CONTRACTIBILITY OF THE MAXIMAL IDEAL SPACE OF ALGEBRAS OF MEASURES IN A HALF-SPACE

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Abstract. Let $\mathbb{H}^{[n]}$ be the canonical half space in $\mathbb{R}^n$, that is,

$$\mathbb{H}^{[n]} = \{(t_1, \dots, t_n) \in \mathbb{R}^n \setminus \{0\} \mid \forall j,\ [t_j \neq 0 \text{ and } t_1 = t_2 = \cdots = t_{j-1} = 0] \Rightarrow t_j > 0\} \cup \{0\}.$$

Let $\mathcal{M}(\mathbb{H}^{[n]})$ denote the Banach algebra of all complex Borel measures with support contained in $\mathbb{H}^{[n]}$, with the usual addition and scalar multiplication, and with convolution $*$, and the norm being the total variation of μ. It is shown that the maximal ideal space $X(\mathcal{M}(\mathbb{H}^{[n]}))$ of $\mathcal{M}(\mathbb{H}^{[n]})$, equipped with the Gelfand topology, is contractible as a topological space. In particular, it follows that $\mathcal{M}(\mathbb{H}^{[n]})$ is a projective free ring. In fact, for all subalgebras R of $\mathcal{M}(\mathbb{H}^{[n]})$ that satisfy a certain mild condition, it is shown that the maximal ideal space $X(R)$ of R is contractible. Several examples of such subalgebras are also given.

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1. Introduction

The aim of this paper is to show that the maximal ideal space $X(R)$ of some Banach subalgebras (possessing a certain mild property) of the convolution algebra $\mathcal{M}(\mathbb{H}^{[n]})$ of all complex Borel measures with support in the half space $\mathbb{H}^{[n]}$, is contractible. It follows then that such Banach algebras are projective free rings. All the notation and precise definitions are explained below.

In particular, our result can be viewed as a two-fold generalization:

(1) of the result in [9], from the *one* dimensional case (of the half space $[0, +\infty)$ of $\mathbb{R}$) to the n-dimensional case (the half space $\mathbb{H}^{[n]}$ of $\mathbb{R}^n$).
(2) of the result in [8], from the *specific* subalgebra of almost periodic measures of $\mathcal{M}(\mathbb{H}^{[n]})$ to all subalgebras of $\mathcal{M}(\mathbb{H}^{[n]})$ satisfying a certain condition. (The result in [8] was in turn a generalization of a *one*-dimensional result of A. Brudnyi [2] to the *multi*-dimensional setting.)

Although the current result is a generalization of the result from the conference paper [9], it does not follow automatically.

Definition 1.1. Let $\mathbb{H}^{[n]} \subset \mathbb{R}^n$ be the *canonical half space* defined by

$$\mathbb{H}^{[n]} = \{(t_1, \dots, t_n) \in \mathbb{R}^n \setminus \{0\} \mid \forall j,\ [t_1 = t_2 = \cdots = t_{j-1} = 0,\ t_j \neq 0] \Rightarrow t_j > 0\} \cup \{0\}.$$

$\mathcal{M}(\mathbb{H}^{[n]})$ denotes the set of all complex Borel measures with support contained in $\mathbb{H}^{[n]}$. Then $\mathcal{M}(\mathbb{H}^{[n]})$ is a complex vector space with addition and scalar multiplication defined in the pointwise manner as usual. The space $\mathcal{M}(\mathbb{H}^{[n]})$ becomes a complex algebra if convolution

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of measures (denoted henceforth by $*$) is taken as the operation of multiplication in the algebra. With the norm of μ taken as the total variation of μ, $\mathcal{M}(\mathbb{H}^{[n]})$ is a Banach algebra. Recall that the *total variation* $\|\mu\|$ *of* μ is defined by

$$\|\mu\| = \sup \sum_{k=1}^{\infty} |\mu(E_k)|,$$

the supremum being taken over all *partitions of* $\mathbb{H}^{[n]}$, that is over all countable collections $(E_k)_{k\in\mathbb{N}}$ of Borel subsets of $\mathbb{H}^{[n]}$ such that $E_k \bigcap E_m = \emptyset$ whenever $m \neq k$ and $\bigcup_{k\in\mathbb{N}} E_k = \mathbb{H}^{[n]}$. The identity with respect to convolution in $\mathcal{M}(\mathbb{H}^{[n]})$ is the *Dirac measure* δ_0^n in $\mathbb{R}^n$ supported at 0, given by

$$\delta_0^n(E) = \begin{cases} 1 & \text{if } 0 \in E, \\ 0 & \text{if } 0 \notin E, \end{cases}$$

where E is any Borel subset of $\mathbb{H}^{[n]}$.

