TOPOLOGICAL STABLE RANK OF $H^\infty(\Omega)$ FOR CIRCULAR DOMAINS Ω

by

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Abstract. — Let Ω be a circular domain, that is, an open disk with finitely many closed disjoint disks removed. Denote by $H^\infty(\Omega)$ the Banach algebra of all bounded holomorphic functions on Ω, with pointwise operations and the supremum norm. We show that the topological stable rank of $H^\infty(\Omega)$ is equal to 2. The proof is based on Suarez's theorem that the topological stable rank of $H^\infty(\mathbb{D})$ is equal to 2, where $\mathbb{D}$ is the unit disk. We also show that for domains symmetric to the real axis, the Bass and topological stable ranks of the real symmetric algebra $H^\infty_{\mathbb{R}}(\Omega)$ are 2.

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1. Introduction

The aim of this short note is to prove that the topological stable rank of the Banach algebra $H^\infty(\Omega)$ of all bounded analytic functions on Ω is equal to 2, where Ω denotes a circular domain. By conformal equivalence, the same assertion will hold for any finitely connected, proper domain in $\mathbb{C}$ whose boundary does not contain any one-point components. We shall also show that for circular domains Ω that are symmetric to the real axis, the real algebra

$$H^\infty_{\mathbb{R}}(\Omega) = \{f \in H^\infty(\Omega) \ : \ (f(z^*))^* = f(z)\ (z \in \Omega)\}$$

has the Bass and topological stable rank 2. Here z^* denotes the complex conjugate of z. The precise definitions are given below.

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The notion of the topological stable rank of a Banach algebra was introduced by M. Rieffel in [**6**], in analogy with the notion of the (Bass) stable rank of a ring defined by H. Bass [**1**]. We recall these definitions now.

Definition 1.1. — Let R be a commutative ring with identity element 1. An n-tuple $a := (a_1, \dots, a_n) \in R^n$ is said to be *invertible* or *unimodular*, (for short $a \in U_n(R)$), if there exists a solution $(x_1, \dots, x_n) \in R^n$ of the Bezout equation $\sum_{j=1}^{n} a_j x_j = 1$. We say that $a = (a_1, \dots, a_n, a_{n+1}) \in U_{n+1}(R)$ is *reducible* if there exist $h_1, \dots, h_n \in R$ such that $(a_1 + h_1 a_{n+1}, \dots, a_n + h_n a_{n+1}) \in U_n(R)$.

The *Bass stable rank of* R (denoted by *bsr* R) is the least $n \in \mathbb{N}$ such that every element $a = (a_1, \dots, a_n, a_{n+1}) \in U_{n+1}(R)$ is reducible, and it is infinite if no such integer n exists.

Let A be a commutative Banach algebra with unit element 1. The least integer n for which $U_n(A)$ is dense in A^n is called the *topological stable rank* of A (denoted by *tsr* A) and we define *tsr* $A = \infty$ if no such integer n exists.

It is well known that $\operatorname{bsr} A \le \operatorname{tsr} A$; see [**6**, Corollary 2.4].

In the case of the classical algebra $H^\infty(\mathbb{D})$ of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, D. Suarez [**9**] showed that the topological stable rank is 2. We will use this result in order to derive our result for $H^\infty(\Omega)$ when Ω is a circular domain.

Theorem 1.1 **(Suarez** [**9**]**).** — *The topological stable rank of $H^\infty(\mathbb{D})$ is 2.*

Let us recall that previously Tolokonnikov [**10**] showed that the Bass stable rank of $H^\infty(\Omega)$ is 1. That was based on S. Treil's [**11**] fundamental result that $H^\infty(\mathbb{D})$ has the Bass stable rank 1.

In [**5**] Mortini and Wick showed that the Bass and topological stable ranks of the real symmetric algebra

$$H^\infty_{\mathbb{R}}(\mathbb{D}) = \{f \in H^\infty(\mathbb{D}) : \ (f(z^*))^* = f(z) \ (z \in \mathbb{D})\}$$

are 2. Using this we will show that $\mathbb{D}$ can be replaced by an arbitrary circular domain symmetric to the real axis.

