The Complexity of Toughness in Regular Graphs

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Abstract

Let $t \geq 1$ be an integer. We show that it is NP-hard to determine if an r-regular graph is t-tough for any fixed integer $r \geq 3\,t$. We also discuss the complexity of recognizing if an r-regular graph is t-tough, for any rational $t \geq 1$.

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1 Introduction

We begin with a few definitions and some notation. A good reference for any undefined terms in graph theory is [7] and for computational complexity is [12]. We consider only undirected graphs with no loops or multiple edges. Let G be a graph. Then G is *hamiltonian* if it has a *Hamilton cycle*, i.e., a cycle containing all of its vertices. It is *traceable* if it has a path containing all of its vertices. Let $\omega(G)$ denote the number of components of G. We say G is *t-tough* if $|S| \geq t \cdot \omega(G-S)$ for every subset S of the vertex set $V(G)$ of G with $\omega(G-S) > 1$. The *toughness* of G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough (taking $\tau(K_n) = \frac{n-1}{2}$ for all $n \geq 1$). A *k-factor* is a k-regular spanning subgraph. Of course, a Hamilton cycle is a 2-factor.

Toughness was introduced by Chvátal in [9]. In recent years much work has been done to understand the relationship between the toughness of a graph and its cycle structure. For survey papers in this area, see [4, 5, 6]. An obvious connection between toughness and hamiltonicity is that being 1-tough is a necessary condition for a graph to be hamiltonian. Chvátal conjectured that there exists a finite constant t_0 such that every t_0-tough graph is hamiltonian. This conjecture is still open. Until recently it was believed that the smallest value of t_0 for which this conjecture would be true was $t_0 = 2$. This belief, known as the "2-tough conjecture", stemmed in large part by letting $k = 2$ in the two results below. The first of these results showed that Chvátal was correct in his conjecture that every k-tough graph on n vertices with $n \geq k+1$ and kn even has a k-factor.

Theorem 1.1 [11]. *Let G be a k-tough graph on n vertices with $n \geq k+1$ and kn even. Then G has a k-factor.*

Theorem 1.2 [11]. *Let $k \geq 1$. For any $\varepsilon > 0$, there exists a $(k-\varepsilon)$-tough graph G on n vertices with $n \geq k+1$ and kn even which has no k-factor.*

We now know that the 2-tough conjecture is false.

Theorem 1.3 [1]. *For every $\varepsilon > 0$, there exists a $(\frac{9}{4}-\varepsilon)$-tough nontraceable graph.*

In this note we continue to study the complexity of recognizing t-tough graphs. This problem was first raised by Chvátal [8] and later appeared in [19] and [10, p. 429]. Consider the following decision problem, where $t > 0$ is a rational number.

> ***t*-TOUGH**
> INSTANCE: Graph G.
> QUESTION: Is $\tau(G) \geq t$?

The following was established in [2].

Theorem 1.4 [2]. *For any rational $t > 0$, t-TOUGH is NP-hard.*

It seems natural to inquire whether the problem of recognizing t-tough graphs remains NP-hard for various subclasses of graphs. A number of results in this direction are discussed in [3]. In this note we focus on the complexity of recognizing tough regular graphs, and begin with the class of cubic graphs. In [9], Chvátal showed that a necessary and sufficient condition on an integer n for the existence of an n-vertex, 3/2-tough cubic graph is that $n = 4$ or $n \equiv 0 \pmod 6$. Later, Jackson and Katerinis [15] strengthened this by characterizing 3/2-tough, cubic graphs.

Theorem 1.5 [15]. *Let G be a cubic graph. Then G is 3/2-tough if and only if $G = K_4$, $G = K_2 \times K_3$, or G is obtained from a 3-connected cubic graph by replacing all the vertices of this graph by triangles.*

The above characterization allows 3/2-tough, cubic graphs to be recognized in polynomial time. However, the arguments used in [15] seem very dependent on the particular constant 3/2, and it seems unlikely that a polynomial time algorithm exists to recognize t-tough, cubic graphs for any t such that $1 \le t < 3/2$. In [3] we established this for $t = 1$.

Theorem 1.6 [3]. *1-TOUGH remains NP-hard for cubic graphs.*

In [3] we were also able to use Theorem 1.6 to establish the following result.

