WEAK BEURLING PROPERTY AND EXTENSIONS TO INVERTIBILITY

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Abstract. Let $\mathcal{E}, \mathcal{E}_*$ be Hilbert spaces. Given a Hilbert space $\mathbb{H}$ of holomorphic functions in a domain Ω in $\mathbb{C}^d$, consider the multiplier space $\mathcal{M}_{\mathbb{H}}(\mathcal{E}, \mathcal{E}_*)$. It is shown that for "nice enough" $\mathbb{H}$, the following statements are equivalent for $f \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}, \mathcal{E}_*)$:

(1) There exists a $g \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}_*, \mathcal{E})$ such that $g(z)f(z) = I_{\mathcal{E}}$ for all $z \in \Omega$.

(2) There exists a Hilbert space $\mathcal{E}_c$ and $f_c \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}_c, \mathcal{E}_*)$ such that

$$F(z) := \begin{bmatrix} f(z) & f_c(z) \end{bmatrix} : \mathcal{E} \oplus \mathcal{E}_c \to \mathcal{E}_*$$

is invertible for every $z \in \Omega$, that is, there exists a function $G \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}_*, \mathcal{E} \oplus \mathcal{E}_c)$ such that $G(z)F(z) = I_{\mathcal{E}\oplus\mathcal{E}_c}$ and $F(z)G(z) = I_{\mathcal{E}_*}$ for all $z \in \Omega$.

Moreover, we characterize the spaces $\mathbb{H}$ for which (1) and (2) above are equivalent. Since this characterization has a close relation with Beurling's theorem for shift invariant subspaces of H^2, we call this property of $\mathbb{H}$ the *weak Beurling property*. We show that all reproducing kernel Hilbert spaces with complete Nevanlinna-Pick kernels have the weak Beurling property. This produces a large class of examples for which (1) and (2) are equivalent. We also give an example of another space with the weak Beurling property.

CDAM Research Report LSE-CDAM-2009-05

1. Introduction

The classical result of Tolokonnikov says that if $\mathcal{E} \subset \mathcal{E}_*$ are Hilbert spaces and $\dim \mathcal{E} < \infty$, then the following two statements are equivalent :

(1) *(Left invertibility)* There exists $g \in H^\infty(\mathbb{D}, \mathcal{L}(\mathcal{E}_*, \mathcal{E}))$ such that $g(z)f(z) = I_{\mathcal{E}}$ for all $z \in \mathbb{D}$.

(2) *(Completing to an isomorphism)* There exists $F \in H^\infty(\mathbb{D}, \mathcal{L}(\mathcal{E}_*, \mathcal{E}_*))$ such that $F|_{\mathcal{E}} = f$ and $F^{-1} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathcal{E}_*, \mathcal{E}_*))$.

Here $H^\infty(\mathcal{E}, \mathcal{E}_*)$ denotes the Banach space of functions $f : \mathbb{D} \to \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ that are holomorphic and bounded, equipped with the supremum norm $\|f\|_\infty := \sup_{z\in\mathbb{D}} \|f(z)\|_{\mathcal{L}(\mathcal{E},\mathcal{E}_*)}$. This surprising fact has far reaching generalizations and has a relation to Nevanlinna-Pick kernels, which is the content of this paper.

When $\mathcal{E}$ and $\mathcal{E}_*$ are both finite dimensional, then this lemma simply says that the ring H^∞ (of scalar functions, with pointwise addition and multiplication) is a Hermite ring. For background on Hermite rings, see [7]. For a proof of Tolokonnikov's result, see [14] or [15]. Tolokonnikov's Lemma was generalized to the case when $\mathcal{E}$ is not necessarily finite dimensional by Sergei Treil in [16].

1991 *Mathematics Subject Classification.* Primary 54C40; Secondary 46J10, 32A38, 93D15.

Key words and phrases. Hermite ring, Tolokonnikov's lemma, weak Beurling property.

For $d \geq 1$, consider a domain Ω in $\mathbb{C}^d$. Given a Banach space X, denote by $\mathcal{O}(\Omega, X)$ the vector space of holomorphic functions from Ω to X. Suppose $\mathbb{H}$ is a Hilbert space whose elements are holomorphic functions on Ω and which contains the constant functions. There is a natural way of identifying $\mathbb{H} \otimes \mathcal{E}$ with a Hilbert space $\mathbb{H}(\mathcal{E})$ which consists of $\mathcal{E}$-valued holomorphic functions on Ω. Let

$$\mathcal{M}_{\mathbb{H}}(\mathcal{E}, \mathcal{E}_*) = \{\varphi \in \mathcal{O}(\Omega, \mathcal{L}(\mathcal{E}, \mathcal{E}_*)) \mid \varphi f \in \mathbb{H}(\mathcal{E}_*) \text{ for all } f \in \mathbb{H}(\mathcal{E})\}. \tag{1}$$