Definition 1.2. For $\mu \in \mathcal{M}(\mathbb{H}^{[n]})$, define the measures $\mu^{[k]} \in \mathcal{M}(\mathbb{H}^{[k]})$, $k = n, n-1, \dots, 2, 1$, inductively as follows. Set $\mu^{[n]} = \mu$. Suppose $\mu^{[k]} \in \mathcal{M}(\mathbb{H}^{[k]})$ has been defined. Then $\mu^{[k-1]} \in \mathcal{M}(\mathbb{H}^{[k-1]})$ is defined by

$$\mu^{[k-1]}(E) = \mu(\{0\} \times E),$$

where E is any Borel subset of $\mathbb{H}^{[k-1]}$.

Given $\theta \in [0, 1)$ and $\mu \in \mathcal{M}(\mathbb{H}^{[k]})$, the measure $\mu_\theta \in \mathcal{M}(\mathbb{H}^{[k]})$ is defined by

$$\mu_\theta(E) = \int_E (1-\theta)^{t_1} d\mu(t), \tag{1}$$

where E is any Borel subset of $\mathbb{H}^{[k]}$. If $\theta = 1$, and $k > 1$, then

$$\mu_1 := \delta_0^1 \otimes \mu^{[k-1]},$$

while if $k = 1$, then set $\mu_1 = \mu(\{0\})\delta_0^1$.

Notation 1.3. If R is a complex commutative unital Banach algebra, then $X(R)$ denotes the maximal ideal space of R. Thus $X(R)$ is the set of all nonzero complex homomorphisms from R to $\mathbb{C}$. $X(R)$ is endowed with the *Gelfand topology*, that is, the weak-$*$ topology induced from the dual space $\mathcal{L}(R; \mathbb{C})$ of the Banach space R.

If R is any Banach subalgebra of $\mathcal{M}(\mathbb{H}^{[n]})$ which satisfies a mild assumption, namely Property (P) in Theorem 1.5 below, then we will show that $X(R)$ is contractible. The notion of contractibility of a topological space is recalled below.

Definition 1.4. A topological space X is said to be *contractible* if there exists a continuous map $H : X \times [0, 1] \to X$ and an $x_0 \in X$ such that for all $x \in X$, $H(x, 0) = x$ and $H(x, 1) = x_0$.

Our main result is the following:

Theorem 1.5. *Suppose that R is a Banach subalgebra of $\mathcal{M}(\mathbb{H}^{[n]})$ satisfying the property*

(P) For all $\mu \in R$ and all $\theta \in [0, 1]$, $\quad \mu_\theta,\ \delta_0^1 \otimes \mu_\theta^{[n-1]},\ \dots,\ \delta_0^{n-1} \otimes \mu_\theta^{[1]} \in R.$

Then the maximal ideal space $X(R)$ equipped with the Gelfand topology is contractible.

In particular, by a result proved in [3], the above implies that R is a projective free ring. The definition of a projective free ring is given below.

Definition 1.6. A commutative ring R with identity is said to be *projective free* if every finitely generated projective R-module is free. Recall that if M is an R-module, then

(1) M is *free* if $M \cong R^d$ for some integer $d \geq 0$;

(2) M is *projective* if there is an R-module N and an integer $d \geq 0$ such that $M \oplus N \cong R^d$.

In terms of matrices (with entries from R), the ring R is projective free iff for every square matrix P satisfying $P^2 = P$, there exists an invertible matrix G such that

$$GPG^{-1} = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix};$$

see [4, Proposition 2.6].

For example, it can be seen from the matricial definition that any field $\mathbb{F}$ is projective free, since matrices P satisfying $P^2 = P$ are diagonalizable over $\mathbb{F}$. Quillen and Suslin independently proved, that the polynomial ring over a projective free ring is again projective free (see [5]), and so in particular, the polynomial ring $\mathbb{F}[x_1, \dots, x_n]$ is projective free, settling Serre's conjecture from 1955. In the context of Banach algebras, the following result was shown recently [3, Corollary 1.4.(1)]:

Proposition 1.7. *Let R be a semisimple complex commutative unital Banach algebra. If the maximal ideal space $X(R)$ (equipped with the Gelfand topology) of the Banach algebra R is contractible, then R is a projective free ring.*

Recall that a commutative unital Banach algebra is said to be *semisimple* if its *radical* (that is, the intersection of all maximal ideals) is 0.

Proposition 1.8. *Every Banach subalgebra R of $\mathcal{M}(\mathbb{H}^{[n]})$ is semisimple.*

This will be proved at the end of Section 2. In light of Proposition 1.7, the main result given in Theorem 1.5 then implies the following.