We now give the precise definition of a circular domain, and also fix some convenient notation.

Notation. Let Ω be a *circular domain*, of connectivity n, that is, an open disk, D, with $n-1$ closed disjoint disks removed [(1)]. Then Ω is the intersection of n simply connected domains, $\Omega = \Omega_0 \cap \Omega_1 \cap \cdots \cap \Omega_{n-1}$, where $\Omega_i = \overline{\mathbb{C}} \setminus \overline{D_i}$, the D_i being open disks in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We assume that $\infty \in D_0$. The boundary of a set $\Omega \subset \mathbb{C}$ is denoted by $\partial\Omega$.

[(1)] We tacitly assume that the closures of the removed disks are contained within D.

Let $H(\Omega)$ denote the set of all holomorphic functions on Ω, and let $H^\infty(\Omega)$ be the Banach algebra of all bounded holomorphic functions on Ω, with pointwise operations and the supremum norm.

If Ω is real symmetric (that is, $z \in \Omega$ if and only if $z^* \in \Omega$), then we use the symbol $H^\infty_{\mathbb{R}}(\Omega)$ to denote the set of functions f belonging to $H^\infty(\Omega)$ that are *real symmetric*, that is, $f(z) = (f(z^*))^*$ $(z \in \Omega)$.

An example of a circular domain is the annulus $\mathbb{A} = \{z \in \mathbb{C} : \ r_1 < |z| < r_2\}$, where $0 < r_1 < r_2$. In this case $\mathbb{A} = \Omega_0 \cap \Omega_1$, where

$$\begin{aligned} \Omega_0 &:= \{z \in \mathbb{C} : \ |z| < r_2\}, \\ \Omega_1 &:= \{z \in \overline{\mathbb{C}} : \ |z| > r_1\}. \end{aligned}$$

Thus $\Omega_0 = \overline{\mathbb{C}} \setminus \overline{D_0}$ and $\Omega_1 = \overline{\mathbb{C}} \setminus \overline{D_1}$, where

$$\begin{aligned} D_0 &:= \{z \in \overline{\mathbb{C}} : \ |z| > r_2\}, \\ D_1 &:= \{z \in \mathbb{C} : \ |z| < r_1\}. \end{aligned}$$

Our main results are the following:

Theorem 1.2. — *Let Ω be a circular domain. The topological stable rank of $H^\infty(\Omega)$ is 2.*

Theorem 1.3. — *Let Ω be circular domain symmetric to the real axis. Then the topological and Bass stable rank of $H^\infty_{\mathbb{R}}(\Omega)$ is 2.*

2. Preliminaries

The following Cauchy decomposition is well known (for $H^p(\Omega)$ functions, $1 \le p \le \infty$) [**4**, Proposition 4.1, p. 86] or [**3**, Theorem 10.12, p.181].

Lemma 2.1. — *Let $\Omega = \bigcap_{j=0}^{n-1} \Omega_j$ be a circular domain of connectivity n. Then any $f \in H(\Omega)$ can be decomposed as $f = f_0 + f_1 + \cdots + f_{n-1}$, where $f_j \in H(\Omega_j)$. If additionally the real part of f is bounded above on Ω, then the same is true for the f_j.*

Proof. — Apply Cauchy's integral formula for a null homologic cycle, close to the boundary of Ω, and use the principle of analytic continuation. Now let us assume that the real part of f is bounded above on Ω. Fix $k \in \{0, 1, \ldots, n-1\}$. Since $f_j(\infty) = 0$ for $j = 1, 2, \ldots, n-1$ and $\sum_{j \neq k} f_j$ is holomorphic in a neighborhood of the set $\overline{\mathbb{C}} \setminus \Omega_k$, we see that the real part of each f_j is bounded above on Ω_j, for $j = 0, 1, \ldots, n-1$. ☐

We will use the following factorization result; the non-symmetric version appears in [**10**, Lemma 1]. Since in our viewpoint, the proof of the annulus-case by Tolokonnikov is not complete, we give a more general proof, that includes also the symmetric case.

