Tutte Polynomials of Bracelets

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Abstract

The identity linking the Tutte polynomial with the Potts model on a graph implies the existence of a decomposition resembling that previously obtained for the chromatic polynomial. Specifically, let $\{G_n\}$ be a family of bracelets in which the base graph has b vertices. Then the Tutte polynomial of G_n can be written as a sum of terms, one for each partition π of a non-negative integer $\ell \leq b$:

$$(x-1)T(G_n; x, y) \ = \ \sum_{\pi} m_\pi(x, y)\, \mathrm{tr}(N_\pi(x, y))^n.$$

The matrices $N_\pi(x, y)$ are (essentially) the constituents of a 'Potts transfer matrix', and their 'multiplicities' $m_\pi(x, y)$ are obtained by substituting $k = (x-1)(y-1)$ in the expressions $m_\pi(k)$ previously obtained in the chromatic case. As an illustration, we shall give explicit calculations for bracelets in which b is small, obtaining (for example) an exact formulae for the Tutte polynomials of the quartic plane ladders.

Tutte Polynomials of Bracelets

1. Introduction

The *Tutte polynomial* of a graph G is a two-variable polynomial $T(G;x,y)$ that provides a great deal of useful information about G. It has applications to the study of colourings and flows on G [**4**], as well as reliability theory [**7**, **12**], knot theory [**14**], and statistical physics [**5**, **6**, **10**].

A paper [**1**] published in 1972 discussed families of graphs $\{G_n\}$ for which the Tutte polynomials $T_n = T(G_n;x,y)$ satisfy a linear recursion of the form

$$T_{n+r} + a_1 T_{n+r-1} + \cdots + a_r T_n = 0,$$

where the a_i are polynomials in x and y with integer coefficients. The families were of the kind that we now call *bracelets* (see Section 2 for the definitions). More examples of this kind were studied by D.A. Sands in his thesis [**9**], but work on these lines soon ground to a halt. This was due partly to the primitive state of computer algebra at the time, and partly to the fact that the only theoretical method available was the deletion-contraction algorithm, which is very inefficient.

However, around the same time it became clear [**13**] that the Tutte polynomial is closely related to the 'interaction models' that occur in statistical physics, and that tools from that field can be applied. One such tool is the *transfer matrix*. Much later it transpired that, in the special case of the chromatic polynomial, the transfer matrix could be analysed by applying the theory of representations of the symmetric group, and this led to explicit formulae for the chromatic polynomials of bracelets [**3**].

In this paper similar methods will be applied to the Tutte polynomial. The key is the identity linking the Tutte polynomial with the Potts model on a graph, which implies the existence of a decomposition resembling that used in the chromatic case. Specifically, let $\{G_n\}$ be a family of bracelets in which the base graph has b vertices. Then the Tutte polynomial of G_n can be written as a sum of terms, one for each partition π of a non-negative integer $\ell \leq b$:

$$(x-1)T(G_n;x,y) \;=\; \sum_{\pi} m_\pi(x,y)\,\mathrm{tr}(N_\pi(x,y))^n.$$

The matrices $N_\pi(x,y)$ are (essentially) the constituents of a 'Potts transfer matrix' (Section 2), and their 'multiplicities' $m_\pi(x,y)$ are obtained by substituting $k=(x-1)(y-1)$ in the expressions $m_\pi(k)$ previously obtained in the chromatic case (Section 3). As an illustration, we shall give explicit calculations for bracelets in which b is small, obtaining (for example) an exact formulae for the Tutte polynomials of the quartic plane ladders.