Corollary 1.9. *Let R be a Banach subalgebra of $\mathcal{M}(\mathbb{H}^{[n]})$ satisfying the property* (P) *from Theorem 1.5. Then R is projective free.*

At the end of this article, we give examples of subalgebras R of $\mathcal{M}(\mathbb{H}^{[n]})$ which satisfy the property (P), which include several well-known classical convolution algebras of measures. Thus we have (with the notation explained in Section 4):

Corollary 1.10. *Let R be one of the Banach algebras $L^1(\mathbb{H}^{[n]}) + \mathbb{C}\delta_0^n$, $\mathcal{A}(\mathbb{H}^{[n]})$, APW_Σ^n or AP_Σ^n. Then the maximal ideal space $X(R)$ is contractible. In particular, R is projective free.*

The motivation for investigating whether or not convolution algebras of measures are projective free rings also arises from control theory, in the problem of stabilization of linear systems, since if R is a projective free ring, then every stabilizable plant with a transfer function over the field of fractions of R has a doubly coprime factorization. The reader is referred to [7], [3] for details.

The proof of Theorem 1.5 is given in Section 3, while examples are given in Section 4. But first, a few technical results used in the sequel are proved in Section 2.

2. Preliminaries

In this section, a few auxiliary facts needed to prove the main result are shown.

Lemma 2.1. *Let $\mu \in \mathcal{M}(\mathbb{H}^{[k]})$. Then*

(1) $\mu_\theta \in \mathcal{M}(\mathbb{H}^{[k]})$.
(2) $\|\mu_\theta\| \leq \|\mu\|$.
(3) $(\delta_0^k)_\theta = \delta_0^k$ *for all* $\theta \in [0,1]$ *and all* $k = 1, \dots, n$.

Proof. (1) and (3) follow immediately from the definitions. The inequality in (2) is shown below. Note that $\|\mu_\theta\| = \sup \sum |\mu_\theta(E_i)|$, the supremum being taken over all partitions $(E_i)_{i\in\mathbb{N}}$ of $\mathbb{H}^{[k]}$. There exists a Borel measurable function w such that $d|\mu|(t) = e^{iw(t)}d\mu(t)$. So

$$\begin{aligned} |\mu_\theta(E_i)| &= \left|\int_{E_i}(1-\theta)^{t_1}d\mu(t)\right| = \left|\int_{E_i} e^{-iw(t)}(1-\theta)^{t_1}e^{iw(t)}d\mu(t)\right| \\ &= \left|\int_{E_i} e^{-iw(t)}(1-\theta)^{t_1}d|\mu|(t)\right| \leq \int_{E_i} 1d|\mu|(t) = |\mu|(E_i). \end{aligned}$$

Hence $\sum |\mu_\theta(E_i)| \leq \sum |\mu|(E_i) = |\mu|(\mathbb{H}^{[k]}) = \|\mu\|$. $\square$

Lemma 2.2. *If* $\mu, \nu \in \mathcal{M}(\mathbb{H}^{[k+1]})$ *and* $k \geq 1$, *then* $(\mu * \nu)^{[k]} = \mu^{[k]} * \nu^{[k]}$.

Proof. Let $E \subset \mathbb{H}^{[k]}$ be a Borel set. Then

$$\begin{aligned} (\mu*\nu)^{[k]}(E) &= (\mu*\nu)(\{0\}\times E) = \int_{\{0\}\times E} \mu((\{0\}\times E)-t)d\nu(t) \\ &= \int_{\{0\}\times E} \mu(\{0\}\times(E-\tau))d\nu^{[k]}(\tau) \\ &= \int_E \mu^{[k]}(E-\tau)d\nu^{[k]}(\tau) = (\mu^{[k]}*\nu^{[k]})(E). \end{aligned}$$

This completes the proof. $\square$

Lemma 2.3. *If* $\mu, \nu \in \mathcal{M}(\mathbb{H}^{[k+1]})$ *where* $k \geq 1$, *then* $(\delta_0^1 \otimes \mu^{[k]}) * (\delta_0^1 \otimes \nu^{[k]}) = \delta_0^1 \otimes (\mu^{[k]} * \nu^{[k]})$.

Proof. (The notation $\mathcal{F}\mu$ is used for the Fourier transform of μ: $(\mathcal{F}\mu)(w) = \int e^{iwt}d\mu(t)$, $w \in \mathbb{R}$). For $w_1 \in \mathbb{R}$ and $\omega \in \mathbb{R}^k$,

$$\begin{aligned} \mathcal{F}((\delta_0^1\otimes\mu^{[k]})*(\delta_0^1\otimes\nu^{[k]}))(w_1,\omega) &= (\mathcal{F}(\delta_0^1\otimes\mu^{[k]}))(w_1,\omega)\cdot(\mathcal{F}(\delta_0^1\otimes\nu^{[k]}))(w_1,\omega) \\ &= (\mathcal{F}\delta_0^1)(w_1)\cdot(\mathcal{F}\mu^{[k]})(\omega)\cdot(\mathcal{F}\delta_0^1)(w_1)\cdot(\mathcal{F}\nu^{[k]})(\omega) \\ &= 1\cdot(\mathcal{F}\mu^{[k]})(\omega)\cdot 1\cdot(\mathcal{F}\nu^{[k]})(\omega) = (\mathcal{F}\mu^{[k]})(\omega)\cdot(\mathcal{F}\nu^{[k]})(\omega) \\ &= (\mathcal{F}(\mu^{[k]}*\nu^{[k]}))(\omega) = 1\cdot(\mathcal{F}(\mu^{[k]}*\nu^{[k]}))(\omega) \\ &= (\mathcal{F}\delta_0^1)(w_1)\cdot(\mathcal{F}(\mu^{[k]}*\nu^{[k]}))(\omega) \\ &= (\mathcal{F}(\delta_0^1\otimes(\mu^{[k]}*\nu^{[k]})))(w_1,\omega). \end{aligned}$$