Theorem 1.7 [3]. *For any integer $t \ge 1$, t-TOUGH remains NP-hard for $3t$-regular graphs.*

In this note we extend both Theorem 1.6 and Theorem 1.7. We first extend Theorem 1.6 by proving Theorem 1.8 below. The proof is modelled after the proof of Theorem 1.6 given in [3].

Theorem 1.8 *For any fixed integer $r \ge 3$, 1-TOUGH remains NP-hard for r-regular graphs.*

We then extend Theorem 1.7 by proving Theorem 1.9 below. The proof, which uses Theorem 1.8, is modelled after the proof of Theorem 1.7 given in [3].

Theorem 1.9 *For any integer $t \ge 1$ and any fixed integer $r \ge 3t$, t-TOUGH remains NP-hard for r-regular graphs.*

The truth of Theorems 1.8 and 1.9 is not surprising in light of Theorems 1.6 and 1.7, respectively. Nevertheless, it is not always easy to establish this type of intuitively obvious result. For example, after Holyer [14] established that it is NP-hard to decide if a cubic graph is class one (relative to edge-coloring), it required a surprisingly difficult proof by Levin and Galil [16] to show the problem is also NP-hard for r-regular graphs for fixed $r > 3$.

We now consider the problem of recognizing t-tough r-regular graphs for any rational $t \geq 1$. Note that if $r < 2t$, an r-regular graph cannot be t-tough since removing the neighborhood of any vertex yields at least two components. If $r = 2t$, then t-TOUGH would be polynomial if the following conjecture of Goddard and Swart [13] is true.

Conjecture 1.10 [13]. *Let G be an r-regular graph. Then G is $\frac{r}{2}$-tough if and only if G is r-connected and $K_{1,3}$-free.*

Note that Conjecture 1.10 is true for $r = 3$ by Theorem 1.5. In addition, the "if" direction of Conjecture 1.10 was established by a result of Matthews and Sumner [17].

Theorem 1.11 [17]. *Let G be a noncomplete $K_{1,3}$-free graph. Then $\tau(G)$ is equal to one half of the connectivity of G.*

We now make the following conjecture.

Conjecture 1.12 *For any rational $t \geq 1$ and any fixed integer $r \geq 1$, t-TOUGH remains NP-hard for the class of r-regular graphs if and only if $r > 2t$.*

In the next two sections we present the proofs of Theorems 1.8 and 1.9. Finally, in the last section we conclude with a discussion of Conjecture 1.12.

2 Proof of Theorem 1.8

We already know that recognizing 1-tough cubic graphs is NP-hard [3]. For fixed $r \geq 4$, we now prove that 1-TOUGH is NP-hard for r-regular graphs by reducing 1-TOUGH for cubic graphs.

Let G be any cubic graph. For $r \geq 4$, construct the corresponding r-regular graph $H_r = H_r(G)$ as follows. Each vertex $v \in V(G)$ will correspond to a graph $H_{v,r}$ as in Figures 1 and 2 below. The structure of $H_{v,r}$ will depend on the parity of r. In each case the three black vertices on each side of the connection graph will have degree $r-1$ in $H_{v,r}$.

Case 1: $r = 2k$, $k \geq 2$.

In this case there are four basic graphs, B_1^a, B_1^b, B_2^a, B_2^b, and one connection graph C_3.

For $j \in \{a, b\}$, the graph B_1^j is a complete graph on $2k$ vertices. Let $V(B_1^j) = R_1^j \cup S_1^j$, where $R_1^j = \{r_{1,1}^j, r_{1,2}^j, \dots, r_{1,k}^j\}$ and $S_1^j = \{s_{1,1}^j, s_{1,2}^j, \dots, s_{1,k}^j\}$.

For $j \in \{a,b\}$, the graph B_2^j is a complete graph on $2k+1$ vertices with certain edges removed. Let $V(B_2^j) = R_2^j \cup S_2^j \cup \{w\}$, where $R_2^j = \{r_{2,1}^j, r_{2,2}^j, \ldots, r_{2,k}^j\}$ and $S_2^j = \{s_{2,1}^j, s_{2,2}^j, \ldots, s_{2,k}^j\}$. Now remove the edges $\{r_{2,1}^j s_{2,1}^j, r_{2,2}^j s_{2,2}^j, \ldots, r_{2,k-1}^j s_{2,k-1}^j\}$.

