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Discounted optimal stopping for diffusions: free-boundary versus martingale approach

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The free-boundary and the martingale approach are competitive methods of solving discounted optimal stopping problems for one-dimensional time-homogeneous regular diffusion processes on infinite time intervals. We provide a missing link showing the equivalence of these approaches for a problem, where the optimal stopping time is equal to the first exit time of the underlying process from a region restricted by two constant boundaries. We also consider several illustrating examples including the rational valuation of the perpetual American strangle option.

1 Introduction

Optimal stopping problems have as objective to search for random times at which the underlying stochastic processes should be stopped, with the aim to optimize the expected values of given reward functionals. The majority of explicitly solvable stopping problems in the field are

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essentially those for one-dimensional diffusion processes with infinite time horizon. The optimal stopping times are then the first times when the underlying processes exit certain regions restricted by constant boundaries. There exist two basic and competitive methods of finding explicit expressions for the value functions and boundaries in such optimal stopping problems.

In common use is the *free-boundary* approach. There, an equivalent ordinary differential free-boundary problem is formulated for the unknown value function and stopping boundaries. Since such a free-boundary problem usually has a non-unique solution, it is often assumed that the appropriate solution should satisfy certain additional conditions (e.g. smooth fit, normal boundary, normal reflection, etc.). Then, by means of standard verification arguments from stochastic analysis (e.g. change-of-variable formula, optional sampling theorem), it is shown that the resulting solution of the free-boundary problem provides the solution of the initial optimal stopping problem (see, e.g. Peskir and Shiryaev [14]).

In the *martingale* approach, the initial discounted reward process is decomposed into a product of a positive martingale and a gain function of the current state of the underlying diffusion process. It is shown that the optimization of the gain function over all admissible stopping boundaries gives the value of the initial optimal stopping problem whenever the optimal stopping time is finite almost surely, with respect to the probability measure constructed by means of the positive martingale (see, e.g. Beibel and Lerche [1] and [2] as well as Lerche and Urusov [11]). It can easily be verified, using the strong Markov property, that the solution of the optimal stopping problem obtained by means of the martingale approach satisfies the associated free-boundary problem. It can also be shown directly that the optimization of the resulting gain function yields the smooth-fit condition for the value function at the optimal stopping boundaries.

In the present paper, we show how the gain function and the positive martingale from the reward decomposition, related to the martingale approach, can be explicitly identified from the solution of the associated free-boundary problem. We will assume that the optimal stopping time has a structure of the first time at which the underlying one-dimensional regular diffusion process exits a region restricted by two constant boundaries. We illustrate our results on several examples related to the rational valuation of perpetual American options.

The paper is organized as follows. In Section 2, we introduce the setting of an optimal stopping problem for a one-dimensional time-homogeneous regular diffusion process and formulate an equivalent free-boundary problem. We derive a closed form solution of the latter problem and decompose it into a form which is appropriate for the comparison of the two approaches. In Section 3, we verify that the solution of the free-boundary problem provides the solution of the initial optimal stopping problem with the reward function which coincides with that of the martingale approach. In Section 4, we discuss several examples of optimal stopping problems for one-dimensional diffusion processes arising from the rational valuation of perpetual American options.

2 Preliminaries

In this section, we give a formulation of the optimal stopping problem for a one-dimensional diffusion process and an equivalent ordinary differential free-boundary problem. We also provide a decomposition of the reward function related to the martingale approach.

2.1. For a precise formulation of the problem, let us consider a probability space $(\Omega, \mathcal{F}, P)$ with a standard Brownian motion $W = (W_t)_{t \geq 0}$. Let $X = (X_t)_{t \geq 0}$ be a process solving the stochastic differential equation

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dW_t \quad (X_0 = x) \tag{2.1}$$

where the coefficients $\mu(x)$ and $\sigma(x)$ are some Lipschitz continuous functions on $(0, \infty)$. The latter assumption guarantees the existence of a pathwise unique solution of the equation in (2.1), for a given starting point $x > 0$ (see, e.g. [8; Chapter V, Theorem 2.5]). It follows that X is a regular diffusion process, in the sense of [9; Chapter XV], on its state space which is assumed to be the positive half line $(0, \infty)$. Let us consider an optimal stopping problem with the value function

$$V_*(x) = \sup_{\tau} E_x \left[e^{-r\tau}\, H(X_\tau) \right] \tag{2.2}$$

where the supremum is taken over all stopping times τ, with respect to the natural filtration $(\mathcal{F}_t)_{t \geq 0}$ of the process X. Here, $H(x)$ is a payoff function, and E_x denotes the expectation under the assumption that $X_0 = x$, for some $x > 0$. Such optimal stopping problems have been considered in [13], [17] and [2] (see also [3; Theorem 3.19]) for regular diffusion processes with general payoffs and infinite time horizon.

2.2. It follows from the general theory of optimal stopping for Markov processes (see, e.g. [14; Chapter I, Section 2.2]) that the optimal stopping time in the problem (2.2) is given by

$$\tau_* = \inf\{t \geq 0 \,|\, V_*(X_t) \leq H(X_t)\}. \tag{2.3}$$

Throughout the paper, we assume that the payoff function $H(x)$ is positive and convex, and we search for an optimal stopping time τ_* of the form

$$\tau_* = \inf\{t \geq 0 \,|\, X_t \notin (a_*, b_*)\} \tag{2.4}$$

for some numbers $0 < a_* < b_* < \infty$ to be determined.