Recall that a Blaschke product B with zeros (z_j) in the disk

$$D(a,r) = \{z \in \mathbb{C} : \ |z-a| < r\}$$

has the form $B(z) = b(\frac{z-a}{r})$, where b is the usual Blaschke product of the unit disk with zeros $w_j = \frac{z_j - a}{r}$. Similarly, the Blaschke product B_e with zeros (z_j) in the exterior of the disk $D(a,r)$ has the form $B_e(z) = b(\frac{r}{z-a})$ where b is the usual Blaschke product of the unit disk with zeros $w_j = \frac{r}{z_j - a}$. We call these functions *generalized Blaschke products.*

Proposition 2.2. — *Let Ω be a circular domain of connectivity n, $n \in \mathbb{N}$, and let $\overline{D_j}$ denote the bounded components of $\overline{\mathbb{C}} \setminus \Omega$, $(j = 1, \cdots, n-1)$, that is, D_j is the open disk $D(a_j, r_j)$. Define*

$$\begin{aligned} \Omega_j &= \overline{\mathbb{C}} \setminus \overline{D}_j, \quad j = 1, \dots, n-1, \\ \Omega_0 &= \Omega \cup \left(\bigcup_{j=1}^{n-1} D_j \right). \end{aligned}$$

Then every function f in $H^\infty(\Omega)$, $f \not\equiv 0$, can be decomposed as:

$$f = f_0 \cdot f_1 \cdot f_2 \cdots f_{n-1} \cdot r,$$

where

$$f_j \in H^\infty(\Omega_j) \cap \left(H^\infty\Big(\bigcup_{k \neq j} D_k \Big) \right)^{-1}, \quad j = 0, 1, 2, \dots, n-1,$$

and where r is a rational function with poles and zeros contained in the set $\{a_1, \cdots, a_{n-1}\}$.

If Ω is a domain symmetric to the real axis, and $f \in H^\infty_{\mathbb{R}}(\Omega)$, then each of the functions f_j and r above can be taken to be real symmetric themselves.

Proof. — We may assume that Ω is the circular domain

$$\Omega = D(a_0, r_0) \setminus \bigcup_{j=1}^{n-1} \overline{D(a_j, r_j)},$$

where $\overline{D}_j = \overline{D(a_j, r_j)} \subseteq D(a_0, r_0)$ and where the closures of the D_j $(j = 1, \dots, n-1)$ are disjoint.

Let $D_0 := D(a_0, r_0)$. Set $\Omega_j = \overline{\mathbb{C}} \setminus \overline{D}_j$, $(j = 0, 1, \dots, n-1)$. It is well known that the sequence (z_k) of zeros of f satisfies the generalized Blaschke condition; that is $\sum_k \mathrm{dist}\ (z_k, \partial\Omega)$ converges (see [**4, 8**]). Split (z_k) into n sequences

$(z_{k,j})_k$, $j = 0, 1, \dots n-1$, so that the cluster points of $(z_{k,j})_k$ are exactly those of (z_k) that belong to ∂D_j, $j = 0, 1, \dots, n-1$. Let B_j be the generalized Blaschke product formed with the zeros $(z_{k,j})_k$ of f, $j = 0, 1, \dots, n-1$. It is clear that the zeros of B_j cluster only at ∂D_j, $0 \le j \le n-1$.

Then f can be written as $f = B_0 \cdot B_1 \cdots B_{n-1} \cdot g$, where $g \in H^\infty(\Omega)$ and g has no zeros in Ω (note that here we have used the fact that divison by B_j does not change the relative supremum of f on the boundary of Ω_j).