Taking the inverse Fourier transform, the claim follows. $\square$

Proposition 2.4. *If* $\mu, \nu \in \mathcal{M}(\mathbb{H}^{[k]})$, *then for all* $\theta \in [0,1]$, $(\mu * \nu)_\theta = \mu_\theta * \nu_\theta$.

Proof. Let us first suppose that $\theta \in [0,1)$. If E is a Borel subset of $\mathbb{H}$, then

$$(\mu*\nu)_\theta(E) = \int_E (1-\theta)^{t_1}d(\mu*\nu)(t) = \iint_{\substack{\sigma+\tau\in E\\ \sigma,\tau\in\mathbb{H}^{[k]}}} (1-\theta)^{\sigma_1+\tau_1}d\mu(\sigma)d\nu(\tau).$$

On the other hand,

$$\begin{aligned}(\mu_\theta * \nu_\theta)(E) &= \int_{\tau\in\mathbb{H}^{[k]}} \mu_\theta(E-\tau)d\nu_\theta(\tau) = \int_{\tau\in\mathbb{H}^{[k]}} \left(\int_{\substack{\sigma\in E-\tau \\ \sigma\in\mathbb{H}^{[k]}}} (1-\theta)^{\sigma_1} d\mu(\sigma)\right) d\nu_\theta(\tau) \\ &= \iint_{\substack{\sigma+\tau\in E \\ \sigma,\tau\in\mathbb{H}^{[k]}}} (1-\theta)^{\sigma_1+\tau_1} d\mu(\sigma)d\nu(\tau).\end{aligned}$$

Now consider the case when $\theta = 1$. If $k = 1$, the claim follows immediately, since

$$(\mu*\nu)_1 = (\mu*\nu)(\{0\})\delta_0^1 = \mu(\{0\})\cdot\nu(\{0\})\delta_0^1 = (\mu(\{0\})\delta_0^1)*(\nu(\{0\})\delta_0^1) = \mu_1*\nu_1.$$

If $k > 1$, then

$$\mu_1*\nu_1 = (\delta_0^1\otimes\mu^{[k-1]})*(\delta_0^1\otimes\nu^{[k-1]}) = \delta_0^1\otimes(\mu^{[k-1]}*\nu^{[k-1]}) = \delta_0^1\otimes(\mu*\nu)^{[k-1]} = (\mu*\nu)_1.$$

This completes the proof. □

The following result says that for a fixed $\mu\in\mathcal{M}(\mathbb{H}^{[k]})$, the map $\theta\mapsto\mu_\theta : [0,1]\to\mathcal{M}(\mathbb{H}^{[k]})$ is continuous.

Proposition 2.5. *If $\mu\in\mathcal{M}(\mathbb{H}^{[k]})$ and $\theta_0\in[0,1]$, then $\lim\limits_{\theta\to\theta_0}\mu_\theta = \mu_{\theta_0}$ in $\mathcal{M}(\mathbb{H}^{[k]})$.*

Proof. $\underline{1}^\circ$ Consider first the case when $\theta_0\in[0,1)$. Given an $\epsilon>0$, first choose an $R>0$ large enough so that $|\mu|(B)<\epsilon$, where $B=\{t\in\mathbb{R}^k \mid \|t\|_2\le R\}$. Let $\theta\in[0,1)$. There exists a Borel measurable function w such that $d(\mu_\theta-\mu_{\theta_0})(t) = e^{-iw(t)}d|\mu_\theta-\mu_{\theta_0}|(t)$. Thus

$$\begin{aligned}\|\mu_\theta-\mu_{\theta_0}\| &= |\mu_\theta-\mu_{\theta_0}|(\mathbb{H}^{[k]}) = \int_{\mathbb{H}^{[k]}} e^{iw(t)}d(\mu_\theta-\mu_{\theta_0})(t) \\ &= \left|\int_{\mathbb{H}^{[k]}} e^{iw(t)}d(\mu_\theta-\mu^{\theta_0})(t)\right| = \left|\int_{\mathbb{H}^{[k]}} e^{iw(t)}((1-\theta)^{t_1}-(1-\theta_0)^{t_1})d\mu(t)\right|.\end{aligned}$$