2.3. By means of standard arguments (see, e.g. [8; Chapter V, Section 5.1]), it can be shown that the infinitesimal operator $\mathbb{L}$ of the process X acts on an arbitrary twice continuously differentiable locally bounded function $F(x)$ according to the rule

$$(\mathbb{L}F)(x) = \mu(x)\, F'(x) + \frac{\sigma^2(x)}{2}\, F''(x) \tag{2.5}$$

for all $x > 0$. In order to find explicit expressions for the unknown value function $V_*(x)$ in (2.2) and the unknown boundaries a_* and b_* in (2.4), we may use the results of general theory of optimal stopping problems for continuous time Markov processes (see, e.g. [6] and [14; Chapter IV, Section 8]). We formulate the associated free-boundary problem

$$(\mathbb{L}V - rV)(x) = 0 \quad \text{for} \quad a < x < b \tag{2.6}$$

$$V(a+) = H(a) \quad \text{and} \quad V(b-) = H(b) \quad (\textit{instantaneous stopping}) \tag{2.7}$$

$$V(x) = H(x) \quad \text{for} \quad x < a \quad \text{and} \quad x > b \tag{2.8}$$

$$V(x) > H(x) \quad \text{for} \quad a < x < b \tag{2.9}$$

$$V'(a+) = H'(a) \quad \text{and} \quad V'(b-) = H'(b) \quad (\textit{smooth fit}) \tag{2.10}$$

for some $0 < a < b < \infty$ fixed. Note that the superharmonic characterization of [4] implies that the value function in (2.2) is the smallest function satisfying the system in (2.6)–(2.9). Such assumptions as well as the smooth-fit conditions in (2.10) are naturally used for the value functions at the optimal stopping boundaries for underlying regular diffusions (see [14; Chapter IV, Section 9] for an extensive overview).

2.4. We will now look for functions which solve the stated free-boundary problem (2.6)–(2.10). Let $U_+(x)$ and $U_-(x)$ be two independent positive solutions of the second order ordinary differential equation in (2.6). Without loss of generality, we may assume that $U_+(x)$ is increasing and $U_-(x)$ is decreasing on $(0, \infty)$. Note that the functions $U_+(x)$ and $U_-(x)$ can be represented as moment generating functions of the first passage times of the process X on constant boundaries (see, e.g. [7; Chapter IV, Section 4.6] or [16; Chapter V, Section 46]). The general solution of the second order equation in (2.6) is thus given by

$$V(x) = C_+ \, U_+(x) + C_- \, U_-(x) \tag{2.11}$$

where C_+ and C_- are some arbitrary constants. Hence, applying the instantaneous-stopping conditions from (2.7) to the function in (2.11), we get that the equalities

$$C_+ \, U_+(a) + C_- \, U_-(a) = H(a) \tag{2.12}$$

$$C_+ \, U_+(b) + C_- \, U_-(b) = H(b) \tag{2.13}$$

hold for some $0 < a < b < \infty$. Solving the system of equations in (2.12)–(2.13), we obtain the function

$$V(x; a, b) = C_+(a, b) \, U_+(x) + C_-(a, b) \, U_-(x) \tag{2.14}$$

which satisfies the system in (2.6)–(2.7) when we put

$$C_+(a, b) = \frac{H(a)U_-(b) - H(b)U_-(a)}{U_+(a)U_-(b) - U_+(b)U_-(a)} \tag{2.15}$$

$$C_-(a, b) = \frac{H(b)U_+(a) - H(a)U_+(b)}{U_+(a)U_-(b) - U_+(b)U_-(a)} \tag{2.16}$$

for $0 < a < b < \infty$. Therefore, applying the smooth-fit conditions from (2.10) to the function in (2.14), we obtain the equalities

$$C_+(a, b) \, U'_+(a) + C_-(a, b) \, U'_-(a) = H'(a) \tag{2.17}$$

$$C_+(a, b) \, U'_+(b) + C_-(a, b) \, U'_-(b) = H'(b) \tag{2.18}$$

with $C_+(a, b)$ and $C_-(a, b)$ given by (2.15) and (2.16), respectively.

2.5. It also follows from (2.14) that the function $V(x;a,b)$ admits the representation

$$V(x;a,b) = G(a,b)\Big(p(a,b)\, U_+(x) + (1-p(a,b))\, U_-(x)\Big) \tag{2.19}$$

for any $a < x < b$ fixed, where the gain function $G(a,b)$ is defined by

$$G(a,b) = C_+(a,b) + C_-(a,b). \tag{2.20}$$

Then the function $p(a,b)$ is given by

$$p(a,b) = \frac{C_+(a,b)}{C_+(a,b) + C_-(a,b)} \tag{2.21}$$

for $0 < a < b < \infty$. By means of straightforward computations, it is shown that the system of equations in (2.17)–(2.18) is equivalent to the system

$$\frac{\partial G}{\partial a}(a,b) = 0 \quad \text{and} \quad \frac{\partial G}{\partial b}(a,b) = 0. \tag{2.22}$$

It means that solutions of the former system are critical points of the gain function $G(a,b)$.

3 Main result

In this section, we formulate and prove the main result of the paper. It builds a missing link between the free-boundary and the martingale approach of [1] and [2].

Theorem 3.1 *Let the process X be a unique pathwise solution of the stochastic differential equation in (2.1). Assume that the payoff function $H(x)$ is positive and convex, and that the optimal stopping time τ_* has the structure (2.4), where the couple a_* and b_* is a solution of the system of equations in (2.17)–(2.18). Then, the value function of the optimal stopping problem in (2.2) has the form*

$$V_*(x) = \begin{cases} V(x;a_*,b_*), & \text{if } a_* < x < b_* \\ H(x), & \text{if } x \le a_* \text{ or } x \ge b_* \end{cases} \tag{3.1}$$

where $V(x;a,b)$ is given by (2.19). Moreover, the function $V(x;a_*,b_*)$ admits the representation

$$V(x;a_*,b_*) = G(a_*,b_*)\, E_x\left[M^*_{\tau_*}\right] \tag{3.2}$$

for $a_* < x < b_*$, where the process $M^* = (M^*_t)_{t \ge 0}$ defined by

$$M^*_t = e^{-r(\tau_* \wedge t)} \Big(p(a_*, b_*)\, U_+(X_{\tau_* \wedge t}) + (1 - p(a_*, b_*))\, U_-(X_{\tau_* \wedge t}) \Big) \tag{3.3}$$

is a uniformly integrable martingale, and the functions $G(a,b)$ and $p(a,b)$ are given by (2.20) and (2.21), respectively.