By [**2**, p. 111-112], there exist $k_j \in \mathbb{Z}$ and h holomorphic in Ω, such that

$$g(z) = \prod_{j=1}^{n-1} (z - a_j)^{k_j} e^{h(z)}.$$

Note that the real part of h is bounded above on Ω.

By Lemma 2.1, there exist $h_j \in H(\Omega_j)$ such that $h = h_0 + h_1 + \cdots + h_{n-1}$ and the real part of each h_j is bounded above on Ω_j, for $j = 0, 1, \cdots, n-1$. Hence the functions $e^{h_j} \in H^\infty(\Omega_j)$.

Now $f{=}r\prod_{j=0}^{n-1} B_j e^{h_j}$, where $r(z){=}\prod_{j=1}^{n-1}(z-a_j)^{k_j}$ gives the desired factorization.

In case of a symmetric domain Ω and $f \in H^\infty_{\mathbb{R}}(\Omega)$, we can choose a_j to be real if the disk $D(a_j, r_j)$ meets the real line, and the other a_j in pairs (a, a^*). Thus we can ensure that r is real symmetric, because the exponents k_j are the same for a_j and a_j^* due to the fact that

$$k_j = \frac{1}{2\pi i} \oint_\Gamma \frac{g'(z)}{g(z)} dz\ ,$$

where Γ denotes a suitable small circle around a_j.

The Blaschke products above are easily seen to be choosable in a real symmetric fashion. Hence, since f is real symmetric, we conclude that g is real symmetric as well. Therefore, e^h is real symmetric; that is

$$e^{h(z)} = (e^{h(z^*)})^* = e^{(h(z^*))^*}.$$

Since Ω is a domain, $h(z) - (h(z^*))^*$ equals a constant $2k\pi i$ for some $k \in \mathbb{Z}$. Therefore

$$h(z) = \frac{h(z) + (h(z^*))^*}{2} + \frac{h(z) - (h(z^*))^*}{2} = \frac{h(z) + (h(z^*))^*}{2} + k\pi i$$

Now in Cauchy's decomposition, we simply consider the symmetric functions $H_j(z) := \frac{h_j(z)+(h_j(z^*))^*}{2}$, and derive

$$h(z) = \sum_{j=0}^{n-1} H_j(z) + k\pi i.$$

Thus we have one of the following cases

$$e^{h(z)} = e^{\sum_{j=0}^{n-1} H_j(z)} \quad (z \in \Omega)$$

or

$$e^{h(z)} = -e^{\sum_{j=0}^{n-1} H_j(z)} \quad (z \in \Omega).$$

In the latter case we take $-r$ instead of r. Thus all the factors in

$$f = r \prod_{j=0}^{n-1} B_j e^{H_j}$$

are symmetric. □

We recall that the corona theorem holds for $H^\infty(\Omega)$ when Ω is a circular domain; see for example [**4**, Theorem 6.1, p.195].

Proposition 2.3. — *Let Ω be a circular domain. Then $(f_1, \dots, f_n)$ is invertible in $H^\infty(\Omega)$ if and only if there exists a $\delta > 0$ such that*

$$\sum_{j=1}^{n} |f_j(z)| \geq \delta \quad (z \in \Omega).$$

This corona-theorem is of course true for $H^\infty_{\mathbb{R}}(\Omega)$. Indeed, if $f_j \in H^\infty_{\mathbb{R}}(\Omega)$ and $(g_1, \dots, g_n)$ is a solution of $\sum_{j=1}^n g_j f_j = 1$ in $H^\infty(\Omega)$, then $(\widetilde{g}_1, \dots, \widetilde{g}_n)$ is a solution of the Bezout equation $\sum_{j=1}^n \widetilde{g}_j f_j = 1$ in $H^\infty_{\mathbb{R}}(\Omega)$, where $\widetilde{g}_j(z) := \frac{g(z)+(g_j(z^*))^*}{2}$ $(z \in \Omega)$.

