically transitive.*

The significance of this result will of course depend the strength of the notion of periodic transitivity. In fact, it is equivalent to what is known as 'chaos, in the sense of Devaney'. To see this, first observe that the definition is the same as that we gave for transitivity, with the added requirement that the orbit which goes through U and V must be periodic. So it implies (i) transitivity. Furthermore, it obviously implies that (ii) the set of periodic points are dense. It has recently been shown [7] that on infinite sets properties (i) and (ii) imply property (iii) *sensitive dependence on initial conditions*. (A map $g : X \to X$ has this property if $\exists \delta > 0 : \forall x \in X, \forall$ neighborhood U of $x, \exists y \in U, n \in N, |g^n(x) - g^n(x)| \geq \delta$.) Devaney's definition of chaos [10] requires all three conditions (i), (ii), and (iii). Since it is easy to see that (i) and (ii) imply periodic transitivity, it follows that periodic transitivity is the same as chaos, according to Devaney.

In [9], Brown and Cassler showed that any compact connected n-manifold can be obtained from I^n by making an identification on the boundary. Hence, by Theorem 2 with h =identity, we obtain a periodically transitive homeomorphism f which equals the identity on the boundary of I^n and so induces a periodically transitive homeomorphism on any compact manifold. Specifically, we have the following.

Corollary 1. *Every compact connected n-manifold X, $n \geq 2$, admits a periodically transitive homeomorphism.*

In fact, Theorem 2 may be completely adapted to the manifold context, if the manifold is endowed with a finite nonatomic Borel measure which is positive on open sets and zero on the manifold boundary. This requires a measure preserving version of Brown's result, which is proved in [6]. V. Prasad and the author intend to put the full generalization of Theorem 2 to compact manifolds in that book, as well as some extensions to the noncompact case.

Oxtoby's 1937 result on the prevalence of transitive homeomorphisms of compact manifolds has been generalized in some other directions, which we mention briefly. In 1941 Oxtoby and Ulam strengthened the notion of transitivity to that of ergodicity. (Note: while ergodicity implies transitivity in this context, it does not imply periodic transitivity.) In 1979, V. Prasad extended Oxtoby's result to R^n (also with the stronger notion of ergodicity). In 1995, the author and V. Prasad [5] extended Oxtoby's result on the annulus to transformations with a transitive lift (a strictly stronger result), thereby obtaining a simple proof of the Poincare-Birkhoff fixed point theorem. A final result which deserves mention, although it is not strictly speaking a generalization of transitivity, is due to Kato [11]. He proved in 1996 that every compact n-manifold, $n \geq 2$, admits what he calls an *everywhere chaotic* homeomorphism. This condition requires both sensitive dependence on initial conditions and the following: 'For any nonempty open sets U, V and each $\epsilon > 0$, there are points $x \in U$ and $y \in V$, and a $k \geq 1$, such that $d\left(f^k(x), f^k(y)\right) \leq \epsilon$.'

All that remains to justify the discussion given in this section is a proof of Theorem 1, which we present along with some further background in section 2. In section 3 we show how Theorem 2 could be proved as a corollary to Oxtoby's result on the denseness of transitivity.

2. Proof of Theorem 1

The automorphism f claimed by Theorem 1 will be obtained as the limit of a Cauchy sequence, so we need to state that the space $\mathcal{M}$ is complete with respect to the metric $\rho(f,g) = \max\left(\max_{x\in I^n}|f(x)-g(x)|, \max_{x\in I^n}|f^{-1}(x)-g^{-1}(x)|\right)$, which is topologically equivalent to the simpler 'uniform topology' metric $\max_{x\in I^n}|f(x)-g(x)|$. (See [15].) We shall also need an 'extension theorem' in the space $\mathcal{M}$ which was proved by Oxtoby and Ulam [15, Lemma 13], so we state it now. Oxtoby and Ulam prove this result as a corollary of a deep result on equivalence of measures under homeomorphisms - however a simple constructive proof is given in [4, Lemma 3.3].

Lemma 1. *Let $p_1, p_2, \ldots, p_k$ and $q_1, q_2, \ldots, q_k$ be two sets of k distinct points in the interior of I^n, with $|p_i - q_i| < \varepsilon$ for $i = 1, \ldots, k$. Then there is a automorphism $h \in \mathcal{M}_0$ with $||h|| < \varepsilon$ and $h(p_i) = q_i$ for $i = 1, \ldots, k$.*