Proof: In order to verify the assertions stated above, it remains to show that the function defined in (3.1) coincides with the value function in (2.2), and that the stopping time τ_* from (2.4) is optimal with the boundaries a_* and b_* specified above. For this, let us denote by $V(x)$ the right-hand side of the expression in (3.1). Since the functions $U_+(x)$ and $U_-(x)$ are twice continuously differentiable as solutions of the second order differential equation in (2.6), taking into account the smooth-fit conditions in (2.10), we may conclude that the derivative $V'(x)$ is of bounded variation on the closed interval $[a_*, b_*]$. It thus follows from the assumption of convexity of $H(x)$ that $V(x)$ can be represented as a difference of two convex functions on $(0, \infty)$. Hence, applying a generalized Itô's formula (see, e.g. [8; Chapter III, Section 3.7] or [15; Chapter IV, Theorem 70]), we get

$$\begin{aligned} e^{-rt}\, V(X_t) = V(x) + M_t & \\ + \int_0^t e^{-rs}\, (\mathbb{L}V - rV)(X_s)\, I(X_s \neq a_*, X_s \neq b_*, X_s \neq c_i, i \in \mathbb{N})\, ds & \\ + \sum_{i \in \mathbb{N}} \frac{1}{2} \int_0^t e^{-rs} \Big(V'(c_i+) - V'(c_i-) \Big)\, I(X_s = c_i)\, d\ell^i_s & \end{aligned} \tag{3.4}$$

for all $t \ge 0$, where the process $M = (M_t)_{t \ge 0}$ defined by

$$M_t = \int_0^t e^{-rs}\, V'(X_s)\, \sigma(X_s)\, I(X_s \neq a_*, X_s \neq b_*, X_s \neq c_i, i \in \mathbb{N})\, dW_s \tag{3.5}$$

is a local martingale with respect to P_x. This is a probability measure under which the process X starts at $x > 0$. Here, the process $\ell^i = (\ell^i_t)_{t \ge 0}$ defined by

$$\ell^i_t = P_x - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I(c_i - \varepsilon < X_s < c_i + \varepsilon)\, \sigma^2(X_s)\, ds \tag{3.6}$$

is the local time of the process X at the point $c_i > 0$, $i \in \mathbb{N}$, at which the derivative $H'(x)$ can have a discontinuity, and $I(\cdot)$ denotes the indicator function. We also observe that the assumption of convexity of $H(x)$ implies the fact that $H'(x)$ is increasing on $(0, \infty)$, so that, the sum in the last line of (3.4) forms an increasing process.

Suppose that at some $x' > 0$, $x' \neq c_i$, $i \in \mathbb{N}$, such that $x' < a_*$ or $x' > b_*$, we have $(\mathbb{L}H - rH)(x') > 0$. Then, there exists some $\varepsilon > 0$ such that $x' + \varepsilon < a_*$ or $x' - \varepsilon > b_*$ and $(\mathbb{L}H - rH)(x) \geq \varepsilon$ holds, for all $x \in (x' - \varepsilon, x' + \varepsilon)$. Using the fact that the process in the last line of (3.4) is increasing, we obtain that the first time when the process X exits the neighborhood $(x' - \varepsilon, x' + \varepsilon)$ is optimal in (2.2). However, this contradicts the assumption about the structure of the optimal stopping time in (2.4). We may therefore conclude that $(\mathbb{L}H - rH)(x) \leq 0$ holds for any $0 < x < a_*$ and $x > b_*$ such that $x \neq c_i$, $i \in \mathbb{N}$. Suppose now that $V(x'') \leq H(x'')$ at some $x'' \in (a_*, b_*)$. It thus follows from (2.3) that the process X started at x'' should be stopped instantly. Hence, the resulting contradiction with the assumption of the structure of the optimal stopping time in (2.4) shows that the inequality $V(x) > H(x)$ should hold for $a_* < x < b_*$.

Getting these arguments together with the fact proved above that the function $V(x)$ and the boundaries a_* and b_* satisfy the system (2.6)–(2.10), we conclude that $(\mathbb{L}V - rV)(x) \leq 0$ holds, for any $x > 0$ such that $x \neq a_*$, $x \neq b_*$, $x \neq c_i$, $i \in \mathbb{N}$, as well as $V(x) \geq H(x)$ holds, for all $x > 0$. It follows from the regularity of the diffusion process in (2.1) that the time spent by X at the points a_*, b_* and c_i, $i \in \mathbb{N}$, is of Lebesgue measure zero, and thus, the indicators appearing in the integrals in the second line of (3.4) and in (3.5) can be ignored. Hence, the expression in (3.4) and the structure of the stopping time in (2.4) yield that the inequalities

$$e^{-r(\tau_* \wedge \tau)} H(X_{\tau_* \wedge \tau}) \leq e^{-r(\tau_* \wedge \tau)} V(X_{\tau_* \wedge \tau}) \leq V(x) + M_{\tau_* \wedge \tau} \tag{3.7}$$

hold for any stopping time τ of the process X started at $x > 0$.

Let $(\tau_n)_{n \in \mathbb{N}}$ be an arbitrary localizing sequence of stopping times for the process M. Taking in (3.7) the expectation with respect to the measure P_x, by means of the optional sampling theorem (see, e.g. [8; Chapter I, Theorem 3.22]), we get that the inequalities

$$\begin{aligned} &E_x \left[e^{-r(\tau_* \wedge \tau \wedge \tau_n)} H(X_{\tau_* \wedge \tau \wedge \tau_n}) \right] \leq E_x \left[e^{-r(\tau_* \wedge \tau \wedge \tau_n)} V(X_{\tau_* \wedge \tau \wedge \tau_n}) \right] \\ &\leq V(x) + E_x \left[M_{\tau_* \wedge \tau \wedge \tau_n} \right] = V(x) \end{aligned} \tag{3.8}$$

hold for all $x > 0$. Hence, letting n go to infinity and using Fatou's lemma, we obtain

$$E_x \left[e^{-r(\tau_* \wedge \tau)} H(X_{\tau_* \wedge \tau}) \right] \leq E_x \left[e^{-r(\tau_* \wedge \tau)} V(X_{\tau_* \wedge \tau}) \right] \leq V(x) \tag{3.9}$$

for any stopping time τ and all $x > 0$. By virtue of the structure of the stopping time in (2.4), it is readily seen that the equalities in (3.9) hold when $0 < x < a_*$ or $x > b_*$.