We will need two technical results, which are proved below. In the following, the notation $M(R)$ is used to denote the maximal ideal space of the unital commutative Banach algebra R. Also the complex homomorphism from $H^\infty(\Omega)$ to $\mathbb{C}$ of point evaluation at a point $z \in \Omega$ will be denoted by φ_z, that is, $\varphi_z(f) = f(z)$, $f \in H^\infty(\Omega)$.

Let $z_0 \in \overline{\Omega}$. The set

$$M_{z_0}(H^\infty(\Omega)) = \{\varphi \in M(H^\infty(\Omega)) : \varphi(z) = z_0\}$$

is called the *fiber of $M(H^\infty(\Omega))$ over z_0*. It is well known (see [**4**]), that we have $\varphi(f) = 0$ for some $\varphi \in M_{z_0}(H^\infty(\Omega))$ if and only if $\liminf_{z \to z_0} |f(z)| = 0$. The *zero set of $f \in H^\infty(\Omega)$* is the set $\{\varphi \in M(H^\infty(\Omega)) : \varphi(f) = 0\}$.

We need a Lemma that lets us decompose two functions that live on different circular domains. To this end, let D_1, D_2 be open disks in $\overline{\mathbb{C}}$ such that $\overline{D_1} \cap \overline{D_2} = \emptyset$. Next, define $\Omega_j := \overline{\mathbb{C}} \setminus \overline{D_j}$ for $j = 1, 2$. Suppose that $f_j \in H^\infty(\Omega_j)$ for $j = 1, 2$ are non-zero functions. Next, set

$$\begin{aligned} Z_1 &= \left\{ \xi \in \partial D_1 = \partial\Omega_1 : f_2(\xi) = 0 \ \text{ and } \ \liminf_{\substack{z \to \xi \\ z \in \Omega_1 \cap \Omega_2}} |f_1(z)| = 0 \right\}, \\ Z_2 &= \left\{ \xi \in \partial D_2 = \partial\Omega_2 : f_1(\xi) = 0 \ \text{ and } \ \liminf_{\substack{z \to \xi \\ z \in \Omega_1 \cap \Omega_2}} |f_2(z)| = 0 \right\}, \ \text{and} \\ Z_3 &= \left\{ a \in \Omega_1 \cap \Omega_2 : f_1(a) = f_2(a) = 0 \right\}, \end{aligned}$$

Lemma 2.4. — *Let D_1, D_2 be open disks in $\overline{\mathbb{C}}$ such that $\overline{D_1} \cap \overline{D_2} = \emptyset$. Define $\Omega_j := \overline{\mathbb{C}} \setminus \overline{D_j}$. Let $f_j \in H^\infty(\Omega_j)$ be nonzero functions. Then the zero sets of f_1 and f_2 meet in at most a finite number of fibers of $H^\infty(\Omega_1 \cap \Omega_2)$. In other words, there exist at most finitely many $z_j \in \overline{\Omega_1 \cap \Omega_2}$ for which*

$$\liminf_{z \to z_j} |f_1(z)| = \liminf_{z \to z_j} |f_2(z)| = 0.$$

Moreover, f_1 and f_2 can be written as

$$\begin{aligned} f_1 &= \prod_{z_j \in Z_2 \cup Z_3} (z - z_j)^{m_j} \widetilde{F}_1, \ \text{and} \\ f_2 &= \prod_{z'_j \in Z_1 \cup Z_3} (z - z'_j)^{m'_j} \widetilde{F}_2 \end{aligned}$$

where $\widetilde{F}_j$ is in $H^\infty(\Omega_1 \cap \Omega_2)$ and has the property that for any element $\varphi \in M(H^\infty(\Omega_1 \cap \Omega_2))$ either $\varphi(\widetilde{F}_1) \neq 0$ or $\varphi(\widetilde{F}_2) \neq 0$.