The graph C_3 is a complete graph on $2k$ vertices with certain edges removed. First let $V(C_3) = R_3 \cup S_3$, where $R_3 = \{r_{3,1}, r_{3,2}, \ldots, r_{3,k}\}$ and $S_3 = \{s_{3,1}, s_{3,2}, \ldots, s_{3,k}\}$. Now remove the edges $\{r_{3,1}s_{3,1}, r_{3,2}s_{3,2}, \ldots, r_{3,k-1}s_{3,k-1}\}$.

Finally, consider the 6 black vertices $\{u_1, u_2, \ldots, u_6\}$ in Figure 1. The vertex

u_1 is joined to $(R_1^a \cup R_2^a) - \{r_{2,k}^a\}$,
u_2 is joined to $(S_2^a \cup R_3) - \{s_{2,k}^a\}$,
u_3 is joined to $(S_1^a \cup R_3) - \{r_{3,k}\}$,
u_4 is joined to $(R_2^b \cup S_3) - \{r_{2,k}^b\}$,
u_5 is joined to $(R_1^b \cup S_3) - \{s_{3,k}\}$ and
u_6 is joined to $(S_1^b \cup S_2^b) - \{s_{2,k}^b\}$.

The graph $H_{v,r}$ for r even is shown in Figure 1.

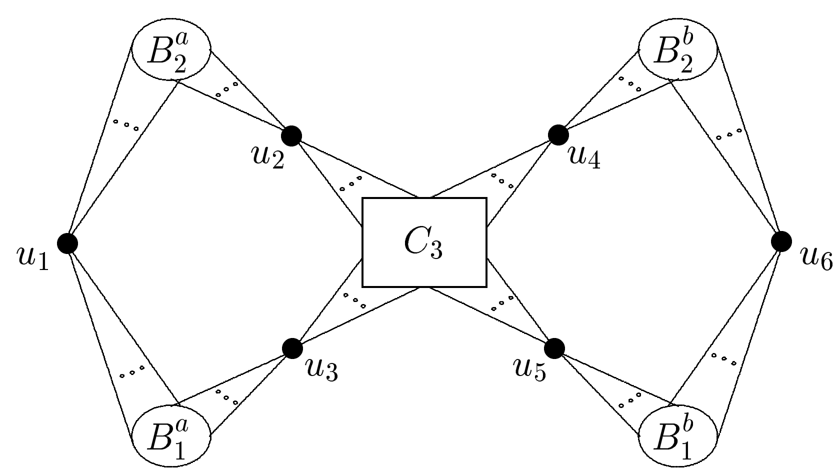

Fig. 1: $H_{v,r}$ for r even.

Case 2: $r = 2k+1$, $k \geq 2$.

In this case there are four basic graphs, B_4^a, B_4^b, B_4^c, B_4^d, and one connection graph C_5.

For $j \in \{a,b,c,d\}$, the graph B_4^j is a complete graph on $2k+2$ vertices with certain edges removed. Let $V(B_4^j) = R_4^j \cup S_4^j$, where $R_4^j = \{r_{4,1}^j, r_{4,2}^j, \ldots, r_{4,k+1}^j\}$ and $S_4^j = \{s_{4,1}^j, s_{4,2}^j, \ldots, s_{4,k+1}^j\}$. Now remove the edges $\{r_{4,1}^j s_{4,1}^j, r_{4,2}^j s_{4,2}^j, \ldots, r_{4,k}^j s_{4,k}^j\}$.

The graph C_5 is a complete graph on $2k+2$ vertices with certain edges removed. Let $V(C_5) = R_5 \cup S_5$, where $R_5 = \{r_{5,1}, r_{5,2}, \ldots, r_{5,k+1}\}$ and $S_5 = \{s_{5,1}, s_{5,2}, \ldots, s_{5,k+1}\}$. Now remove the edges $\{r_{5,1}s_{5,2}, r_{5,2}s_{5,3}, \ldots, r_{5,k}s_{5,k+1}\}$ and $\{r_{5,2}s_{5,1}, r_{5,3}s_{5,2}, \ldots, r_{5,k+1}s_{5,k}\}$.