It remains to show that the equalities are attained in (3.9) when τ_* replaces τ, for $a_* < x < b_*$. By virtue of the fact that the function $V(x; a_*, b_*)$ and the boundaries a_* and b_* satisfies the conditions in (2.6) and (2.7), it follows from the expression in (3.4) and the structure of the stopping time in (2.4) that the equalities

$$e^{-r(\tau_* \wedge \tau_n)} \, V(X_{\tau_* \wedge \tau_n}; a_*, b_*) = G(a_*, b_*) \, M^*_{\tau_* \wedge \tau_n} = V(x) + M_{\tau_* \wedge \tau_n} \tag{3.10}$$

hold for all $a_* < x < b_*$ and any localizing sequence $(\tau_n)_{n \in \mathbb{N}}$ of M, where the process M^* is defined in (3.3). Observe that the assumption of convexity of $H(x)$ also yields the property

$$E_x \Big[\sup_{t \geq 0} e^{-r(\tau_* \wedge t)} \, H(X_{\tau_* \wedge t}) \Big] < \infty \tag{3.11}$$

for all $a_* < x < b_*$. Hence, letting n go to infinity and taking into account the fact that the variable $e^{-r\tau_*} \, H(X_{\tau_*})$ is equal to zero on the set $\{\tau_* = \infty\}$, we can apply the Lebesgue dominated convergence theorem for (3.10) to obtain the equalities

$$E_x \left[e^{-r\tau_*} \, H(X_{\tau_*}) \right] = E_x \left[e^{-r\tau_*} \, V(X_{\tau_*}; a_*, b_*) \right] = G(a_*, b_*) \, E_x \left[M^*_{\tau_*} \right] = V(x) \tag{3.12}$$

for all $a_* < x < b_*$, which together with the expressions in (3.9) imply the fact that $V(x)$ coincides with the value function in (2.2). Note that these arguments also yield the expression in (3.2) as well as the fact that M^* from (3.3) is a uniformly integrable martingale. □

Remark 3.2 The assertion of Theorem 3.1 implies the fact that the function $G(a, b)$ defined in (2.20) attains a *local* maximum at the couple a_* and b_*. In order to prove this claim, we observe that, for every $x_* \in (a_*, b_*)$ fixed, we may put $U_+(x_*) = U_-(x_*) = 1$, without loss of generality, since the functions $U_+(x)$ and $U_-(x)$ were chosen as arbitrary independent solutions of the second order differential equation in (2.6). Suppose now that there exist $a' \in (a_* - \varepsilon, a_* + \varepsilon)$ and $b' \in (b_* - \varepsilon, b_* + \varepsilon)$ such that $G(a', b') > G(a_*, b_*)$ holds, for some $\varepsilon > 0$ with $a_* + \varepsilon < x_* < b_* - \varepsilon$. In this case, it follows from the expression in (2.19) that $V(x_*; a', b') > V(x_*; a_*, b_*)$ holds. This contradicts the assertion of Theorem 3.1 stating that τ_* of the type (2.4) is optimal. Therefore, we have that $V(x_*; a, b) \leq V(x_*; a_*, b_*)$ holds for all $a \in (a_* - \varepsilon, a_* + \varepsilon)$ and $b \in (b_* - \varepsilon, b_* + \varepsilon)$, thus proving the claim. Note again that, since a_* and b_* is a couple solving the smooth-fit equations in (2.17)–(2.18), it is then automatically a critical point of the function $G(a,b)$.

Remark 3.3 It follows from [19; Chapter VIII, Proposition 1.13] that there exists a probability measure Q being locally equivalent to P on the filtration $(\mathcal{F}_t)_{t\geq 0}$ and such that its density process is given by

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t} = \frac{M_t^*}{E[M_t^*]} \tag{3.13}$$

for all $t \geq 0$, so that, the restrictions $Q \,|\, \mathcal{F}_{\tau_*}$ and $P \,|\, \mathcal{F}_{\tau_*}$ are equivalent on the set $\{\tau_* < \infty\}$. By virtue of the fact that $e^{-r\tau_*} H(X_{\tau_*})$ is equal to zero on the set $\{\tau_* = \infty\}$, we see from (3.10) that M_∞^* is also zero on $\{\tau_* = \infty\}$. We may therefore conclude from (3.2) that the expressions

$$\begin{aligned} V(x; a_*, b_*) &= G(a_*, b_*)\, E_x\left[M_{\tau_*}^* \, I(\tau_* < \infty) + M_{\tau_*}^* \, I(\tau_* = \infty)\right] \\ &= G(a_*, b_*)\, Q_x(\tau_* < \infty) \end{aligned} \tag{3.14}$$

hold, for $a_* < x < b_*$, where Q_x denotes the appropriate probability measure under which the process X starts at $x > 0$.

Remark 3.4 Note that the assertion of Theorem 3.1 remains valid also for the case, where the optimal stopping time is the first hitting time of only one constant boundary. In other words, for an appropriate function $H(x)$, we should assume from the beginning that either $a_* \equiv 0$ or $b_* \equiv \infty$ in (2.4), so that, for the expressions in (2.19)–(2.21), we have $G(0,b) \equiv C_+(0,b)$ and $p(0,b) \equiv 1$, or $G(a,\infty) \equiv C_-(a,\infty)$ and $p(a,\infty) \equiv 0$, respectively.