Hence

$$\begin{aligned}\|\mu_\theta-\mu_{\theta_0}\| &\le \left|\int_{B\cap\mathbb{H}^{[k]}} e^{iw(t)}((1-\theta)^{t_1}-(1-\theta_0)^{t_1})d\mu(t)\right| + \left|\int_{\mathbb{H}^{[k]}\setminus B} e^{iw(t)}((1-\theta)^{t_1}-(1-\theta_0)^{t_1})d\mu(t)\right| \\ &\le \left(\max_{t\in B\cap\mathbb{H}^{[k]}} \left|(1-\theta)^{t_1}-(1-\theta_0)^{t_1}\right|\right)|\mu|(B) + 2|\mu|(\mathbb{H}^{[k]}\setminus B) \\ &\le \left(\max_{t\in B\cap\mathbb{H}^{[k]}} \left|(1-\theta)^{t_1}-(1-\theta_0)^{t_1}\right|\right)|\mu|(\mathbb{H}^{[k]}) + 2\epsilon.\end{aligned}$$

But by the mean value theorem applied to the function $\theta\mapsto(1-\theta)^{t_1}$,

$$(1-\theta)^{t_1}-(1-\theta_0)^{t_1} = (\theta-\theta_0)\cdot t_1\cdot(1-c)^{t_1-1} = (\theta-\theta_0)\cdot t_1\cdot\frac{(1-c)^{t_1}}{1-c},$$

for some c (depending on $t=t_1$, θ and θ_0) in between θ and θ_0. Since c lies between θ and θ_0, and since both θ and θ_0 lie in $[0,1)$, and $0\le t_1\le R$, it follows that $(1-c)^{t_1}\le 1$ and

$$\frac{1}{1-c}\le\max\left\{\frac{1}{1-\theta},\frac{1}{1-\theta_0}\right\}.$$

Thus using the above, and the fact that $0 \le t_1 \le R$,

$$\begin{aligned}\max_{t\in B\cap\mathbb{H}^{[k]}} \left|(1-\theta)^{t_1}-(1-\theta_0)^{t_1}\right| &= \max_{t\in B\cap\mathbb{H}^{[k]}} |\theta-\theta_0|\cdot|t_1|\cdot|(1-c)^{t_1}|\cdot\frac{1}{|1-c|}\\ &\le |\theta-\theta_0|\cdot R\cdot 1\cdot\max\left\{\frac{1}{1-\theta},\frac{1}{1-\theta_0}\right\}.\end{aligned}$$

Hence

$$\begin{aligned}\limsup_{\theta\to\theta_0}\|\mu_\theta-\mu_{\theta_0}\| &\le \limsup_{\theta\to\theta_0}\left(\left(\max_{t\in B\cap\mathbb{H}^{[k]}}\left|(1-\theta)^{t_1}-(1-\theta_0)^{t_1}\right|\right)|\mu|(\mathbb{H}^{[k]})+2\epsilon\right)\\ &\le \limsup_{\theta\to\theta_0}\left(|\theta-\theta_0|\cdot R\cdot\max\left\{\frac{1}{1-\theta},\frac{1}{1-\theta_0}\right\}\cdot|\mu|(\mathbb{H}^{[k]})\right)+2\epsilon\\ &= 0\cdot R\cdot\frac{1}{1-\theta_0}|\mu|(\mathbb{H}^{[k]})+2\epsilon=0+2\epsilon=2\epsilon.\end{aligned}$$

Since $\epsilon>0$ was arbitrary, it follows that $\limsup\limits_{\theta\to\theta_0}\|\mu_\theta-\mu_{\theta_0}\|=0$. Also $\|\mu_\theta-\mu_{\theta_0}\|\ge 0$, and so $\lim\limits_{\theta\to\theta_0}\|\mu_\theta-\mu_{\theta_0}\|=0$.

$\underline{2}^\circ$ Now consider the case when $\theta_0=1$. Assume for the moment that $k>1$ and $\mu^{[k-1]}=0$. We will show that $\lim\limits_{\theta\to 1}\mu_\theta=0$ in $\mathcal{M}(\mathbb{H}^{[k]})$. Given an $\epsilon>0$, first choose a $r>0$ small enough so that $|\mu|(B)<\epsilon$, where $B=\{t\in\mathbb{R}^k \mid \|t\|_2\le r\}$. (This is possible, since $\mu^{[k-1]}=0$.) There exists a Borel measurable function w such that $d\mu_\theta(t)=e^{-iw(t)}d|\mu_\theta|(t)$. Thus

$$\begin{aligned}\|\mu_\theta\| &= |\mu_\theta|(\mathbb{H}^{[k]})=\int_{\mathbb{H}^{[k]}}e^{iw(t)}d\mu_\theta(t)=\int_{\mathbb{H}^{[k]}}e^{iw(t)}(1-\theta)^{t_1}d\mu(t)=\left|\int_{\mathbb{H}^{[k]}}e^{iw(t)}(1-\theta)^{t_1}d\mu(t)\right|\\ &\le \left|\int_{B\cap\mathbb{H}^{[k]}}e^{iw(t)}(1-\theta)^{t_1}d\mu(t)\right|+\left|\int_{\mathbb{H}^{[k]}\setminus B}e^{iw(t)}(1-\theta)^{t_1}d\mu(t)\right|\\ &\le |\mu|(B)+(1-\theta)^r\cdot|\mu|(\mathbb{H}^{[k]}\setminus B)\le\epsilon+(1-\theta)^r\cdot|\mu|(\mathbb{H}^{[k]}).\end{aligned}$$