Additionally, when $\lambda \in \overline{\Omega_1 \cap \Omega_2}$, each $\varphi \in M_\lambda(H^\infty(\Omega_1 \cap \Omega_2))$ is such that $\varphi = \varphi_\lambda \in M(H^\infty(\Omega_1))$ whenever $\lambda \in \Omega_2$, or $\varphi = \varphi_\lambda \in M(H^\infty(\Omega_2))$ whenever $\lambda \in \Omega_1$.

Proof. — It is clear that if $\varphi \in M(H^\infty(\Omega_1 \cap \Omega_2))$, then $\varphi \in M(H^\infty(\Omega_1))$ and $\varphi \in M(H^\infty(\Omega_2))$.

Now the set $Z_3 = \{z \in \Omega_1 \cap \Omega_2 \mid f_1(z) = f_2(z) = 0\}$ is finite, for otherwise, there is an accumulation point of zeros in $\partial\Omega_1$ or in $\partial\Omega_2$. But $\partial\Omega_1$ is contained in Ω_2, and $\partial\Omega_2$ is contained in Ω_1. So either f_1 or f_2 is identically 0, a contradiction.

Consider the set Z_2 and let $\lambda \in Z_2$. There are only finitely many zeros of f_1 on the circle $\partial D_2 \subset \Omega_1$, since f_1 is not identically zero. Similarly, we can can argue in the case when $\lambda \in Z_1$. Thus, Z_1 is finite as well. This completes the proof. $\square$

It is clear that an analogous version holds true for the symmetric case.

Lemma 2.5. — *Let D_1, D_2 be open disks in $\overline{\mathbb{C}}$ such that $\overline{D_1} \cap \overline{D_2} = \emptyset$. Define $\Omega_1 := \overline{\mathbb{C}} \setminus \overline{D_1}$ and $\Omega_2 := \overline{\mathbb{C}} \setminus \overline{D_2}$. Let $f_1, g_1 \in H^\infty(\Omega_1)$ and $f_2, g_2 \in H^\infty(\Omega_2)$ be nonconstant functions such that there exists $\delta > 0$ such that the following hold:*

(P1) *For all $z \in \Omega_1$, $|f_1(z)| + |g_1(z)| \geq \delta$.*
(P2) *For all $z \in \Omega_2$, $|f_2(z)| + |g_2(z)| \geq \delta$.*

Then, for every $\varepsilon > 0$, there exist $F_1, G_1 \in H^\infty(\Omega_1), F_2, G_2 \in H^\infty(\Omega_2)$ such that

(C1) *(F_1, G_2) is invertible in $H^\infty(\Omega_1 \cap \Omega_2)$,*
(C2) *(G_1, F_2) is invertible in $H^\infty(\Omega_1 \cap \Omega_2)$*
(C3) *(F_1, G_1) is invertible in $H^\infty(\Omega_1)$,*
(C4) *(F_2, G_2) is invertible in $H^\infty(\Omega_2)$, and*
(C5) *$||f_1 - F_1|| + ||g_1 - G_1|| + ||f_2 - F_2|| + ||g_2 - G_2|| < \varepsilon$.*

In particular, (F_1F_2, G_1G_2) is invertible in $H^\infty(\Omega_1 \cap \Omega_2)$.

Proof. — Consider the pair $(f_1, g_2) \in H^\infty(\Omega_1) \times H^\infty(\Omega_2)$. By Lemma 2.4 we may perturb the finitely many zeros of f_1 belonging to $S_2 \cup S_3$ and those of g_2 that lie in S_1 so that the new functions F_1 and G_2 form an invertible pair in $H^\infty(\Omega_1 \cap \Omega_2)$. Now we do the same with the pair (g_1, f_2) in $H^\infty(\Omega_1) \times H^\infty(\Omega_2)$. This gives an invertible pair $(G_1, F_2) \in H^\infty(\Omega_1 \cap \Omega_2)$. By choosing these perturbations sufficiently small, we see that the pairs (F_1, G_1) and (F_2, G_2) stay invertible in the associated space $H^\infty(\Omega_1)$, respectively $H^\infty(\Omega_2)$. This yields that (F_1F_2, G_1G_2) is invertible in $H^\infty(\Omega_1 \cap \Omega_2)$. □

It is clear that an analogous version holds true for the symmetric case.