Remark 3.5 Assuming the uniqueness of the stopping boundaries a_* and b_* as solutions of the system in (2.17)–(2.18), also for the one-sided cases where either $a_* = 0$ or $b_* = \infty$, we can compare and identify the outcomes of both approaches. In that case, the value of $G(a_*, b_*)$ from (3.3) is equal to C^* in [1; Theorem 4] and [2; Theorem 3], while the process M^* from (3.3) coincides with the martingale M from [1; Subsection 2.3] and [2; Section 3] under $x = 1$. Hence, the value of $p(a_*, b_*)$ from (2.21) coincides with p^* from [1; Subsection 2.3] or [2; Section 3]. It therefore follows that the result of [1; Lemma 1] naturally yields

$$G(a_*, b_*) = \sup_{a \leq 1} G_{p^*}(a) = \sup_{b \geq 1} G_{p^*}(b) \tag{3.15}$$

where $G_p(x) = H(x)/[pU_+(x) + (1-p)U_-(x)]$, $x > 0$.

4 Some examples

In this section, we consider several examples of optimal stopping problems related to the rational valuation of perpetual American options. For the most of the examples below, we assume that $\mu(x) = (r-\delta)x$ and $\sigma(x) = \theta x$, $x > 0$, for some $0 < \delta < r$ and $\theta > 0$.

Example 4.1 (Perpetual American put and call.) Let us first consider the function $H(x) = (L-x)^+$, $x > 0$, where $L > 0$ is a given constant (see [12] and [1; Subsection 2.1]). In this case, the assertion of Theorem 3.1 holds with $p(a,\infty) \equiv 0$, $M_t^* = e^{-r(\tau_* \wedge t)} \, X_{\tau_* \wedge t}^{\gamma_-}$, where

$$\gamma_- = \frac{1}{2} - \frac{r-\delta}{\theta^2} - \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\theta^2}\right)^2 + \frac{2r}{\theta^2}} \tag{4.1}$$

and the gain function $G(a,\infty) \equiv C_-(a,\infty) = (L-a)^+/a^{\gamma_-}$ which attains its unique maximum value at

$$\widetilde{a} = \frac{\gamma_- L}{\gamma_- - 1}. \tag{4.2}$$

This is the unique (left) optimal stopping boundary for the process X.

We also consider the function $H(x) = (x-K)^+$, $x > 0$, where $K > 0$ is a given constant. In this case, the assertion of Theorem 3.1 holds with $p(0,b) \equiv 1$, $M_t^* = e^{-r(\tau_* \wedge t)} \, X_{\tau_* \wedge t}^{\gamma_+}$, where

$$\gamma_+ = \frac{1}{2} - \frac{r-\delta}{\theta^2} + \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\theta^2}\right)^2 + \frac{2r}{\theta^2}} \tag{4.3}$$

and the gain function $G(0,b) \equiv C_+(0,b) = (b-K)^+/b^{\gamma_+}$ which attains its unique maximum value at

$$\widetilde{b} = \frac{\gamma_+ K}{\gamma_+ - 1}. \tag{4.4}$$

This is the unique (right) optimal stopping boundary for the process X.

Example 4.2 (Perpetual American strangle option.) Let us now consider the function $H(x) = (L-x)^+ \vee (x-K)^+$, $x > 0$, where $0 < L < K$ are given constants (see [1; Subsection 2.4]). In this case, the assertion of Theorem 3.1 holds with $U_-(x) = x^{\gamma_-}$ and $U_+(x) = x^{\gamma_+}$, where γ_- and γ_+ are given by (4.1) and (4.3), respectively, and the gain function $G(a,b)$ admits the representation

$$G(a,b) = \frac{(b-K)(a^{\gamma_+} - a^{\gamma_-}) - (L-a)(b^{\gamma_+} - b^{\gamma_-})}{a^{\gamma_+} b^{\gamma_-} - b^{\gamma_+} a^{\gamma_-}}. \tag{4.5}$$

It attains its maximum value at the couple a_* and b_* being the *unique* solution of the system in (2.17)–(2.18) with

$$C_+(a,b) = \frac{(L-a)b^{\gamma_-} - (b-K)a^{\gamma_-}}{a^{\gamma_+}b^{\gamma_-} - b^{\gamma_+}a^{\gamma_-}} \tag{4.6}$$

$$C_-(a,b) = \frac{(b-K)a^{\gamma_+} - (L-a)b^{\gamma_+}}{a^{\gamma_+}b^{\gamma_-} - b^{\gamma_+}a^{\gamma_-}} \tag{4.7}$$

so that, the function $p(a,b)$ takes the form

$$p(a,b) = \frac{(L-a)b^{\gamma_-} - (b-K)a^{\gamma_-}}{(b-K)(a^{\gamma_+} - a^{\gamma_-}) - (L-a)(b^{\gamma_+} - b^{\gamma_-})} \tag{4.8}$$

for some $0 < a < L < K < b < \infty$. We also note that the value of $G(a_*, b_*)$ coincides with the value of C^* in [1; Subsection 2.4] when $0 < L < 1 < K$, under $x = 1$.

In order to prove that the system of equations in (2.17)–(2.18) with (4.6)–(4.7) admits a unique solution a_* and b_*, we see that the former admits the representation

$$[\gamma_-(L-a)+a]\Big(\frac{b}{a}\Big)^{\gamma_+} - [\gamma_+(L-a)+a]\Big(\frac{b}{a}\Big)^{\gamma_-} = (\gamma_- - \gamma_+)(b-K) \tag{4.9}$$

$$[\gamma_-(b-K)-b]\Big(\frac{a}{b}\Big)^{\gamma_+} - [\gamma_+(b-K)-b]\Big(\frac{a}{b}\Big)^{\gamma_-} = (\gamma_- - \gamma_+)(L-a) \tag{4.10}$$

which is equivalent to

$$\Big(\frac{b}{a}\Big)^{\gamma_+} = \frac{\gamma_-(b-K)-b}{\gamma_-(L-a)+a} \quad \text{and} \quad \Big(\frac{b}{a}\Big)^{\gamma_-} = \frac{\gamma_+(b-K)-b}{\gamma_+(L-a)+a} \tag{4.11}$$

or in another form

$$\frac{(\gamma_- - 1)a - \gamma_- L}{a^{\gamma_+}} = \frac{(1-\gamma_-)b + \gamma_- K}{b^{\gamma_+}} \tag{4.12}$$

$$\frac{(1-\gamma_+)a + \gamma_+ L}{a^{\gamma_-}} = \frac{(\gamma_+ - 1)b - \gamma_+ K}{b^{\gamma_-}} \tag{4.13}$$

for some $0 < a < L < K < b < \infty$.