2. The Potts transfer matrix

Let G be a connected graph with vertex set V and edge-set E. In the *Potts model* on G we assign to each 'state' $\sigma : V \to \{1, 2, \dots, k\}$, a weight defined in terms of interactions $i_\sigma(e)$ $(e \in E)$. Specifically, if e has vertices v and w, then

$$i_\sigma(e) = \begin{cases} 1 & \text{if } \sigma(v) = \sigma(w); \\ 0 & \text{otherwise.} \end{cases}.$$

The *partition function* for the Potts model is defined to be

$$ZP(G; k, y) = \sum_{\sigma:V\to\{1,\dots,k\}} \prod_{e\in E} y^{i_\sigma(e)}.$$

Clearly ZP is a polynomial function of the two parameters k and y. For example, suppose $G = K_3$, with $V = \{1, 2, 3\}$ and $E = \{12, 13, 23\}$. Then the interaction weight is y^3 for k states, y for $3k(k-1)$ states, and 1 for the remaining $k(k-1)(k-2)$ states. Hence

$$ZP(K_3; k, y) = ky^3 + 3k(k-1)y + k(k-1)(k-2).$$

The result holds only for integer values of k, but we can consider ZP as a polynomial in $\mathbb{Z}[k, y]$, given by the same formula. Now it turns out that if we introduce a new indeterminate x defined by the substitution $k = (x-1)(y-1)$, then $ZP(G; (x-1)(y-1), y)$ is divisible by $(x-1)(y-1)^{|V|}$ in $\mathbb{Z}[x, y]$ For example,

$ZP(K_3; (x-1)(y-1), y)$

$= (x-1)(y-1)y^3+3(x-1)(y-1)(xy-x-y)y+(x-1)(y-1)(xy-x-y)(xy-x-y-1)$

$= (x-1)(y-1)^3(x^2 + x + y).$

The general result is established by a calculation that also identifies the complementary factor as the Tutte polynomial $T(G; x, y)$.

Theorem 1 The following identity holds in $\mathbb{Z}[x, y]$:

$$(x-1)(y-1)^{|V|}T(G; x, y) = ZP(G; (x-1)(y-1), y).$$

Proof This result is quite old [**2**, **13**]; a recent and authoritative account is given by Sokal [**10**]. □

A *bracelet* G_n is formed by taking n copies of a graph B and joining each copy to the next by a set of links L (with $n+1=1$ by convention). For each choice of B and L we get a family of graphs for which the transfer matrix method is appropriate.

Let V be the vertex-set of the base graph B. Construct the graph J which has two disjoint copies of V for its vertex-set, say V_1 and V_2, and the following edges. The vertices in V_1 are joined among themselves by edges as in B, the vertices in V_1 are joined to those in V_2 by the links L, and the vertices in V_2 are not joined among themselves. Suppose J is 'coloured' by assigning to V_1 the state σ, and V_2 the state τ. Then we say that an edge of J is *monochromatic* if both its vertices receive the same colour. The matrix $M = M(k, y)$ has $k^{|V|}$ rows and columns, indexed by the 'states' $\sigma, \tau : V \to \{1, 2, \ldots, k\}$. The entries of M are given by

$$M_{\sigma\tau} = y^{\mu(\sigma,\tau)},$$

where $\mu(\sigma, \tau)$ is the number of monochromatic edges, as defined above.

Theorem 2

$$ZP(G_n; k, y) \ = \ \mathrm{tr}\, M(k, y)^n.$$

Proof The proof is standard. A version directly applicable to the present formulation can be found in [**2**, p.26]. □

The essence of the transfer matrix method is that the eigenvalues of M^n are the nth powers of the eigenvalues of M, so the spectrum of M determines $ZP(G_n; k, y)$ for all values of n. Furthermore, M commutes with the obvious action of the symmetric group Sym_k on the states, and hence the spectral decomposition of M can be deduced from the corresponding Sym_k-module.

3. The Specht decomposition

In this section b and k are fixed positive integers, and $[b]$, $[k]$ are sets of the corresponding sizes (for convenience $k > 2b$).