A straightforward application of the closed graph theorem shows that for each $\varphi \in \mathcal{M}(\mathcal{E}, \mathcal{E}_*)$, the induced multiplication operator $M_\varphi : H(\mathcal{E}) \to H(\mathcal{E}_*)$, $f \mapsto \varphi f$ is continuous. Equip $\mathcal{M}_{\mathbb{H}}(\mathcal{E}, \mathcal{E}_*)$ with a norm by defining $\|\varphi\|$ to be the operator norm of M_φ. This makes $\mathcal{M}_{\mathbb{H}}(\mathcal{E}, \mathcal{E}_*)$ a Banach space which we shall call the *multiplier space.*

Since throughout the paper all Hilbert spaces are assumed to be separable, we always assume that in any Hilbert space an orthonormal basis is fixed, and so any operator A in $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ can be identified with its (possibly infinite) matrix. Thus besides the usual involution

$$A \mapsto A^* : \mathcal{L}(\mathcal{E}, \mathcal{E}_*) \to \mathcal{L}(\mathcal{E}_*, \mathcal{E})$$

(here A^* is the adjoint of A), we also have the following operations:

$$\begin{aligned} A &\mapsto A^\top \quad \text{(transpose of the matrix) and} \\ A &\mapsto \overline{A} \quad \text{(complex conjugation entrywise of the matrix).} \end{aligned}$$

So $A^\top = \overline{(A^*)}$. Note that if $\varphi \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}, \mathcal{E}_*)$, then $\varphi^\top$ defined by

$$\varphi^\top(z) = (\varphi(z))^\top = \overline{(\varphi(z))^*}, \quad z \in \Omega,$$

is holomorphic in Ω and $\varphi^\top \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}_*, \mathcal{E})$.

Definition 1.1. A map $f \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}, \mathcal{E}_*)$ is said to be *left-invertible* if there exists a function $g \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}_*, \mathcal{E})$ such that $g(z)f(z) = I_{\mathcal{E}}$ for all $z \in \Omega$.

Definition 1.2. The Hilbert space $\mathbb{H}$ is said to have the *weak Beurling property* if for every $g \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}_*, \mathcal{E})$ such that $g^\top \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}, \mathcal{E}_*)$ is left-invertible, there exists a Hilbert space $\mathcal{E}_c$ and a function $\Theta \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}_c, \mathcal{E}_*)$ such that $\ker M_g = \operatorname{ran} M_\Theta$.

The terminology of "weak Beurling" property is motivated by the terminology used by Tolokonnikov in [15]. Our main result is the following:

Theorem 1.3. *Let $\mathbb{H}$ be a Hilbert space. Then the following statements are equivalent:*

(1) *$\mathbb{H}$ satisfies the weak Beurling property.*
(2) *For every $f \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}, \mathcal{E}_*)$, the following are equivalent:*
   (a) *There exists a function $g \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}_*, \mathcal{E})$ such that $g(z)f(z) = I_{\mathcal{E}}$ for all $z \in \Omega$.*
   (b) *There exists a Hilbert space $\mathcal{E}_c$ and a function $f_c \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}_c, \mathcal{E}_*)$ such that*

$$F(z) := \begin{bmatrix} f(z) & f_c(z) \end{bmatrix} : \mathcal{E} \oplus \mathcal{E}_c \to \mathcal{E}_*$$

   is invertible for each $z \in \Omega$, that is, there exists a function $G \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}_*, \mathcal{E} \oplus \mathcal{E}_c)$ such that such that $G(z)F(z) = I_{\mathcal{E} \oplus \mathcal{E}_c}$ and $F(z)G(z) = I_{\mathcal{E}_*}$ for all $z \in \Omega$.

Our result is a generalization of Tolokonnikov's lemma because in the case when Ω is the open unit disk $\mathbb{D}$ in the complex plane and $\mathbb{H}$ is the Hardy space H^2, then $\mathbb{H}$ has the weak-Beurling property by virtue of Beurling-Lax-Halmos theorem. In this case, $\mathcal{M}_{\mathbb{H}}(\mathcal{E}, \mathcal{E}_*)$ is $H^\infty(\mathbb{D}, (\mathcal{E}, \mathcal{E}_*))$.

Tolokonnikov's Lemma for algebras of holomorphic functions is relevant in *control theory*, where it plays an important role in the problem of stabilization of linear systems. Indeed, Tolokonnikov's Lemma implies that if a transfer function G has a right (or left) coprime factorization, then G has a doubly coprime factorization, and the standard Youla parameterization yields all stabilizing controllers for G. For background on the relevance of Tolokonnikov's Lemma in control theory and further details, we refer the reader to Vidyasagar [17].