In order to show the existence and uniqueness of a solution of the system of equations in (4.12)–(4.13), we use the idea of proof of the existence and uniqueness of solutions of the systems of equations in (4.73)–(4.74) from [20; Chapter IV, Section 2] and (3.16)–(3.17) from [5; Section 3]. Let us introduce the functions

$$I_+(a) = \frac{(\gamma_- - 1)a - \gamma_- L}{a^{\gamma_+}} \quad \text{and} \quad I_-(a) = \frac{(1-\gamma_+)a + \gamma_+ L}{a^{\gamma_-}} \tag{4.14}$$

$$J_+(b) = \frac{(1-\gamma_-)b + \gamma_- K}{b^{\gamma_+}} \quad \text{and} \quad J_-(b) = \frac{(\gamma_+ - 1)b - \gamma_+ K}{b^{\gamma_-}} \tag{4.15}$$

hold for all $0 < a < L < K < b < \infty$. Observe that, for the derivatives of the functions in (4.14)–(4.15), the expressions

$$I'_+(a) = -\frac{(\gamma_+ - 1)(\gamma_- - 1)a - \gamma_+\gamma_- L}{a^{\gamma_+ + 1}} \equiv -\frac{(\gamma_+ - 1)(\gamma_- - 1)(a - \overline{L})}{a^{\gamma_+ + 1}} < 0 \tag{4.16}$$

$$I'_-(a) = \frac{(\gamma_+ - 1)(\gamma_- - 1)a - \gamma_+\gamma_- L}{a^{\gamma_- + 1}} \equiv \frac{(\gamma_+ - 1)(\gamma_- - 1)(a - \overline{L})}{a^{\gamma_- + 1}} > 0 \tag{4.17}$$

$$J'_+(b) = \frac{(\gamma_+ - 1)(\gamma_- - 1)b - \gamma_+\gamma_- K}{b^{\gamma_+ + 1}} \equiv \frac{(\gamma_+ - 1)(\gamma_- - 1)(b - \overline{K})}{b^{\gamma_+ + 1}} < 0 \tag{4.18}$$

$$J'_-(b) = -\frac{(\gamma_+ - 1)(\gamma_- - 1)b - \gamma_+\gamma_- K}{b^{\gamma_- + 1}} \equiv -\frac{(\gamma_+ - 1)(\gamma_- - 1)(b - \overline{K})}{b^{\gamma_- + 1}} > 0 \tag{4.19}$$

hold for all $0 < a < \overline{L} < \overline{K} < b < \infty$ with

$$\overline{L} = \frac{\gamma_+\gamma_- L}{(\gamma_+ - 1)(\gamma_- - 1)} \equiv \frac{rL}{\delta} \quad \text{and} \quad \overline{K} = \frac{\gamma_+\gamma_- K}{(\gamma_+ - 1)(\gamma_- - 1)} \equiv \frac{rK}{\delta}. \tag{4.20}$$

Hence, the function $I_+(a)$ decreases on the interval $(0, \overline{L})$ with $I_+(0+) = \infty$ and $I_+(\overline{L}) = \gamma_- L/[(\gamma_+ - 1)\overline{L}^{\gamma_+ + 1}] < 0$, so that, the range of its values is given by the interval $(I_+(\overline{L}), \infty)$. The function $J_+(b)$ decreases on the interval $(\overline{K}, \infty)$ with $J_+(\overline{K}) = -\gamma_- K/[(\gamma_+ - 1)\overline{K}^{\gamma_+ + 1}] > 0$ and $J_+(\infty) = 0$, so that, the range of its values is given by the interval $(0, J_+(\overline{K}))$. The function $I_-(a)$ increases on the interval $(0, \overline{L})$ with $I_-(0+) = 0$ and $I_-(\overline{L}) = -\gamma_+ L/[(\gamma_- - 1)\overline{L}^{\gamma_- + 1}] > 0$, so that, the range of its values is given by the interval $(0, I_-(\overline{L}))$. The function $J_-(b)$ increases on the interval $(\overline{K}, \infty)$, so that, the range of its values is given by the interval $(J_-(\overline{K}), \infty)$.

It thus follows from (4.12) that, for each $b \in (\overline{K}, \infty)$, there exists a unique number $a \in (\widehat{a}, \widetilde{a})$ with $\widetilde{a}$ given by (4.2) and such that $\widehat{a}$ is uniquely determined by the equation $I_+(\widehat{a}) = J_+(\overline{K})$. It also follows from (4.13) that, for each $a \in (0, \overline{L})$, there exists a unique number $b \in (\widetilde{b}, \widehat{b})$ with $\widetilde{b}$ given by (4.4) and such that $\widehat{b}$ is uniquely determined by the equation $I_-(\overline{L}) = J_-(\widehat{b})$. We may therefore conclude that the equations in (4.12) and (4.13) uniquely define the function $b_+(a)$ on $(\widehat{a}, \widetilde{a})$ with the range $(\overline{K}, \infty)$, and the function $b_-(a)$ on $(0, \overline{L})$ with the range $(\widetilde{b}, \widehat{b})$, respectively. This fact directly implies that, for each point $a \in (\widehat{a}, \widetilde{a})$, there exist unique values $b_+(a)$ and $b_-(a)$ belonging to $(\widetilde{b}, \widehat{b})$, that together with the inequalities $\overline{K} < b_-(0) \equiv \widetilde{b} < b_-(\overline{L}) < \infty \equiv b_+(\widetilde{a})$ guarantees the existence of exactly one intersection point with coordinates a_* and b_* of the curves associated with the functions $b_+(a)$ and $b_-(a)$ on the interval $(\widehat{a}, \widetilde{a})$ such that $\widetilde{b} < b_+(a_*) \equiv b_-(a_*) \equiv b_* < \widehat{b}$. This completes the proof of the claim.