Proof of Theorem 1: We begin by showing that if h already satisfies the cyclic permutation condition for $k = 1, \ldots, K$, then for any $\delta > 0$ we can find a $g \in \mathcal{M}_0$ with $||g|| < \delta$ which is the identity on the dyadic centers of ranks $m_1, \ldots, m_k$ and such that gh satisfies the condition for $k = 1, \ldots, K+1$. (We intend this to include the case $K = 0$, which covers the original automorphism h.) We begin by choosing the next rank $r = m_{k+1}$ so that their diameters $d = d_r$ are less than some positive number to be determined later. Let $b_1, b_2, \ldots, b_R$ be an enumeration of all the $R = 2^{nr}$ dyadic cubes of rank r (which we call simply 'r-cubes') and number them so that consecutively numbered cubes have an $n-1$ dimensional face in common. Let c_i denote the center of the cube b_i. We say that the r-cube b_i 'knows' the r-cube b_j if $h(b_i) \cap b_j \neq \emptyset$. Observe that if B is the union of l distinct r-cubes, it has the volume of l r-cubes and therefore cannot be a subset of any set consisting of fewer than l distinct r-cubes. In other words, any set of l r-cubes 'knows' at least l r-cubes. According to the well known Marriage Theorem this condition implies that there is a permutation σ of the set $\{1, \ldots, R\}$ such that the r-cube b_i knows the r-cube $b_{\sigma(i)}$, or equivalently,

$$h(b_i) \cap b_{\sigma(i)} \neq \emptyset, \text{ for all } i = 1, \ldots, R. \tag{1}$$

We wish to approximate the permutation σ by another permutation τ (of the same set) which is cyclic and for which $b_{\sigma(i)}$ is close to $b_{\tau(i)}$ for all i. The simplest way is to apply the algorithm given by the author in [3] which approximates any permutation of $\{1, \ldots, R\}$ (for any integer R) by a cylic one which differs uniformly by at most 2. In particular,

$$\text{there is a cyclic permutation } \tau \text{ with } |\sigma(i) - \tau(i)| \leq 2, i = 1, \ldots, R. \tag{2}$$

The algorithm is easy: We modify σ by consecutively looking at each of the pairs $(1,2), (3,4), \ldots, (R-1, R), (2,3), (4,5), \ldots, (R-2, R-1)$, and in each case transposing the two numbers iff they lie in different cycle of the current permutation. Eventually there is only one cycle and in the process any number is moved at most twice, both times to an adjacent number. For example, if (in cycle notation) $\sigma = [(1), (2,7), (3), (4), (56), (8), (9)]$, then $\tau = [(1,3,5,6,4,2,9,8,7)]$.

It follows from (1) that for every center c_i of the r-cube b_i, we have $\left|h(c_i) - c_{\sigma(i)}\right| \leq \omega(d_r) + d_r$, where ω is the uniform modulus of continuity of h on I^n, since the diameter of $h(b_i)$ is bounded above by $\omega(d_r)$. It follows from (2) that $\left|c_{\sigma(i)} - c_{\tau(i)}\right| \leq 2d_r$. Consequently we have

$$\left|h(c_i) - c_{\tau(i)}\right| \leq \omega(d_r) + 3d_r < \delta,$$

if we choose $m_{K+1} = r > m_K$ sufficiently large. For $i = 1, \ldots, r$, set $p_i = h(c_i)$ and $q_i = c_{\tau(i)}$. For $i = R + j$, where $j = 1, \ldots, J$, and $J = \sum\limits_{k=1}^{K} 2^{nm_k}$ is the number of the centers of ranks $m_{1,}\ldots,m_K$ permuted by h, define $p_i = q_i$ to be some enumeration of these centers. Then $|p_i - q_i| < \delta$ for $i = 1,\ldots,R+J$, so Lemma 1 gives us a automorphism $g \in \mathcal{M}_0$ with $||g|| < \delta$ with $g(p_i) = q_i$. Hence g is the identity on the centers already permuted by h and $gh(c_i) = c_{\tau(i)}$ for the centers c_i, $i = 1,\ldots,R$, of the cubes of rank $r = m_{K+1}$. Thus g satisfies the requirements claimed in the first sentence of the proof.

Returning to the proof of the full theorem, we use the claim already establised to construct a automorphism $g = g_0 \in \mathcal{M}_0$ with $||g_0|| < \varepsilon/2$ so that $g_0 h$ permutes the dyadic centers of some fixed rank m_1. We then choose $\delta = \delta_1 < \varepsilon/4$ so that any automorphism $\hat{g} \in \mathcal{M}_0$ with $||\hat{g}|| < \delta_1$ satisfies $\rho(\hat{g}g_0, g_0) < \varepsilon/4$, and use the established claim to find a $g_1 \in \mathcal{M}_0$ with $||g_1|| < \delta_1$ which is the identity on the dyadic centers of rank m_1 and such that $g_1 g_0 h$ cyclically permutes the dyadic centers of some rank $m_2 > m_1$. We continue in this fashion and define $f = \lim_k g_k \circ g_{k-1} \circ \ldots \circ g_1$. This limit exists in $\mathcal{M}_0$ because the construction gives $\rho(g_k \circ g_{k-1} \circ \ldots \circ g_{1,} g_{k-1} \circ \ldots \circ g_1) < \varepsilon/2^{k+1}$, and has the required properties by construction. ■

This proof uses Peter Lax's idea [12] of applying the Marriage Theorem to approximate an automorphism by a dyadic permutation, and the author's method [3] of approximating a permutation by a cyclic one with error two. The notion of an infinite family of permuted centers is related to a similar idea for families of permuted *subcubes* obtained in an unpublished result from the author's thesis, which generalizes (topologically) a similar idea of Oxtoby and Ulam [15].