The plan of the paper is the following. In Section 2, we prove the main result. In Section 3, we show that any reproducing kernel Hilbert space with a complete Nevanlinna-Pick kernel has the weak Beurling property. Section 4 shows that a weighted Hardy space of the half plane has the weak Beurling property. Hence an application of our main result shows that analogues of Tolokonnikov's lemma hold for the multiplier algebras of these spaces.

2. Proof of Theorem 1.3

Proof. (1)$\Rightarrow$(2): That the statement (b) implies statement (a) is obvious. We prove (a)$\Rightarrow$(b) below.

So suppose that f is left-invertible, that is, there exists a $g \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}_*, \mathcal{E})$ such that $gf \equiv I_{\mathcal{E}}$ on Ω. By assumption, we have that for the subspace

$$\ker M_g = \{\varphi \in \mathbb{H}(\mathcal{E}) \mid g(z)\varphi(z) \equiv 0 \text{ on } \Omega\},$$

there exists a Hilbert space $\mathcal{E}_c$ and a function $\Theta \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}_c, \mathcal{E}_*)$ such that $\ker M_g = \text{ran } M_\Theta$. Define

$$F := \begin{bmatrix} f & \Theta \end{bmatrix} : \mathcal{E} \oplus \mathcal{E}_c \to \mathcal{E}_*.$$

(Thus we take $f_c = \Theta$ in (b).)

We now define for each $z \in \Omega$, the operator $\Theta(z)^{-1}$ acting on ran $\Theta(z)$. Let $z \in \Omega$, and suppose that $e_* \in \text{ran}(\Theta(z)) \subset \mathcal{E}_*$. Let $(\Theta(z))^{-1}e_*$ be the unique vector in $\text{ran}(\Theta(z))^* \subset \mathcal{E}_c$ such that $\Theta(z)((\Theta(z))^{-1}e_*) = e_*$.

For a vector $e_* \in \mathcal{E}_*$, let $\mathbf{e}_* \in \mathbb{H}(\mathcal{E}_*)$ be the constant function taking the value e_* everywhere on Ω. It is clear that $(1 - fg)\mathbf{e}_* \in \ker M_g$. But $\ker M_g = \text{ran } M_\Theta$, and so there exists a $\psi \in \mathbb{H}(\mathcal{E}_c)$ such that

$$(1 - f(z)g(z))e_* = \Theta(z)\psi(z), \quad z \in \Omega.$$

This means that $(1-f(z)g(z))e_* \in \text{ran}(\Theta(z))$ for every $z \in \Omega$, and so $(\Theta(z))^{-1}(1-f(z)g(z))e_*$ is well-defined.

Define the linear transformation $G(z) : \mathcal{E}_* \to \mathcal{E} \oplus \mathcal{E}_c$ $(z \in \Omega)$ as follows:

$$G(z)e_* = \begin{bmatrix} g(z)e_* \\ (\Theta(z))^{-1}(1 - f(z)g(z))e_* \end{bmatrix}.$$

Then we have for $e \in \mathcal{E}$ and $e_c \in \mathcal{E}_c$ that:

$$\begin{aligned} G(z)F(z)\begin{bmatrix} e \\ e_c \end{bmatrix} &= G(z)(f(z)e+\Theta(z)e_c) \\ &= \begin{bmatrix} g(z)f(z)e+g(z)\Theta(z)e_c \\ (\Theta(z))^{-1}(I-f(z)g(z))f(z)+(\Theta(z))^{-1}(I-f(z)g(z))\Theta(z)e_c \end{bmatrix} \\ &= \begin{bmatrix} I_{\mathcal{E}}e \\ I_{\mathcal{E}_c}e_c \end{bmatrix}, \end{aligned}$$

where we have used in the last step the fact that $g(z)f(z) \equiv I_{\mathcal{E}}$ and $g(z)\Theta(z)e_c = 0$ (because $\Theta \mathbf{e_c} \in \operatorname{ran} M_\Theta = \ker M_g$). Thus $G(z)F(z) = I_{\mathcal{E}\oplus\mathcal{E}_c}$ for all $z \in \Omega$. Furthermore, for every $e_* \in \mathcal{E}_*$, we obtain

$$\begin{aligned} F(z)G(z)e_* &= F(z)\begin{bmatrix} g(z)e_* \\ (\Theta(z))^{-1}(1-f(z)g(z))e_* \end{bmatrix} \\ &= f(z)g(z)e_*+\Theta(z)(\Theta(z))^{-1}(1-f(z)g(z))e_* = e_* = I_{\mathcal{E}_*}e_*. \end{aligned}$$

Hence $F(z)G(z) = I_{\mathcal{E}_*}$ for all $z \in \Omega$.