Example 4.3 (Perpetual American bear and bull spreads.) Let us consider the function $H(x) = (L-x)^+ \wedge (L-K)$, $x > 0$, where $0 < K < L$ are given constants. In this case, the assertion of Theorem 3.1 holds with $p(a,\infty) \equiv 0$, $M_t^* = e^{-r(\tau_* \wedge t)} X_{\tau_* \wedge t}^{\gamma_-}$, and the gain function $G(a,\infty) \equiv C_-(a,\infty) = [(L-a)^+ \wedge (L-K)]/a^{\gamma_-}$ which attains its unique maximum value at $a_* = \widetilde{a} \vee K$. This is the unique (left) optimal stopping boundary for the process X, with $\widetilde{a}$ given by (4.2).

We also consider the function $H(x) = (x-K)^+ \wedge (L-K)$, $x > 0$, where $0 < K < L$ are given constants. In this case, the assertion of Theorem 3.1 holds with $p(0,b) \equiv 1$, $M_t^* = e^{-r(\tau_* \wedge t)} X_{\tau_* \wedge t}^{\gamma_+}$, and the gain function $G(0,b) \equiv C_+(0,b) = [(b-K)^+ \wedge (L-K)]/b^{\gamma_+}$ which attains its unique maximum value at $b_* = \widetilde{b} \wedge L$. This is the unique (right) optimal stopping boundary for the process X, with $\widetilde{b}$ given by (4.4).

Example 4.4 (Perpetual American barrier down-and-out put and up-and-out call.) Let us consider the function $H(x) = (L-x)^+ I(x > K)$, $x > 0$, where $0 < K < L$ are given constants. In this case, in (2.19) we have $p(a,\infty) \equiv 0$, $U_-(x) = x^{\gamma_-}$, and the gain function $G(a,\infty) \equiv C_-(a,\infty) = [(L-a)^+ I(a > K)]/a^{\gamma_-}$ which attains its unique maximum at $a_* = \widetilde{a}$ whenever $\widetilde{a} > K$, being the unique (left) optimal stopping boundary for the process X. There exists no optimal stopping time otherwise.

We also consider the function $H(x) = (x-K)^+ I(x < L)$, $x > 0$, where $0 < K < L$ are given constants. In this case, in (2.19) we have $p(0,b) \equiv 1$, $U_+(x) = x^{\gamma_+}$, and the gain function $G(0,b) \equiv C_+(0,b) = [(b-K)^+ I(b < L)]/b^{\gamma_+}$ which attains its unique maximum at $b_* = \widetilde{b}$ whenever $\widetilde{b} < L$, being the unique (right) optimal stopping boundary for the process X. There exists no optimal stopping time otherwise.

Example 4.5 (Perpetual American integral call option with floating strike.) Suppose now that $\mu(x) = 1 - \rho x$ and $\sigma(x) = \theta x$, $x > 0$, and $H(x) = (x-K)^+$, where $K > 0$ and $0 < r < \rho$ are some given constants (see [10]). In this case, the assertion of Theorem 3.1 holds with $p(0,b) \equiv 1$ and the function

$$U_+(x) = \int_0^\infty \exp\left(-\frac{2z}{\theta^2}\right) z^{-(\beta_- + 1)} (1 + xz)^{\beta_+} \, dz \tag{4.21}$$

where

$$\beta_{\pm} = \frac{1}{2} + \frac{\rho}{\theta^2} \pm \sqrt{\left(\frac{1}{2} + \frac{\rho}{\theta^2}\right)^2 + \frac{2r}{\theta^2}} \tag{4.22}$$

and the gain function $G(0,b) \equiv C_+(0,b) = (b-K)^+/U_+(b)$, which attains its unique maximum value at some b_* on the interval (K,∞), with $U_+(b)$ given by (4.21).

Example 4.6 (Perpetual American lookback call option with floating strike.) Let us consider the process X solving the stochastic differential equation

$$dX_t = -\rho\, X_t\, dt + \theta\, X_t\, dW_t + I(X_t = 1)\, dN_t \quad (X_0 = x) \tag{4.23}$$

where $N = (N_t)_{t\ge 0}$ is an increasing process changing its value only when X started at $x > 1$ arrives at the point 1 being the instantaneously reflecting boundary, and $0 < r < \rho$ are some given constants (see [18], [21; Chapter VIII, Section 2d] or [2; Section 4.3]). Let us consider $H(x) = (x-K)^+$, $x > 1$, with some $K > 0$. In this case, in (2.19) we have $p(0,b) \equiv \gamma_-/(\gamma_- - \gamma_+)$, $U_-(x) = x^{\gamma_-}$ and $U_+(x) = x^{\gamma_+}$, where γ_- and γ_+ are given by (4.1) and (4.3), and the gain function $G(0,b) = (b-K)^+/[p(0,b)b^{\gamma_+} + (1-p(0,b))b^{\gamma_-}]$ attains its maximum at some point b_* on the interval (K,∞). Note that, in this setting, the maximization of the gain function is equivalent to the condition

$$V'(1+) = 0 \quad (\textit{normal reflection}) \tag{4.24}$$

for the value function of the appropriate optimal stopping problem, instead of the smooth-fit condition in (2.10).

