A shift measure for the thickness of tails

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Abstract

By analogy with the hazard-rate, we propose to measure tail thickness of a cumulative distribution function $F(x)$ by means of a shift function $p(x) \geq 0$ defined by

$$f(x) = F(-x + p(x)),$$

where $f(x) = F'(x)$ is the density function. We show that in the case of the standard normal distribution $p(x)$ is asymptotic to $\frac{\ln x}{x}$, i.e. $\lim_{x\to\infty} \frac{\exp xp(x)}{x} = 1$

1. Introduction

Let $f(x) > 0$ be a positive, symmetric (density) differentiable function so that $f'(0) = 0$ (by the Mean Value Theorem) which is decreasing for $x > 0$ (to zero). The size of the tail

$$F(-x) = \int_{-\infty}^{-x} f(t)dt$$

by comparison to $f(x)$ ($= f(-x)$) may conveniently be measured by the function

$$h(x) = \frac{-f'(x)}{f(x)} \geq 0,$$

since by L'Hopital's rule

$$\lim_{x\to\infty} \frac{f(x)}{F(-x)} = \lim_{x\to\infty} \frac{-f'(x)}{f(x)}.$$

Evidently the density function may be recovered from $h(x)$ as

$$f(x) = f(0) \exp - \int_0^x h(t)dt.$$

Of course $h(0) = 0$.

Examples. We note that evidently the normal distribution is characterised by $h(x) = x$; asymptotic values for $h(x) \simeq \alpha x^{-1}$ ($\alpha > 0$) characterise distributions with tails asymptotic to $x^{-\alpha}$ (eg $f(x) = \frac{1}{\pi(1+x^2)}$), whereas asymptotic values for $h(x) \simeq k\alpha x^{\alpha-1}$ ($k, \alpha > 0$) characterise distributions with exponential asymptotes of $\exp -kx^\alpha$ (in effect the gamma distributions, since $\int_0^\infty \exp(-kx^\alpha)dx = k^{-1/\alpha}\Gamma(1+1/\alpha)$). The point of these examples is that $h(x)$ may be either increasing or decreasing.

To motivate an alternative measure of thickness we verify the following lemma which is routine but of interest to us as a starting point for a more involved argument.

Lemma. The hazard-rate function

$$\frac{f(x)}{F(-x)} = \frac{f(x)}{1 - F(x)}$$

is for $x \geq 0$ increasing/decreasing from $2f(0)$, according as $h(x)$ is increasing/decreasing.

Proof. Letting

$$G(x) = \frac{F(-x)}{f(x)}$$

we have

$$G'(x) = -1 + h(x)G(x),$$

so consider that if

$$g(x) = f(x) - h(x)F(-x) = -f(x) \cdot G'(x)$$

then

$$g'(x) = -h'(x)F(-x)$$

and also by l'Hopital's Rule

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} F(-x)\{\frac{f(x)}{F(-x)} - h(x)\} = 0.$$

Thus we have $g(x)$ is positive/negative if h is increasing/decreasing.

Corollary If $h(x)$ is increasing

$$h(x)F(-x) < f(x).$$

In particular

$$\Phi(-1) < \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}),$$

where as usual $\Phi(x)$ denotes the cumulative standard normal.

Proof. By the last argument

$$g(x) = f(x) - h(x)F(-x) > 0.$$

In the case of the normal distribution we have

$$F(-x) < f(x).$$

The following equation is therefore a valid defintion for a function.

Definition. The function $p(x) \geq 0$ such that

$$f(x) = F(-x + p(x))$$

is a measure of tail thickness. This definition may be rewritten as a functional equation for $p(x)$:

$$\{p'(x) - 1\} f(p(x) - x) = f'(x).$$

implying that $p' \leq 1$. In this note we prove that in the case of the normal distribution $p(x)$ is asymptotic to $\frac{\ln x}{x}$, i.e.

$$\lim_{x \to \infty} \frac{\exp xp(x)}{x} = 1.$$

To be precise we prove

Theorem 1 We have for $x > 1$

$$p(x) \leq \frac{\ln x}{x - 1}.$$

Theorem 2 We have for $x > 0$

$$p(x) \geq \frac{\ln x}{x}.$$

2. Proof of Theorem 1

We first showing the following:

Lemma If $q'(x) < 0$, $\lim_{x \to \infty} q(x) = 0$ and $h(x)f(x) < f(x - q(x))$ then

$$F(-x + q(x)) > f(x).$$

Proof Consider

$$H(x) = F(-x + q(x)) - f(x),$$

then

$$H'(x) = f(-x + q(x)) \cdot (-1 + q'(x)) - h(x)f(x).$$

But $q'(x) < 0$, so

$$H'(x) < h(x)f(x) - f(-x + q(x)) < 0,$$

i.e. $H(x)$ is decreasing; on the other hand

$$\lim_{x\to\infty} H(x) = 0$$

since $\lim_{x\to\infty} q(x) = 0$, hence for $x > 0$ we have $H(x) > 0$ and this proves the lemma.