The above calculations show that $F(z)$ is one-to-one and onto, and hence $F(z)$ has a continuous inverse, which must be the linear transformation $G(z)$. So $G(z) \in \mathcal{L}(\mathcal{E}_*, \mathcal{E}\oplus\mathcal{E}_c)$.

We now observe that $z \mapsto G(z) : \Omega \to \mathcal{L}(\mathcal{E}_*, \mathcal{E}\oplus\mathcal{E}_c)$ is analytic. Indeed, by applying the $\overline{\partial}_{z_i}$ operator on the relation $G(z)F(z) \equiv I_{\mathcal{E}\oplus\mathcal{E}_c}$, we obtain that

$$0 = \overline{\partial}_{z_i}(GF) = (\overline{\partial}_{z_i}G)F + G\overline{\partial}_{z_i}F = (\overline{\partial}_{z_i}G)F + G0 = (\overline{\partial}_{z_i}G)F,$$

and so $(\overline{\partial}_{z_i}G)F \equiv 0$. By post-multiplying this relation by G, and using the fact that $F(z)G(z) \equiv I_{\mathcal{E}_*}$, we get $\overline{\partial}_{z_i}G \equiv 0$. Since the choice of $i \in \{1,\dots,d\}$ was arbitrary, G is holomorphic. Finally, since G is the inverse of a multiplier, it is a multiplier itself.

(2)⇒(1): Suppose that $g \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}_*, \mathcal{E})$ such that $f := g^\top \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}, \mathcal{E}_*)$ is left-invertible. From the equivalence in (2), it follows that there exists a Hilbert space $\mathcal{E}_c$ and a function $f_c \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}_c, \mathcal{E}_*)$ such that F given by

$$F(z) = \begin{bmatrix} g^\top(z) & f_c(z) \end{bmatrix} : \mathcal{E}\oplus\mathcal{E}_c \to \mathcal{E}_* \quad (z \in \Omega)$$

is invertible, that is, there exists a function $G \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}_*, \mathcal{E}\oplus\mathcal{E}_c)$ such that such that $G(z)F(z) = I_{\mathcal{E}\oplus\mathcal{E}_c}$ and $F(z)G(z) = I_{\mathcal{E}_*}$ for all $z \in \Omega$. We have

$$F^\top(z) = \begin{bmatrix} g(z) \\ f_c^\top(z) \end{bmatrix} : \mathcal{E}_* \to \mathcal{E}\oplus\mathcal{E}_c.$$

From $F(z)G(z) = I_{\mathcal{E}_*}$ $(z \in \Omega)$, it follows that

$$G^\top(z)F^\top(z) = I_{\mathcal{E}_*} \quad (z \in \Omega). \tag{2}$$

Since $G^\top \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}\oplus\mathcal{E}_c, \mathcal{E}_*)$, we can write

$$G^\top(z) = \begin{bmatrix} h(z) & h_c(z) \end{bmatrix},$$

where $h \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}, \mathcal{E}_*)$, $h_c \in \mathcal{M}_{\mathbb{H}}(\mathcal{E}_c, \mathcal{E}_*)$. We will show that $\ker M_g = \text{ran } M_{h_c}$. The equation (2) gives

$$\left[\begin{array}{cc} h(z) & h_c(z) \end{array}\right] \left[\begin{array}{c} g(z) \\ f_c^\top(z) \end{array}\right] = h(z)g(z) + h_c(a)f_c^\top(z) = I_{\mathcal{E}_*}.$$

If $\varphi \in \ker M_g$, then we have

$$h(z)\underbrace{g(z)\varphi(z)}_{=0} + h_c(z)f_c^\top(z)\varphi(z) = \varphi(z),$$

and so $\varphi \in \text{ran } M_{h_c}$. Hence $\ker M_g \subset \text{ran } M_{h_c}$.

Also,

$$G(z) = \left[\begin{array}{c} h^\top(z) \\ h_c^\top(z) \end{array}\right]$$

and so $G(z)F(z) = I_{\mathcal{E}\oplus\mathcal{E}_c}$ yields

$$\left[\begin{array}{c} h^\top(z) \\ h_c^\top(z) \end{array}\right] \left[\begin{array}{cc} g^\top(z) & f_c(z) \end{array}\right] = I_{\mathcal{E}\oplus\mathcal{E}_c}.$$

In particular, $h_c^\top(z)g^\top(z) = 0$, and so $g(z)h_c(z) = 0$. Consquently, $\text{ran } M_{h_c} \subset \ker M_g$.