Example 4.7 (Perpetual spread option stopping game.) Let us finally consider a stopping game with the value function

$$\begin{aligned} V_*(x) &= \inf_{\eta} \sup_{\tau} E_x\Big[e^{-r(\eta\wedge\tau)}\Big(H_1(X_\eta)\, I(\eta < \tau) + H_2(X_\tau)\, I(\tau \le \eta)\Big)\Big] \\ &= \sup_{\tau} \inf_{\eta} E_x\Big[e^{-r(\eta\wedge\tau)}\Big(H_1(X_\eta)\, I(\eta < \tau) + H_2(X_\tau)\, I(\tau \le \eta)\Big)\Big] \end{aligned} \tag{4.25}$$

where $H_i(x) = (x-L_i)\, I(L_i \le x < K_i) + (K_i - L_i)\, I(x \ge K_i)$, $x > 0$, $i = 1, 2$, are payoff functions with some constants L_i and K_i such that $0 < L_i < K_i$, as well as $L_1 < L_2$, $K_1 < K_2$

and $K_1 - L_1 = K_2 - L_2$. It is shown in [5] that the value function in (4.25) takes the form

$$V_*(x) = \begin{cases} H_1(x), & \text{if } 0 < x \le a_* \\ V(x; a_*, b_*), & \text{if } a_* < x < b_* \\ H_2(x), & \text{if } x \ge b_* \end{cases} \tag{4.26}$$

and the optimal stopping times τ_* and η_* in (4.25) have the form

$$\eta_* = \inf\{t \ge 0 \,|\, X_t \le a_*\} \tag{4.27}$$

$$\tau_* = \inf\{t \ge 0 \,|\, X_t \ge b_*\} \tag{4.28}$$

for some numbers $0 < L_1 \le a_* < b_* \le K_2 < \infty$ to be determined.

Adopting the schema of arguments applied above to the stopping game in (4.25), we obtain that in (2.19) the gain function $G(a, b)$ admits the representation

$$G(a,b) = \frac{(b - L_2)(a^{\gamma_+} - a^{\gamma_-}) - (a - L_1)(b^{\gamma_+} - b^{\gamma_-})}{a^{\gamma_+} b^{\gamma_-} - b^{\gamma_+} a^{\gamma_-}}. \tag{4.29}$$

It attains its saddle value at the couple $a_* = \overline{a} \vee L_1$ and $b_* = \overline{b} \wedge K_2$ with $\overline{a} \in [L_1, L_1(r/\delta))$ and $\overline{b} \in (L_2(r/\delta), K_2]$, being a unique solution of the system of equations in (2.17)–(2.18), whenever it exists (see [5; Section 3]). Then, we have

$$C_+(a,b) = \frac{(a - L_1) b^{\gamma_-} - (b - L_2) a^{\gamma_-}}{a^{\gamma_+} b^{\gamma_-} - b^{\gamma_+} a^{\gamma_-}} \tag{4.30}$$

$$C_-(a,b) = \frac{(b - L_2) a^{\gamma_+} - (a - L_1) b^{\gamma_+}}{a^{\gamma_+} b^{\gamma_-} - b^{\gamma_+} a^{\gamma_-}} \tag{4.31}$$

so that, the function $p(a, b)$ takes the form

$$p(a,b) = \frac{(a - L_1) b^{\gamma_-} - (b - L_2) a^{\gamma_-}}{(b - L_2)(a^{\gamma_+} - a^{\gamma_-}) - (a - L_1)(b^{\gamma_+} - b^{\gamma_-})} \tag{4.32}$$

for some $0 < L_1 \le a < b \le K_2 < \infty$.

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References

[1] Beibel, M. and Lerche, H. R. (1997). A new look at optimal stopping problems related to mathematical finance. *Statistica Sinica* **7** (93–108).

[2] Beibel, M. and Lerche, H. R. (2000). A note on optimal stopping of regular diffusions under random discounting. *Theory of Probability and Its Applications* **45** (657–669).

[3] Bensoussan, A. and Lions, J. L. (1982). *Applications of Variational Inequalities in Stochastic Control.* North Holland, Amsterdam.

[4] Dynkin, E. B. (1963). The optimum choice of the instant for stopping a Markov process. *Soviet Mathematical Doklady* **4** (627–629).

[5] Gapeev, P. V. (2005). The spread option optimal stopping game. *Exotic Option Pricing and Advanced Levy Models.* Kyprianou, A., Schoutens, W. and Wilmott, P. (eds.) Wiley, Chichester (293–305).

[6] Grigelionis, B. I. and Shiryaev, A. N. (1966). On Stefan's problem and optimal stopping rules for Markov processes. *Theory of Probability and Its Applications* **11** (541–558).

[7] Itô, K. and McKean, H. P. (1965). *Diffusion Processes and Their Sample Paths.* Springer, Berlin.

[8] Karatzas, I. and Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus.* (Second Edition) Springer, New York.

[9] Karlin, S. and Taylor, H. M. (1981). *A Second Course in Stochastic Processes.* Academic Press, London – New York.

[10] Kramkov, D. O. and Mordecki, E. (1994). Integral opton. *Theory of Probability and Its Applications* **39**(1) (201–211).

[11] Lerche, H. R. and Urusov, M. (2007). Optimal stopping via measure transformation: The Beibel–Lerche approach. *Stochastics* **3–4** (275–291).

[12] McKean, H. P. (1965). Appendix: A free boundary problem for the heat equation arising from a problem of mathematical economics. *Industrial Management Review* **6** (32–39).

[13] Mucci, A. G. (1978). Existence and explicit determination of optimal stopping times. *Stochastic Processes and Their Applications* **8** (33–58).

[14] Peskir, G. and Shiryaev, A. N. (2006). *Optimal Stopping and Free-Boundary Problems.* Birkhäuser, Basel.

[15] Protter, Ph. E. (2004). *Stochastic Integration and Differential Equations.* (Second Edition) Springer, Berlin.

[16] Rogers, L. C. G. and Williams, D. (1987). *Diffusions, Markov Processes and Martingales II. Itô Calculus.* Wiley, New York.

[17] Salminen, P. (1985). Optimal stopping of one-dimensional diffusions. *Mathematische Nachrichten* **124** (85–101).

[18] Shepp, L. A. and Shiryaev, A. N. (1994). A new look at the pricing of Russian options. *Theory of Probability and Its Applications* **39**(1) (103–119).

[19] Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion.* Springer, Berlin.

[20] Shiryaev, A. N. (1978). *Optimal Stopping Rules.* Springer, Berlin.

[21] Shiryaev, A. N. (1999). *Essentials of Stochastic Finance.* World Scientific, Singapore.