Consequently, $\limsup\limits_{\theta\to 1}\|\mu_\theta-\mu_{\theta_0}\|\le\epsilon$. But $\epsilon>0$ was arbitrary, and so $\limsup\limits_{\theta\to 1}\|\mu_\theta\|=0$. Since $\|\mu_\theta\|\ge 0$, it follows that $\lim\limits_{\theta\to 1}\|\mu_\theta\|=0$.

If $\mu_{k-1}\ne 0$, then define $\nu:=\mu-\delta_0^1\otimes\mu^{[k-1]}\in\mathcal{M}(\mathbb{H}^{[k]})$. It is clear that $\nu^{[k-1]}=0$ and $\nu_\theta=\mu_\theta-\delta_0^1\otimes\mu^{[k-1]}$. From the above, $\lim\limits_{\theta\to 1}\nu_\theta=0$, and so $\lim\limits_{\theta\to 1}\mu_\theta=\delta_0^1\otimes\mu^{[k-1]}=\mu_1$ in $\mathcal{M}(\mathbb{H}^{[k]})$.

$\underline{3}^\circ$ The case when $\theta_0=1$ and $k=1$ is analogous to 2° above. ☐

Finally we prove that every Banach subalgebra R of $\mathcal{M}(\mathbb{H}^{[n]})$ is semisimple.

Proof of Proposition 1.8. If $s\in\mathbb{C}$, $\mathrm{Re}(s)\ge 0$, and $k\in\{1,\dots,n\}$, then $\Phi_s^{[k]}$, given by

$$\Phi_s^{[k]}(\mu)=\int_{\{t\ \mid\ t=(0,\tau)\in\mathbb{R}^k\times\mathbb{H}^{[n-k]}\}}e^{-st_k}d\mu(t)\qquad(\mu\in R),$$

is an element of $X(R)$, and so the kernel of $\Phi_s^{[k]}$ is a maximal ideal in R. But if $\Phi_s^{[k]}(\mu)=0$ for all s and all k, then μ is zero on $\mathbb{H}^{[n]}$. So the radical of R is 0. ☐

3. Contractibility of $X(R)$

In this section we will prove our main result.

Proof of Theorem 1.5. Define $H : X(R) \times [0,1] \to X(R)$ as follows. If $\theta \in [0,1]$, $\Phi \in X(R)$ and $\mu \in R$, then

$$(H(\Phi,\theta))(\mu) = \begin{cases} \Phi(\mu_{n\theta}) & 0 \le \theta < \frac{1}{n}, \\ \Phi(\delta_0^k \otimes \mu_{n\theta-k}^{[n-k]}) & \frac{k}{n} \le \theta < \frac{k+1}{n}, \quad k = 1,\dots,n-1, \\ \Phi(\mu(\{0\})\delta_0^n) = \mu(\{0\}) & \theta = 1. \end{cases}$$

We show that H is well-defined. From the definition, $H(\Phi,1) \in X(R)$ for all $\Phi \in X(R)$. If $\theta \in [0,1)$, then the linearity of $H(\Phi,\theta) : R \to \mathbb{C}$ is obvious. Continuity of $H(\Phi,\theta)$ follows from the fact that Φ is continuous and $\|\mu_\theta\| \le \|\mu\|$ for $\theta \in [0,1]$. That $H(\Phi,\theta)$ is multiplicative is a consequence of Proposition 2.4, and the fact that Φ respects multiplication. Finally $(H(\Phi,\theta))(\delta_0^n) = \Phi((\delta_0^n)_\theta) = \Phi(\delta_0^n) = 1$.

It is obvious that $H(\cdot,0)$ is the identity map and $H(\cdot,1)$ is a constant map.

Finally, we show below that H is continuous. Since $X(\mathcal{M}(\mathbb{H}^{[n]}))$ is equipped with the Gelfand topology, we just have to prove that for every convergent net $(\Phi_i,\theta_i)_{i\in I}$ with limit (Φ,θ) in $X(\mathcal{M}(\mathbb{H}^{[n]})) \times [0,1]$, there holds that $(H(\Phi_i,\theta_i))(\mu) \to (H(\Phi,\theta))(\mu)$. We have

$$|(H(\Phi_i,\theta_i))(\mu) - (H(\Phi,\theta))(\mu)| \le |(H(\Phi_i,\theta_i))(\mu) - (H(\Phi_i,\theta))(\mu)| + |(H(\Phi_i,\theta) - (H(\Phi,\theta))(\mu)|,$$