Thus we have shown that $\ker M_g = \text{ran } M_{h_c}$, and this shows that $\mathbb{H}$ does have the weak Beurling property. $\square$

Remark 2.1.

(1) We have not made any assumptions on the dimensions of $\mathcal{E}$ and $\mathcal{E}_*$. It is clear from the left invertibility of f that we must have $\dim \mathcal{E} \leq \dim \mathcal{E}_*$.

In the case when $\dim \mathcal{E}_* < +\infty$, this implies that $\dim \mathcal{E} < +\infty$. Thus in this case, f can be identified with a finite (tall) matrix-valued holomorphic function. In fact, the pointwise invertibility of $F(z)$ shows that $\dim \mathcal{E} + \dim \mathcal{E}_c = \dim \mathcal{E}_*$. This can be expressed succintly by saying that $\mathcal{M}_{\mathbb{H}}(\mathbb{C}, \mathbb{C})$ is a Hermite ring. See [7].

(2) In [15], it was shown that:

Proposition 2.2. *If R is a commutative unital Banach algebra, then the following are equivalent:*

(a) *R satisfies the* weak Lax-Halmos property*, that is, for every $k, n \in \mathbb{N}$ with $k < n$, and for every $g \in R^{k\times n}$ such that $g^\top \in R^{n\times k}$ is left-invertible, there exists a $\Theta \in R^{n\times(n-k)}$ such that $\ker \mathbf{M}_g =$ ran $\mathbf{M}_\Theta$. (Here for a matrix $\varphi \in R^{p\times q}$, by $\mathbf{M}_\varphi$, we mean the map $v \mapsto \mathbf{M}_\varphi := \varphi v : R^q \to R^p$, given simply by matrix multiplication by the matrix φ.)*

(b) *For every $f \in R^{n\times k}$ $(k < n)$, the following are equivalent:*
   (i) *There exists a matrix $g \in R^{k\times n}$ such that $gf = I$.*
   (ii) *There exists a matrix $f_c \in R^{n\times(n-k)}$ such that $F := \left[\begin{array}{cc} f & f_c \end{array}\right]$ is invertible in $R^{n\times n}$, that is, there exists a matrix $G \in R^{n\times n}$ such that $GF = FG = I$.*

Thus in light of our main result, we have the following corollary:

Corollary 2.3. *The Banach algebra $\mathcal{M}_{\mathbb{H}}(\mathbb{C}, \mathbb{C})$ has the weak Lax-Halmos property iff $\mathbb{H}$ has the weak Beurling property.*

Hence our contribution in this paper can be tied to the above abstract result Proposition 2.2 of Tolokonnikov in the case when the Banach algebra R is a multiplier algebra $\mathcal{M}_{\mathbb{H}}(\mathbb{C},\mathbb{C})$ on a Hilbert space $\mathbb{H}$. Note that in this latter case, we can consider infinite matrices $f \in \mathcal{M}_{\mathbb{H}}(\mathcal{E},\mathcal{E}_*)$ and their matrix completion problems, as opposed to the finite matrices considered in Tolokonnikov's abstract result, so our main result is a generalization of Tolokonnikov's result.

3. Spaces with Nevanlinna-Pick kernel

So far we considered Hilbert spaces of holomorphic functions without explicit consideration of kernels. Let $k(z,w)$ be a positive definite kernel in Ω which is a holomorphic function in z and an anti-holomorphic function in w, and let $\mathbb{H}_k$ be the corresponding Hilbert space of holomorphic function on Ω. The aim of this section is to show that there are many reproducing kernel Hilbert spaces with the weak Beurling property. A kernel k is called *irreducible* if

(1) the functions k_ω, $\omega \in \Omega$ are independent, and
(2) $k(x,y)$ is never zero for all $x, y \in \Omega$ with $x \neq y$.

All the reproducing kernels will be assumed to be irreducible.

The *Nevanlinna-Pick problem* for the reproducing kernel Hilbert space $\mathbb{H}_k$ is the following. Given $w_1,\dots,w_n \in \Omega$ and numbers $\lambda_1,\dots,\lambda_n$, is there a $\varphi \in \mathcal{M}_{\mathbb{H}_k}$ of norm at most one which interpolates the data, i.e., satisfies $\varphi(w_i) = \lambda_i$ for $i = 1,\dots,n$? If $T_{w,\lambda}$ is the operator on the n-dimensional space spanned by $k_{w_1},\dots,k_{w_n}$ which sends $k_{w_i} \to \bar{\lambda}_i k_{w_i}$, then using the fact that $M_\varphi^* k_w = \overline{\varphi(w)} k_w$, it is easy to see that a necessary condition to solve the Nevanlinna-Pick problem is that $\| T_{w,\lambda} \| \leq 1$.