Finally, consider the 6 black vertices $\{w_1, w_2, \ldots, w_6\}$ in Figure 2. The vertex

w_1 is joined to $(R_4^a \cup R_4^c) - \{r_{4,k+1}^a, r_{4,k+1}^c\}$,
w_2 is joined to $(S_4^a \cup R_5) - \{s_{4,k+1}^a, r_{5,k+1}\}$,
w_3 is joined to $(S_4^c \cup R_5) - \{s_{4,k+1}^c, r_{5,1}\}$,
w_4 is joined to $(R_4^b \cup S_5) - \{r_{4,k+1}^b, s_{5,k+1}\}$,
w_5 is joined to $(R_4^d \cup S_5) - \{r_{4,k+1}^d, s_{5,1}\}$ and
w_6 is joined to $(S_4^b \cup S_4^d) - \{s_{4,k+1}^b, s_{4,k+1}^d\}$.

The graph $H_{v,r}$ for r odd is shown in Figure 2.

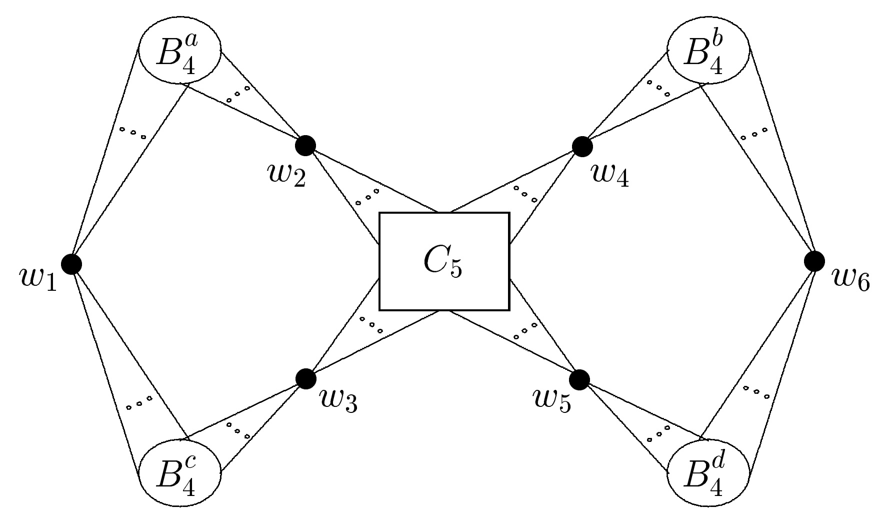

Fig. 2: $H_{v,r}$ for r odd.

For r even, arbitrarily designate the black vertices in $H_{v,r}$ on one side of C_3 by $A_{v,r}$, and those on the other side by $B_{v,r}$. Similarly, for r odd arbitrarily designate the black vertices in $H_{v,r}$ on one side of C_5 by $A_{v,r}$, and those on the other side by $B_{v,r}$. Thus the set of "black" vertices in H_r is $B = \bigcup_{v \in V(G)} A_{v,r} \cup B_{v,r}$. Let W be the remaining set of "white" vertices in H_r. An edge vw in G will be represented in H_r by joining any previously unused vertex in $A_{v,r}$ to any previously unused vertex in $B_{w,r}$, and any previously unused vertex in $A_{w,r}$ to any previously unused vertex in $B_{v,r}$. Note that H_r is an r-regular 2-connected graph.

To complete the proof, it now suffices to show the following.

Claim. *G is 1-tough if and only if $H_r(G)$ is 1-tough.*

Proof of the Claim: Suppose first that G is not 1-tough. Then there exists a nonempty set $X \subseteq V(G)$ with $\omega(G - X) > |X|$. Let $Y \subseteq V(H_r)$ be given by $Y = \bigcup_{v \in X} (A_{v,r} \cup B_{v,r})$. It is easily verified that $\omega(H_r - Y) > |Y|$, and thus H_r is also not 1-tough.

Conversely, suppose H_r is not 1-tough. Then there exists a nonempty set $Y \subseteq V(H_r)$ with $\omega(H_r - Y) > |Y|$. We will now establish a series of properties (Lemmas 2.1 to 2.4) which we may assume Y satisfies, since otherwise we may select a nonempty set $Y' \subseteq V(H_r)$ with $\omega(H_r - Y') > |Y'|$ satisfying the desired properties.

Lemma 2.1 *We may assume that $Y \cap W = \varnothing$.*

Proof: For a given $v \in V(G)$, let F be any basic or connection subgraph of $H_{v,r} \subseteq H_r$, together with the black neighbors of this subgraph (see Figures 1 and 2). Let $Y_F(W)$ (resp., $Y_F(B)$) denote $Y \cap V(F) \cap W$ (resp., $Y \cap V(F) \cap B$), with $Y_F(W) \neq \varnothing$. We first prove

Fact 1: $\omega(\,(F - Y_F(B)) - Y_F(W)\,) \leq |Y_F(W)| + 1$.