3. Proof of tsr($H^\infty(\Omega)$)$= 2$

Proof of Theorem 1.2. — Let $f, g \in H^\infty(\Omega)$. By Proposition 2.2, we can write

$$\begin{aligned} f &= f_0 \cdot f_1 \cdot \dots \cdot f_{n-1} \cdot r, \\ g &= g_0 \cdot g_1 \cdot \dots \cdot g_{n-1} \cdot s. \end{aligned}$$

where f_j and $g_j \in H^\infty(\Omega_j)$. We note that since the rational functions r, s have zeros and poles only in the set $\{a_1, \dots, a_{n-1}\}$, it follows that r, s are invertible in $H^\infty(\Omega)$. Since each Ω_i is simply connected, it follows from the fact that the topological stable rank of $H^\infty(\mathbb{D})$ is 2 and the Riemann mapping theorem, that also the topological stable rank of $H^\infty(\Omega_i)$ is equal to 2. Hence the pairs $(f_0, g_0), \dots, (f_{n-1}, g_{n-1})$ can be replaced by unimodular pairs $(\widetilde{f}_0, \widetilde{g}_0), \dots, (\widetilde{f}_{n-1}, \widetilde{g}_{n-1})$ such that for every $i = 0, 1, \dots, n-1$

$$\|f_i - \widetilde{f}_i\|_\infty + \|g_i - \widetilde{g}_i\|_\infty < \epsilon.$$

By a repeated application of Lemma 2.5 to the pairs $(\widetilde{f}_k, \widetilde{g}_j)$ with $j \neq k$, we get the existence of $F_0, \dots, F_{n-1}$, $G_0, \dots, G_{n-1}$, such that

$$\|F_k - f_k\|_\infty + \|G_k - g_k\|_\infty < \epsilon,$$

and the pair (F_k, G_j) is unimodular in $H^\infty(\Omega_k \cap \Omega_j)$ for all $0 \leq k, j \leq n-1$. By the elementary theory of Banach algebras, it follows that there exists a $\delta > 0$ such that

$$|F_k(z)| + |G_j(z)| \geq \delta \quad (z \in \Omega_k \cap \Omega_j).$$

Thus there exists a $\delta' > 0$ such that with

$$\begin{aligned} \widetilde{f} &:= F_0 \cdot F_1 \cdot \dots \cdot F_{n-1} \cdot r, \\ \widetilde{g} &:= G_0 \cdot G_1 \cdot \dots \cdot G_{n-1} \cdot s, \end{aligned}$$

we have for all $z \in \Omega = \Omega_0 \cap \dots \cap \Omega_{n-1}$,

$$|\widetilde{f}(z)| + |\widetilde{g}(z)| \geq \delta'.$$

By the corona theorem for $H^\infty(\Omega)$, we obtain that $(\widetilde{f}, \widetilde{g})$ is a unimodular pair in $H^\infty(\Omega)$. Also, it can be seen that given $\epsilon' > 0$, we can choose $\epsilon > 0$ small enough at the outset so that

$$\|f - \widetilde{f}\|_\infty + \|f - \widetilde{g}\|_\infty \leq \epsilon'.$$

This completes the proof. □

The same proof shows that the topological stable rank of $H^\infty_{\mathbb{R}}(\Omega)$ is 2 as well. Since the unimodular pair $(z, 1 - z^2)$ is not reducible (here we assume that $]-1,1[\subseteq \Omega$, $-1, 1 \notin \Omega$,) we have that the Bass stable rank κ of $H^\infty_{\mathbb{R}}(\Omega)$ is not one. Since κ is always less than the topological stable rank, we obtain that $\kappa = 2$.

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