Definition 3.1. The kernel k is called a *Nevanlinna-Pick kernel* if the condition $\| T_{w,\lambda} \| \leq 1$ for every n and $\{w_1,\dots,w_n\}$ and $\{\lambda_1,\dots,\lambda_n\}$ is also sufficient to solve the Nevanlinna-Pick Problem.

Let ν be a natural number. Let $\{e_1,\dots,e_\nu\}$ be a basis for $\mathbb{C}^\nu$. Given n points $w_1,\dots,w_n$ in Ω and n matrices $\Lambda_1,\dots,\Lambda_n$ of order $\nu \times \nu$, the *matrix Nevanlinna Pick problem* is that of finding $\varphi \in \mathcal{M}_{H_k}(\mathbb{C}^\nu,\mathbb{C}^\nu)$ of norm at most 1 which satisfies $\varphi(w_i) = \Lambda_i$ for $i = 1,\dots,n$. Again because of

$$M_\varphi^*(k_z \otimes v) = k_z \otimes \varphi(z)^* v, \quad z \in \Omega, \quad v \in \mathbb{C}^\nu,$$

a necessary condition for solvability of the matrix Nevanlinna-Pick problem is that the operator $T_{w,\Lambda}$ sending

$$k_{z_i} \otimes e_j \to k_{z_i} \otimes \Lambda_i^* e_j$$

is a contraction.

Definition 3.2. The kernel k is called a complete Nevanlinna-Pick kernel if $\| T_{w,\Lambda} \| \leq 1$ for all ν and all n and $\{w_1,\dots,w_n\}$ and $\{\Lambda_1,\dots,\Lambda_n\}$ is a sufficient condition to solve the matrix Nevanlinna-Pick problem.

Examples of spaces with complete Nevanlinna-Pick kernels are

(1) *Drury-Arveson space.* This space, denoted by H^2_d is defined to be the Hilbert space of holomorphic functions on the Euclidean unit ball in $\mathbb{C}^d$ with reproducing kernel

$$(1-\langle z,w\rangle)^{-1}.$$

For more on the space H^2_d and the multipliers, see the seminal paper of Arveson [3]. The space H^2_d was first used by Drury [6] who generalized von Neumann's inequality to operator tuples. This kernel is universal among the complete Nevanlinna Pick kernels, see [1].

(2) *Weighted ℓ^2 spaces.* Consider a weight sequence $(w_n)_{n\geq 0}$ satisfying

$$w_{n+1}^2 \geq w_n w_{n+2}$$

and the space of functions $f(z)=\sum_{n\geq 0} a_n z^n$ on $\mathbb{D}$ with

$$\|f\|^2 := \sum_{n\geq 0} |a_n|^2 w_n < \infty.$$

Thus Dirichlet space is a special case, corresponding to $w_n = n+1$. These spaces have complete NP kernels.

(3) *Dirichlet type spaces.* Let μ be a finite positive measure supported in $\overline{\mathbb{D}}$. Define

$$D(\mu)=\left\{ f \;\middle|\; \begin{array}{l} f \text{ is holomorphic in } \mathbb{D} \text{ and} \\ \|f\|^2 := \|f\|^2_{H^2} + \int_{\mathbb{D}} |f'(\zeta)|^2 U_\mu(\zeta)\,dm_2(\zeta) < \infty \end{array} \right\}$$

where dm_2 is the normalized Lebesgue area measure in $\mathbb{D}$, and U_μ is defined as

$$U_\mu(\zeta)=\int_{\mathbb{D}} \log\left|\frac{1-\overline{z}\zeta}{\zeta-z}\right|^2 \frac{d\mu(z)}{1-|z|^2} + \int_{\mathbb{T}} \frac{1-|\zeta|^2}{\zeta-z|^2}\,d\mu(z),\ \zeta\in\mathbb{D}.$$

These spaces were introduced by Richter [11] and Aleman [2] and were shown to have complete Nevanlinna-Pick kernel by Shimorin [13].

(4) *Weighted Sobolev spaces.* Given positive functions $w_0 \in C[x_0,x_1]$ and $w_1 \in C^1[x_0,x_1]$, consider the space

$$W_2^1 = \{ f : [x_0,x_1] \to \mathbb{C} \mid f \text{ is absolutely continuous with } |f'|^2 \text{ integrable } \}$$

with the norm

$$\|f\|^2 := \int_{x_0}^{x_1} |f(x)|^2 w_0(x)\,dx + \int_{x_0}^{x_1} |f'(x)|^2 w_1(x)\,dx.$$

Then W_2^1 with this norm has a complete NP kernel.