Proof of Fact 1: When $r \geq 5$, $F - Y_F(B)$ is hamiltonian, and hence 1-tough, regardless of which vertices are in $Y_F(B)$. Thus $\omega(\,(F - Y_F(B)) - Y_F(W)\,) \leq |Y_F(W)| < |Y_F(W)| + 1$. If $r = 4$, it is easy to verify the inequality directly for each F. □

Let $Y' = Y - Y_F(W)$. We next prove

Fact 2: $\omega(H_r - Y') > |Y'|$.

Proof of Fact 2: By Fact 1 we have

$$\begin{aligned}
\omega(H_r - Y') &\geq \omega(H_r - Y) - [\,\omega(\,(F - Y_F(B)) - Y_F(W)\,) - 1\,] \\
&> |Y| - [\,(|Y_F(W)| + 1) - 1\,] \\
&= |Y| - |Y_F(W)| \\
&= |Y'|.
\end{aligned}$$
□

To complete the proof of Lemma 2.1, simply iterate the above procedure for each basic or connection graph F in H_r having $Y_F(W) \neq \varnothing$. We finally obtain a set $Y' \subseteq B$ with $\omega(H_r - Y') > |Y'|$. This set Y' must be nonempty; otherwise the previous set Y' consists solely of white vertices in a single basic or connection graph F and satisfies $\omega(H_r - Y') > |Y'|$. This clearly is impossible. □

Lemma 2.2 *We may assume, for all $v \in V(G)$, that $A_{v,r} \cap Y$ (resp., $B_{v,r} \cap Y$) is either $A_{v,r}$ (resp., $B_{v,r}$) or $\varnothing$.*

Proof: Suppose $A_{v,r} \cap Y$ is neither $A_{v,r}$ nor $\varnothing$. Then there are two "consecutive" vertices $a, b \in A_{v,r}$ (see Figure 3) with $a \in Y$, $b \notin Y$. Set $Y' = Y \cup \{b\}$.

It is immediate that $|Y'| = |Y| + 1$ and $\omega(H_r - Y') \geq \omega(H_r - Y) + 1$, and thus we have a nonempty $Y' \subseteq V(H_r)$ such that $\omega(H - Y') > |Y'|$. We simply iterate this modification to Y until $A_{v,r} \cap Y = A_{v,r}$. The proof for $B_{v,r}$ is identical. □

We will call $v \in V(G)$ a *split vertex* if exactly one of $A_{v,r}, B_{v,r}$ belongs to Y. In a moment we will show we may assume there are no split vertices. First we need the following result.

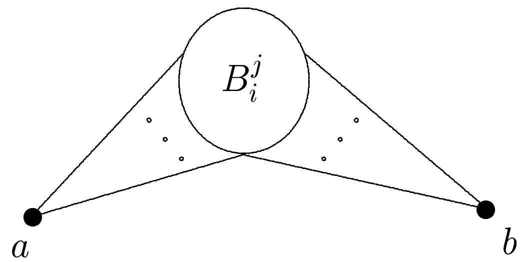

Fig. 3: Consecutive vertices in $A_{v,r}$.

Lemma 2.3 *Let $v \in V(G)$. If $A_{v,r} \nsubseteq Y$ and $B_{v,r} \subseteq Y$, then we may assume, for all $w \in N_G(v)$, that $B_{w,r} \subseteq Y$.*

Proof: Suppose $A_{v,r} \nsubseteq Y$, $B_{v,r} \subseteq Y$, but $B_{w,r} \nsubseteq Y$ for some $w \in N_G(v)$. Set $Y' = Y \cup A_{v,r}$. We find $|Y'| = |Y| + 3$ and $\omega(H_r - Y') \geq \omega(H_r - Y) + 3$, and thus we have a nonempty $Y' \subseteq V(H_r)$ such that $\omega(H_r - Y') > |Y'|$. Now simply iterate the modification of Y until the desired condition holds. □