and from the definition of H, it is immediate that $|(H(\Phi_i,\theta) - (H(\Phi,\theta))(\mu)| \to 0$. So it remains to show that $|(H(\Phi_i,\theta_i))(\mu) - (H(\Phi_i,\theta))(\mu)| \to 0$. There is no loss of generality in assuming that all the θ_i's belong to one of the intervals $\left[0,\frac{1}{n}\right)$, $\left[\frac{1}{n},\frac{2}{n}\right)$, $\dots$, $\left[\frac{n-1}{n},1\right)$. But then Proposition 2.5 yields the desired result: for example if $\theta_i \in [\frac{k}{n},\frac{k+1}{n})$ and $\theta = \frac{k+1}{n}$, then

$$\begin{aligned} |(H(\Phi_i,\theta_i))(\mu) - (H(\Phi_i,\theta))(\mu)| &= |\Phi_i(\delta_0^k \otimes \mu_{n\theta_i-k}^{[n-k]} - \delta_0^k \otimes (\delta_0^1 \otimes \mu^{[n-k-1]}))| \\ &\le \|\Phi_i\| \cdot \|\delta_0^k\| \cdot \|\mu_{n\theta_i-k} - \delta_0^1 \otimes \mu^{[n-k-1]}\| \\ &\le 1 \cdot 1 \cdot \|\mu_{n\theta_i-k} - \delta_0^1 \otimes \mu^{[n-k-1]}\| \to 0. \end{aligned}$$

This completes the proof. □

Our definition of the map H is based on the following consideration, in the case of $n=1$, when $\mathbb{H}^{[n]} = \mathbb{H}^{[1]} = [0,+\infty)$. The result given below can be thought of as a generalization of the Riemann-Lebesgue Lemma for functions $f_a \in L^1(0,+\infty)$ (that the limit as $s \to +\infty$ of the Laplace transform of f_a is 0):

Proposition 3.1. *If $\mu \in \mathcal{M}(\mathbb{H}^{[1]})$, then* $\displaystyle\lim_{s\to+\infty} \int_0^{+\infty} e^{-st} d\mu(t) = \mu(\{0\})$.

The set $X(\mathcal{M}(\mathbb{H}^{[1]}))$ contains the half plane $\mathbb{C}_{\ge 0} := \{s \in \mathbb{C} \mid \mathrm{Re}(s) \ge 0\}$ in the sense that each $s \in \mathbb{C}_{\ge 0}$, gives rise to the corresponding complex homomorphism $\Phi_s : \mathcal{M}(\mathbb{H}^{[1]}) \to \mathbb{C}$, given simply by point evaluation of the Laplace transform of μ at s:

$$\mu \mapsto \Phi_s(\mu) = \int_0^{+\infty} e^{-st} d\mu(t), \quad \mu \in \mathcal{M}(\mathbb{H}^{[1]}).$$

If we imagine s tending to $+\infty$ along the real axis we see from Proposition 3.1, that Φ_s starts looking more and more like the complex homomorphism $\Phi_{+\infty}$ given by

$$\mu \mapsto \Phi_{+\infty}(\mu) := \mu(\{0\}), \quad \mu \in \mathcal{M}(\mathbb{H}^{[1]}).$$

So we may define $H(\Phi_s, \theta) = \Phi_{s-\log(1-\theta)}$, which would suggest that at least the part $\mathbb{C}_{\geq 0}$ of $X(\mathcal{M}(\mathbb{H}^{[1]}))$ is contractible to $\Phi_{+\infty}$. But we see that we can view the action of $H(\Phi_s, \theta)$ defined above as follows:

$$(H(\Phi_s, \theta))(\mu) = \Phi_{s-\log(1-\theta)}(\mu) = \int_0^{+\infty} e^{-(s-\log(1-\theta))t} d\mu(t) = \int_0^{+\infty} e^{-st}(1-\theta)^t d\mu(t) = \Phi_s(\nu),$$

where ν is the measure such that $d\nu(t) = (1-\theta)^t d\mu(t)$. This motivates the definition of μ_θ given in (1), and the definition of H in the proof of Theorem 1.5.

4. Examples

As specific examples of R in Theorem 1.5, we have the following:

4.1. **The algebra $L^1(\mathbb{H}^{[n]})+\mathbb{C}\delta_0^n$.** Consider the Banach subalgebra $L^1(\mathbb{H}^{[n]})+\mathbb{C}\delta_0^n$ of $\mathcal{M}(\mathbb{H}^{[n]})$, consisting of all complex Borel measures of the type $\mu_a + \alpha\delta_0^n$, where μ_a is absolutely continuous (with respect to the Lebesgue measure) and $\alpha \in \mathbb{C}$. It can be checked that this Banach subalgebra of $\mathcal{M}(\mathbb{H}^{[n]})$ has the property (P) in the statement of Theorem 1.5.