The object of study in this section is the multiplier space $\mathcal{M}_{\mathbb{H}_k}(\mathcal{E},\mathcal{E}_*)$, as defined in (1), with $\mathbb{H}$ replaced by $\mathbb{H}_k$.

Theorem 3.3. *If k is an irreducible complete Nevanlinna-Pick kernel, then $\mathbb{H}_k$ satisfies the weak Beurling property.*

Proof. First note that Agler and McCarthy showed that any irreducible complete Nevanlinna-Pick kernel k satisfies the following. Choose a base point w. Then there is a positive definite kernel b on Ω such that $|b(y,x)| < k(w,w)$ and

$$k(y,x)k(w,w) - k(y,w)k(w,x) = b(y,x)k(y,x), \text{ for all } x, y \in \Omega.$$

In fact, this is a characterization, see [1]. Suppose

$$b(y,x) = \sum_j b_j(y)\overline{b_j(x)}.$$

Now let $g \in \mathcal{M}_{\mathbb{H}_k}(\mathcal{E}_*, \mathcal{E})$. The subspace $\ker M_g$ is invariant under multiplication by b_j, that is, $M_{b_j}^{\mathcal{E}_*} \ker M_g \subset \ker M_g$, $j = 1, \dots, d$. McCullough and Trent showed in [8] that for such an invariant subspace, there exists a Hilbert space $\mathcal{E}_c$ and a $\Theta \in \mathcal{M}_{\mathbb{H}_k}(\mathcal{E}_c, \mathcal{E}_*)$ such that the subspace is the range of the operator of multiplication by Θ, i.e., $\ker M_g = \operatorname{ran} M_\Theta$. Thus the weak Beurling property is satisfied. □

In light of Theorem 1.3, we obtain the following corollary:

Corollary 3.4. *If k is a complete Nevanlinna-Pick kernel and if $f \in \mathcal{M}_{\mathbb{H}_k}(\mathcal{E}, \mathcal{E}_*)$, then the following statements are equivalent:*

(1) *There exists a function $g \in \mathcal{M}_{\mathbb{H}_k}(\mathcal{E}_*, \mathcal{E})$ such that $g(z)f(z) = I_{\mathcal{E}}$ for all $z \in \Omega$.*
(2) *There exists a Hilbert space $\mathcal{E}_c$ and a function $f_c \in \mathcal{M}_{\mathbb{H}_k}(\mathcal{E}_c, \mathcal{E}_*)$ such that*

$$F(z) := \begin{bmatrix} f(z) & f_c(z) \end{bmatrix} : \mathcal{E} \oplus \mathcal{E}_c \to \mathcal{E}_*,$$

is invertible for every $z \in \Omega$, that is, there exists a function $G \in \mathcal{M}_{\mathbb{H}_k}(\mathcal{E}_*, \mathcal{E} \oplus \mathcal{E}_c)$ such that such that $G(z)F(z) = I_{\mathcal{E}\oplus\mathcal{E}_c}$ and $F(z)G(z) = I_{\mathcal{E}_*}$ for all $z \in \Omega$.

4. The multiplier algebra of the weighted Hardy space

In this section we prove that the multiplier algebra of a certain control theoretically motivated weighted Hardy space has the weak Beurling property.

Consider the Sobolev space[1]

$$W^{2,n}(0,\infty) = \{w : (0,\infty) \to \mathbb{C} \mid w, w', w'', \dots, w^{(n)} \in L^2(0,\infty)\}.$$

Let $H^{2,n}$ denote the space of Fourier transforms of all elements from $W^{2,n}(0,\infty)$, equipped with the norm

$$\|F\|_{H^{2,n}} := \|y \mapsto (1+y^2)^{\frac{n}{2}} F(iy)\|_{L^2}, \quad F \in H^{2,n}.$$

Then $H^{2,n}$ is a Hilbert space with the norm above induced by an inner product. Clearly $H^{2,n} := H^{2,0}$ is the classical Hardy space of the half plane $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \operatorname{Re}(s) > 0\}$.

Proposition 4.1. *$f \in H^2(\mathcal{E})$ iff $F := \varphi f \in H^{2,n}(\mathcal{E})$, where $\varphi \in H^\infty$ is the outer function given by*

$$\varphi(s) = \exp\left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{ys+i}{y+is} k(iy) \frac{dy}{1+y^2}\right), \quad s \in \mathbb{C}_+, \tag{3}$$

[1]In control engineering, one thinks of this as a signal space of signals with finite energy and such that the first n derivatives of these signals also have finite energy.

and $k(iy) := \log \frac{1}{(1+y^2)^{\frac{n}{2}}}$, $y \in \mathbb{R}$.