Lemma 2.4 *We may assume there are no split vertices.*

Proof: Suppose $S = \{ v \in V(G) \mid v \text{ is a split vertex} \} \neq \varnothing$. Consider any component K of $\langle S \rangle$ in G, and let $V(K) = \{v_1, v_2, \ldots, v_m\}$. By Lemma 2.3 we may assume, without loss of generality, that $A_{v_i,r} \nsubseteq Y$, $B_{v_i,r} \subseteq Y$, for $i = 1, 2, \ldots, m$. Note that if $wv_i \in E(G)$ for some i, $1 \leq i \leq m$, and $w \notin V(K)$, then by Lemma 2.3, $A_{w,r}, B_{w,r} \subseteq Y$. Also note that $\bigcup_{i=1}^{m} (H_{v_i,r} - B_{v_i,r})$ induces a subgraph in H_r with exactly $3m$ components. Now set $Y' = Y \setminus (\bigcup_{i=1}^{m} B_{v_i,r})$. Suppose $Y' = \varnothing$. If $V(K) \neq V(G)$, then G is disconnected by the above observation on edges wv_i. On the other hand, if $V(K) = V(G)$, then $\omega(H_r - Y) = 3|V(G)| = |Y|$, a contradiction. Hence $Y' \neq \varnothing$. Thus we have

(1) $\quad |Y'| \; = \; |Y| - 3m \; \geq \; 1.$

16.226 By Lemma 2.3, we also have

(2) $\quad \omega(H_r - Y') \; = \; \omega(H_r - Y) - (3m - 1).$

Using (1), (2), and $\omega(H_r - Y) > |Y|$, we find that Y' is a nonempty subset of $V(H_r)$ such that

$$\omega(H_r - Y') \; > \; |Y| - 3m + 1 \; = \; |Y'| + 1 \; > \; |Y'|.$$

Now simply iterate the above modification to Y for every component K of $\langle S \rangle$. □

Finally, let $X = \{ v \in V(G) \mid A_{v,r}, B_{v,r} \subseteq Y \}$. Since there are no split vertices, it is easy to check that X is a nonempty subset of $V(G)$ such that $\omega(G - X) > |X|$ and so G is not 1-tough.

This proves the Claim, and completes the proof of Theorem 1.8. □

Note that for every $r \geq 4$, $H_{v,r}$ (and thus H_r) contains a 1-factor. Thus the proof of Theorem 1.8 actually proves the following result: For every fixed integer $r \geq 3$, 1-TOUGH remains NP-hard for the class of 2-connected, r-regular graphs which contain a 1-factor.

3 Proof of Theorem 1.9

Before proving Theorem 1.9, we require the following lemma. The proof is essentially the same as the proof of Theorem 1.7 given in [3].

Lemma 3.1 *Let $t \geq 1$ and $k \geq 3$ be integers. Then t-TOUGH remains NP-hard for the class of kt-regular graphs.*

Proof of Lemma 3.1: We will reduce 1-TOUGH for 2-connected, k-regular graphs containing a 1-factor (see the remark at the end of the previous section). Let G be any such graph, and construct $H = H(G)$ as follows. Each vertex in G is replaced by a K_t in H. Each edge in the 1-factor in G is replaced by a matching (henceforth called an *m-join*) between the corresponding K_t's in H, while each edge not in the 1-factor in G is replaced by a complete bipartite join (henceforth called a *c-join*) between the corresponding K_t's. It is immediate that H is $k\,t$-regular.

We next show that H is $2t$-connected. Since G is 2-connected, disconnecting G requires removing at least two vertices, one vertex and a nonincident edge, or two independent edges. Thus disconnecting H requires removing at least two K_t's, removing one K_t and breaking a nonincident m-join, or breaking two independent m-joins. Note that "breaking an m-join" means removing enough vertices in the m-join to eliminate all edges in the join without completely removing either K_t; obviously this requires removing a total of at least t vertices in the two K_t's. It follows that any cutset in H must contain at least $2t$ vertices, and thus H is $2t$-connected.

To complete the proof, we now show G is 1-tough if and only if H is t-tough. If G is not 1-tough, there exists a cutset $X \subseteq V(G)$ with $\omega(G - X) > |X|$. Let $Y \subseteq V(H)$ consist of the K_t's corresponding to the vertices in X. It is easy to see that Y is a cutset and $\omega(H - Y) = \omega(G - X) > |X| = |Y|/t$, and thus H is not t-tough.

Conversely, suppose H is not t-tough. Then there exists a cutset $Y \subseteq V(H)$ with $\omega(H - Y) > |Y|/t$. We now establish the following.