Now put

$$q(x) = \frac{\ln x}{x - 1},$$

then we have:

Claim 1

$$q'(x) < 0.$$

Proof:

$$\begin{aligned} q'(x) &= \frac{1}{x(x-1)} + (-1)\frac{\ln x}{(x-1)^2} \\ &= \frac{1}{(x-1)^2}\{\frac{x-1}{x} - \ln x\} \\ &= \frac{1}{(x-1)^2}\{1 - \frac{1}{x} - \ln x\} \\ &< q'(1) = 0, \end{aligned}$$

since for $x \geq 1$

$$\frac{d}{dx}\{1 - \frac{1}{x} - \ln x\} = \frac{1}{x^2} - \frac{1}{x} = \frac{1}{x^2}(x - 1) > 0.$$

Claim 2

$$-f'(x) = h(x)f(x) < f(x - q(x)).$$

Proof We are to prove

$$x\exp(-\frac{1}{2}x^2) < \exp(-\frac{1}{2}(x-\frac{\ln x}{x-1})^2),$$

and this is equivalent to

$$-2\ln x + x^2 > x^2 - 2\frac{x\ln x}{x-1} + \left(\frac{\ln x}{x-1}\right)^2,$$

or, since $\ln x > 0$, to

$$2\left\{\frac{x}{x-1} - 1\right\} > \frac{\ln x}{(x-1)^2},$$

which (after multiplying by $(x-1)^2$) is the same as saying

$$2(x-1) > \ln x.$$

but it is easy to check this holds for $x > 1$ (i.e. we have $q(x) \leq 2$).

3. Proof of Theorem 2

This time we let

$$q(x) = \frac{\ln x}{x}$$

and consider

$$G(x) = \Phi(-x + q(x)) - \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}x^2).$$

One checks, since $\Phi(-x) < \frac{1}{x}f(x)$ (i.e. $h(x)F(-x) < f(x)$), that

$$\begin{aligned} G(1) &= F(-1) - f(1) < 0, \\ G(\infty) &= 0, \end{aligned}$$

and we can deduce that $G(x) < 0$ for $x > 1$ if we compute $G'(x)$ and show that it is positive for $x > 1$; this we do by referring to a related function $g(x)$ which has $g'(x)$ of constant negative sign. The latter signature is verified by showing that the equation $g'(x) = 0$ has no solution for $x > 1$ and checking the sign of $g'(1+)$.

This time for $x > 1$

$$q'(x) = \frac{1}{x^2} - \frac{\ln x}{x^2} = \frac{1}{x^2}(1 - \ln x)$$

and

$$q''(x) = -\frac{2}{x^3}(1 - \ln x) - \frac{1}{x^3} = \frac{-1}{x^3}(3 - 2\ln x),$$

and we still have

$$\lim_{x\to\infty} q(x) = \lim_{x\to\infty} q'(x) = 0.$$

Also we have

$$\lim_{x\to 0} q(x) = \lim_{x\to 0} q'(x) = \infty.$$

We observe that

$$G'(x) = xf(x) + f(-x+q)\{-1+q'\} > 0$$

if and only if

$$\ln x - \frac{1}{2}x^2 > -\frac{1}{2}(x-q)^2 + \ln\{1-q'\}$$

if and only if (since $\ln x = xq$)

$$\frac{1}{2}q^2 > \ln\{1-q'\}.$$

So consider

$$g(x) = \frac{1}{2}q^2 - \ln\{1-q'\}.$$

Notice that

$$g(\infty) = 0,$$

so subject to the following claim we have for $x > 1$ that $g(x) > 0$ and hence $G'(x) > 0$. We remark that $G'(0+) = 0$.

So we are left to show the following.

Claim

$$g'(x) = qq' + \frac{q''}{1-q'} < 0.$$

First notice that since $q(1) = 0, q'(1+) = 1 - \ln(1+), q''(1) = -3$ we have

$$g'(1+) = qq' + \frac{q''}{1-q'} = \frac{-3}{\ln(1+)} < 0.$$

So it is enough to show that the following is impossible for $x > 1$:

$$qq' = \frac{-q''}{1-q'},$$

or

$$\frac{\ln x}{x}\frac{1}{x^2}(1-\ln x) = \frac{\frac{1}{x^3}(3-2\ln x)}{1-\frac{1}{x^2}(1-\ln x)},$$

i.e.

$$\ln x(1-\ln x) = \frac{(3-2\ln x)}{1-\frac{1}{x^2}(1-\ln x)}.$$

Put $y = \ln x$ and we have

$$y(1-y) = \frac{(3-2y)}{1-e^{-2y}(1-y)}.$$

This may be rewritten as

$$e^{-2y} = -\frac{3-2y-y(1-y)}{y(1-y)^2} = \frac{3y-3-y^2}{y(1-y)^2} = -\frac{(y-3/2)^2+(3-9/4)}{y(1-y)^2} < 0$$

where $y = \ln x > 0$ and so there is a contradiction; hence we have our claim.

Finally we must argue separately for $x < 1$. Consider

$$t(x) = -x + q(x).$$

Since $q'(1) = 1$ and $q''(1) = -3$ the function $t(x)$ has a local maximum at $x = 1$. Indeed we have

$$\begin{aligned} t'(x) &= -1 + q'(x) \\ &= -1 + \frac{1}{x^2}(1-\ln x) \end{aligned}$$

which has signature change: $+, 0, -$ passing through $x = 1$; hence, since $F(x)$ is increasing in x

$$F(-1) = \max_{x>0} F(t(x)).$$

Thus for $x \leq 1$ we have, since $-f(x) \leq -f(-1)$

$$G(x) \leq F(-1) - f(1) < 0.$$