4.2. **The algebra $\mathcal{A}(\mathbb{H}^{[n]})$.** The Banach subalgebra $\mathcal{A}(\mathbb{H}^{[n]})$ of $\mathcal{M}(\mathbb{H}^{[n]})$ consists of all complex Borel measures that do not have a singular non-atomic part. Then it can be verified that $\mathcal{A}(\mathbb{H}^{[n]})$ also possesses the property (P). (So in the case when $n = 1$, we recover the main result in [10], but this time without recourse to the explicit description of the maximal ideal space.)

4.3. **Algebras of almost periodic functions.** The algebra AP^n of complex valued (uniformly) almost periodic functions is, by definition, the smallest closed subalgebra of $L^\infty(\mathbb{R}^n)$ (with all operations defined pointwise), that contains all the functions $e_\lambda(x) := e^{i\langle\lambda,x\rangle}$. Here the variable $x = (x_1, \dots, x_n)$, the parameter $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, and $\langle\lambda, x\rangle := \sum_{k=1}^n \lambda_k x_k$. For any $f \in AP^n$, its *Bohr-Fourier series* is defined by the formal sum $\sum_\lambda f_\lambda e^{i\langle\lambda,x\rangle}$ ($x \in \mathbb{R}^n$), where

$$f_\lambda := \lim_{N\to\infty} \frac{1}{(2N)^n} \int_{[-N,N]^n} e^{-i\langle\lambda,x\rangle} a(x) dx, \quad \lambda \in \mathbb{R}^n,$$

and the sum $\sum_\lambda f_\lambda e^{i\langle\lambda,x\rangle}$ is taken over the set $\sigma(f) := \{\lambda \in \mathbb{R}^n \mid f_\lambda \neq 0\}$, called the *Bohr-Fourier spectrum* of f. The Bohr-Fourier spectrum of every $f \in AP^n$ is at most a countable set.

The *almost periodic Wiener algebra* APW^n is defined as the set of all AP^n such that the Bohr-Fourier series $\sum_\lambda f_\lambda e^{i\langle\lambda,x\rangle}$ of f converges absolutely. The almost periodic Wiener algebra is a Banach algebra with pointwise operations and the norm $\|f\| := \sum_{\lambda\in\mathbb{R}^n} |f_\lambda|$. Let Δ be a nonempty subset of $\mathbb{R}^n$. Denote

$$\begin{aligned} AP^n_\Delta &= \{f \in AP^n \mid \sigma(f) \subset \Delta\} \\ APW^n_\Delta &= \{f \in APW^n \mid \sigma(f) \subset \Delta\}. \end{aligned}$$

If Δ is an additive subset of $\mathbb{R}^n$, then AP^n_Δ (respectively APW^n_Δ) is a Banach subalgebra of AP^n (respectively APW^n). Moreover, if $0 \in \Delta$, then AP^n_Δ and APW^n_Δ are also unital.

Let $\Sigma \subset \mathbb{H}^{[n]}$ be an *additive semigroup* (if $\lambda, \mu \in \Sigma$, then $\lambda + \mu \in \Sigma$) and suppose $0 \in \Sigma$. The Banach algebra APW_Σ^n is isomorphic to the following Banach subalgebra R of $\mathcal{M}(\mathbb{H}^{[n]})$:

$$R = \left\{ \sum_\lambda f_\lambda \delta_0^n(\lambda) \;\middle|\; \sum_\lambda f_\lambda e^{i\langle \lambda, x\rangle} \in APW_\Sigma^n \right\}.$$

In the above, $\delta_0^n(\lambda) \in \mathcal{M}(\mathbb{H}^{[n]})$ denotes the Dirac measure supported at $\lambda \in \mathbb{H}^{[n]}$, that is,

$$(\delta_0^n(\lambda))(E) = \begin{cases} 1 & \text{if } \lambda \in E, \\ 0 & \text{if } \lambda \notin E, \end{cases}$$

where E is any Borel subset of $\mathbb{H}^{[n]}$. It can be seen that the subalgebra R has the property (P). Thus the maximal ideal space of APW_Σ^n is contractible. The maximal ideal spaces of AP_Σ^n and APW_Σ^n are homeomorphic as topological spaces; see for example [1, Theorem 3.1]. So the maximal ideal space of AP_Σ^n is contractible as well. So we recover the main result from [8]. (In [8], instead of the canonical half space $\mathbb{H}^{[n]}$, more general half spaces S were considered. There a subset S of $\mathbb{R}^n$ was called a *half space* in $\mathbb{R}^n$ if it satisfied the properties $S \cup (-S) = \mathbb{R}^n$, $S \cap (-S) = \{0\}$, $x + y \in S$ for all $x, y \in S$, $\alpha x \in S$ for all $x \in S$ and $\alpha \geq 0$. However, it was shown in [8, Proposition 1.2] that any such half space S is of the form $Z\mathbb{H}^{[n]}$ for an invertible matrix $Z \in \mathbb{R}^{n \times n}$.)

Summarizing the results of this section, we have shown Corollary 1.10 as a particular consequence of our main result Theorem 1.5.

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