3. Dense Transitivity implies Dense Periodic Transitivity

In this section we give a proof of Theorem 2 based on Oxtoby's eairlier result mentioned above, that the transitive automorphisms are dense in $\mathcal{M}$. The result proved by Oxtoby contains our Theorem 2 with the word 'periodically' removed (it also asserts G_δ−ness, which we don't). We will need the following simple lemma.

Lemma 2. *Let F be a finite subset of the interior of I^n, and let U and V be nonempty open subsets. Suppose that F is invariant under some automorphism $h \in \mathcal{M}$ and that $\delta > 0$ is given. Then there is an automorphism $f \in \mathcal{M}_0$ with $||f|| < \delta$ such that f equals the identity on F and fh has a periodic orbit entering both U and V.*

Proof. According to the result of Oxtoby on the denseness of transitivity mentioned above, there exists an $f_1 \in \mathcal{M}_0$ with $||f_1|| < \delta/2$ and $g = f_1 h$ transitive. It follows from the transitivity of g that there is some point $x \in U$ whose orbit under g enters V and returns to within a distance $\delta/2$ of x, while avoiding the finite set $F \cup f_1 F$. That is, $g^l(x) \in U$ and $|g^m(x) - x| < \delta/2$, for some integers $0 < l < m$. For $i = 1,\ldots,m-1$, set $g^i(x) = p_i = q_i$. Set $p_m = g^m(x)$ and $q_m = x$. Next, label the finite set $F = \{q_{m+1}, \ldots, q_k\}$ for some k, and set $p_j = f_1(q_j)$ for $j = m+1, \ldots, k$. Now apply Lemma 1 to obtain an $f_2 \in \mathcal{M}_0$ with $||f_2|| < \delta/2$ such that $f(p_j) = q_j$ for $j = 1, \ldots, k$, and set $f = f_2 f_1$. It follows from the construction that $||f|| < \delta$, f is the identity on F, that $(fh)^l(x) = g^l(x) \in U$ and that

$$(fh)^m(x) = (f_2 g)^m(x) = f_2 g\left[g^{m-1}(x)\right] = f_2(g^m(x)) = x. \blacksquare$$

We now give an alternate proof of Theorem 2.

Proof. Let $\{U_i, V_i\}_{i=1}^{\infty}$ be an enumeration of all open dyadic cubes of all ranks. Setting $F = F_0 = \emptyset$ and choosing $\delta > 0$ so that $||g|| < \delta$ implies that $\rho(h, gh) < \epsilon/2$, use the lemma to find $f_1 \in \mathcal{M}_0$ so that there is an $f_1 h$ orbit O_1 which enters both U_1 and V_1. Set $F_1 = F_0 \cup 0_1$. We continue recursively finding, for each i, an automorphism $f_i \in \mathcal{M}_0$ which equals the identity on $F_{i-1} = F_{i-2} \cup O_{i-1}$, has an $f_i \ldots f_1 h-$orbit entering both U_i and V_i, and satisfies $\rho(f_i \ldots f_1 h, f_{i-1} \ldots f_1 h) < \epsilon/2^i$. Define $f = \lim_i f_i \ldots f_1$. To see that fh is periodically transitive, observe that for any pair U, V of nonempty open sets, there is an i with $U_i \subset U$ and $V_i \subset V$, and consequently an $f_i \ldots f_1 h-$orbit O_i entering both U_i and V_i. Since for $j > i$, f_j equals the identity on the orbit $O_i \subset F_{j-1}$, it follows that O_i is also an orbit of fh. $\blacksquare$

The proof given in above is only superficially useful, because the simplest proof of Oxtoby's transitivity theorem is actually based on a slighly simpler version (with a single cycle of dyadic centers) of Theorem 1. Such a proof is given in the appendix to the second edition of Oxtoby's book [14], based on the author's simplification of the Oxtoby-Ulam proof of generic ergodicity.

The title of this section should not be taken in too broad a sense. For rotations of the circle, transitivity is dense (true for irrational rotations). However, such rotations have no periodic points and certainly are not periodically transitive. What fails is of course Lemma 1.

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