Claim. *We may assume each K_t in H is entirely contained in Y or entirely disjoint from Y (i.e., no K_t in H is "split" by Y).*

Assuming we have established the Claim, let $X \subseteq V(G)$ denote the vertices in G corresponding to the K_t's in Y. Then X is a cutset in G and $\omega(G-X) = \omega(H-Y) > |Y|/t = |X|$, and thus G is not 1-tough as desired.

To prove the claim consider a cutset $Y \subseteq V(H)$ such that $\omega(H-Y) > |Y|/t$ and Y splits as few K_t's in H as possible. If Y splits no K_t's there is nothing to prove, so suppose A is a K_t split by Y, and let B denote the K_t which is m-joined to A. We now consider several cases.

Case 1: B is not split by Y.

Let $Y' = Y - (A \cap Y)$. Then $|Y'| < |Y|$ while $\omega(H-Y') = \omega(H-Y)$, since A is still c-joined to the same K_t's as was $A - Y$. Thus we have $\frac{|Y'|}{\omega(H-Y')} < \frac{|Y|}{\omega(H-Y)} < t$, or $\omega(H-Y') > |Y'|/t$. Since Y is a cutset and $\omega(H-Y') = \omega(H-Y)$, Y' is also a cutset in H. Since Y' splits fewer K_t's than Y, this violates the optimality of Y.

Case 2.1: B is split by Y and $|A \cap Y| + |B \cap Y| < t$.

Set $Y' = Y - (A \cap Y) - (B \cap Y)$. Then $\omega(H-Y') = \omega(H-Y)$, since $A - Y$ and $B - Y$ belong to the same component of $H - Y$, and A (resp, B) is c-joined to its adjacent K_t's besides B (resp, A). Thus we get $\frac{|Y'|}{\omega(H-Y')} < \frac{|Y|}{\omega(H-Y)} < t$, or $\omega(H-Y') > |Y'|/t$. Since Y is a cutset and $\omega(H-Y') = \omega(H-Y)$, Y' is also a cutset in H. Again, Y' splits fewer K_t's than Y, and this violates the optimality of Y.

Case 2.2: B is split by Y and $|A \cap Y| + |B \cap Y| \geq t$.

Let $Y' = Y - (A \cap Y) - Z$, where $Z \subseteq B \cap Y$ is any subset with $|Z| = t - |A \cap Y| > 0$. Since H is $2t$-connected and Y is a cutset in H, we have $|Y'| = |Y| - t \geq 2t - t = t$. Note that $\omega(H-Y') \geq \omega(H-Y) - 1$, since we might still lose one component by pulling together the two components containing $A - Y$ and $B - Y$, but nothing more. Since $\omega(H-Y) > \frac{|Y|}{t}$, we get $\frac{|Y'|}{\omega(H-Y')} \leq \frac{|Y|-t}{\omega(H-Y)-1} < t$, or $\omega(H-Y') > \frac{|Y'|}{t} \geq 1$. Thus Y' is a cutset in H. Since Y' splits fewer K_t's than Y, this violates the optimality of Y.

This proves the Claim, and thereby proves Lemma 3.1. $\square$

Proof of Theorem 1.9: Given any integer $r \geq 3t$, choose the integer $k \geq 3$ such that $kt \leq r \leq (k+1)t - 1$. In the construction of $H(G)$ given in the proof of Lemma 3.1, replace each m-join between a pair of K_t's by $r - kt + 1$ edge-disjoint 1-factors between the pair of K_t's. The resulting graph $H'(G)$ is then r-regular. The remainder of the proof that G is 1-tough if and only if $H'(G)$ is t-tough is essentially the same as in the proof of Lemma 3.1. In particular, if $H'(G)$ is not t-tough, then clearly $H(G)$ is not t-tough. Thus G is not 1-tough as before. $\square$

4 Concluding Remarks

Conjecture 1.12 appears quite difficult, particularly when r is slightly larger than $2t$. In [3] we considered a special case of Conjecture 1.12; namely, that for any integer $t \geq 1$, t-TOUGH remains NP-hard for $(2t+1)$-regular graphs. We thought that a reasonable approach toward attacking this special case was to use the same approach as in the proof of Theorem 1.7 given in [3], except to interchange the roles of the m-joins and c-joins in the construction of $H(G)$. Note that this yields a graph $\hat{H}(G)$ that is $(2t+1)$-regular. Unfortunately, we recently found a 1-tough graph G' for which the graph $\hat{H}(G')$ constructed in this way is not t-tough.

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