Proof. Since

$$\int_{-\infty}^{\infty} |k(iy)| \frac{dy}{1+y^2} = \int_{-\infty}^{\infty} \frac{n}{2} \frac{\log(1+y^2)}{1+y^2} dy = n\pi \log 2 < +\infty,$$

it follows that (3) defines an outer function. Moreover,

$$\log |\varphi(iy)| = k(iy) \quad \text{a.e.},$$

and so $|\varphi(iy)| = \dfrac{1}{(1+y^2)^{\frac{n}{2}}}$ a.e.

If $f \in H^2$, then $F := \varphi \cdot f$ satisfies

$$\|F\|^2_{H^{2,n}} = \int_{\mathbb{R}} (1+y^2)^n \frac{1}{(1+y^2)^n} |f(iy)|^2 dy = \|f\|^2_{H^2} < +\infty,$$

and so $F \in H^{2,n}$.

Conversely, if $F \in H^{2,n}$, then $f := \frac{1}{\varphi} F$ satisfies

$$\|f\|^2_{H^2} = \int_{\mathbb{R}} |f(iy)|^2 dy = \int_{\mathbb{R}} (1+y^2)^n |F(iy)|^2 dy = \|F\|^2_{H^{2,n}} < +\infty.$$

This completes the proof. □

We now prove an analog of the Beurling-Lax-Halmos Theorem for $H^{2,n}(\mathcal{E})$.

Theorem 4.2. *Let $\mathcal{E}$ be a Hilbert space. Suppose that $\mathcal{S}$ is a nonzero subspace of $H^{2,n}(\mathcal{E})$ that is invariant under $M_{e^{-\lambda s}}$ for all $\lambda \geq 0$. Then there exists a Hilbert space $\mathcal{E}_c$ and a function Θ such that*

1. *Θ is a $\mathcal{L}(\mathcal{E}_c, \mathcal{E})$-valued function that is holomorphic in $\mathbb{C}_+$, and*
2. *$\mathcal{S} = M_\Theta H^{2,n}(\mathcal{E})$.*

Proof. This follows immediately from the classical Beurling-Lax-Halmos Theorem for H^2 and Proposition 4.1 above. □

It follows now that $H^{2,n}$ possesses the weak Beurling property.

Theorem 4.3. *$H^{2,n}$ satisfies the weak Beurling property.*

Proof. Clearly $\ker M_g = \{\varphi \in H^{2,n}(\mathcal{E}) \mid g(z)\varphi(z) \equiv 0 \text{ on } \mathbb{C}_+\}$ is invariant under pointwise multiplication by the functions $s \mapsto e^{-\lambda s}$ for all $\lambda \geq 0$, that is, $M^{\mathcal{E}_*}_{e^{-\lambda s}} \ker M_g \subset \ker M_g$. Thus by the Beurling-Lax-Halmos type result proved above, namely Theorem 4.2, there exists a Hilbert space $\mathcal{E}_c$ and a $\Theta \in \mathcal{M}(\mathcal{E}_c, \mathcal{E}_*)$ such that $\ker M_g = \operatorname{ran} M_\Theta$. □

From Theorem 1.3, we obtain:

Corollary 4.4. *The following statements are equivalent for a function $f \in \mathcal{M}_{H^{2,n}}(\mathcal{E}, \mathcal{E}_*)$:*

1. *There exists a $g \in \mathcal{M}_{H^{2,n}}(\mathcal{E}_*, \mathcal{E})$ such that $g(s)f(s) = I_{\mathcal{E}}$ for all $s \in \mathbb{C}_+$.*

(2) *There exists a Hilbert space* $\mathcal{E}_c$ *and a function* $f_c \in \mathcal{M}_{H^{2,n}}(\mathcal{E}_c, \mathcal{E}_*)$ *such that*

$$F(s) := \begin{bmatrix} f(s) & f_c(s) \end{bmatrix} : \mathcal{E} \oplus \mathcal{E}_c \to \mathcal{E}_*,$$

is invertible for every $s \in \mathbb{C}_+$, *that is, there exists a function* $G \in \mathcal{M}_{H^{2,n}}(\mathcal{E}_*, \mathcal{E} \oplus \mathcal{E}_c)$ *such that such that* $G(s)F(s) = I_{\mathcal{E}\oplus\mathcal{E}_c}$ *and* $F(s)G(s) = I_{\mathcal{E}_*}$ *for all* $s \in \mathbb{C}_+$.

Acknowledgements. This work is supported by Ramanna Fellowship of DST and UGC-SAP Phase IV. The second author thanks Serguei Shimorin (Royal Institute of Technology, Stockholm) for a useful discussion.

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