Following the remark at the end of the previous section, we begin by considering certain modules determined by actions of the symmetric group. For a partition π of an integer ℓ we write $|\pi| = \ell$, and denote the parts of π by $\pi_1, \pi_2, \ldots, \pi_\ell$, where zeros are included as necessary. Let $d(\pi)$ be the dimension of the irreducible representation of the symmetric group Sym_ℓ that corresponds to π. There is an explicit formula for $d(\pi)$ (the hook formula) [**8**, p.124]. For $k > 2|\pi|$, let π^k denote the partition of k formed from π by adding a part $k - |\pi|$. The hook formula for $d(\pi^k)$ turns out to be a polynomial function of k [**3**]: we write it as

$$m_\pi(k) = \frac{d(\pi)}{|\pi|!} \prod_{i=1}^{|\pi|} (k - |\pi| - \pi_i + i).$$

For the partitions with $|\pi| \leq 3$ the formulae are

$$m_{[0]}(k) = 1, \quad m_{[1]}(k) = k-1, \quad m_{[20]}(k) = \frac{1}{2}k(k-3), \quad m_{[11]}(k) = \frac{1}{2}(k-1)(k-2),$$

$$m_{[300]}(k) = \frac{1}{6}k(k-1)(k-5), \quad m_{[210]}(k) = \frac{1}{3}(k-2)(k-4)(k-6),$$

$$m_{[111]}(k) = \frac{1}{6}(k-1)(k-2)(k-3).$$

Let $\iota : [b] \to [k]$ be an injection and consider the action $\iota \mapsto \alpha\iota$ of Sym_k on the vector space generated by such injections. In the work on the chromatic case [**3**] it was shown that the resulting module has $e(b, \pi)$ constituents (Specht modules) of dimension $m_\pi(k)$, where

$$e(b, \pi) = \binom{b}{|\pi|} d(\pi).$$

Let $S(b, r)$ denote the number of partitions of a set of size b into r parts (the Stirling number of the second kind).

Theorem 3 The module defined by the action of Sym_k on the vector space generated by all functions $[b] \to [k]$ has $e^*(b, \pi)$ constituents of dimension $m_\pi(k)$, where

$$e^*(b, \pi) = \sum_{r=1}^{b} S(b, r) \binom{r}{|\pi|} d(\pi).$$

Proof The proof depends on an elementary decomposition of the space of functions (see, for example, [**11**, p.35]).

Let $V(b, k), U(b, k)$ be the modules generated by the injections $[b] \to [k]$ and the functions $[b] \to [k]$ respectively. A function $\sigma : [b] \to [k]$ takes exactly r distinct values, for some $1 \leq r \leq b$, and so determines a partition of $[b]$ into r parts. The set of all functions that induce a given partition generates a submodule isomorphic to $V(r, k)$. Hence, for $1 \leq r \leq b$, $U(b, k)$ has $S(b, r)$ submodules $V(r, k)$, and each of these has $e(r, \pi)$ constituents of dimension

$m_\pi(k)$. (The constituents depend on r, but not the partition.) It follows from the decomposition of V quoted above that $U(b,k)$ has

$$\sum_{r=1}^{b} S(b,r)e(r,\pi) \;=\; \sum_{r=1}^{b} S(b,r)\binom{r}{|\pi|}d(\pi)$$

constituents of dimension $m_\pi(k)$. □

Our main result is now at hand. Given a permutation $\alpha \in Sym_k$, let $R(\alpha)$ be the matrix representing the action of α on $U(b,k)$, that is

$$R(\alpha)_{\sigma\tau} = \begin{cases} 1 & \text{if } \alpha\sigma = \tau, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that the transfer matrix $M = M(k,y)$ satisfies $MR(\alpha) = R(\alpha)M$, for all α in Sym_k. In other words, M belongs to the commutant algebra of the module $U(b,k)$. An important result [**8**, Theorem 1.7.8] tells us that the constituents of the commutant algebra correspond to those of the module, but with the dimensions and multiplicities switched. It follows that $M(k,y)$ has $m_\pi(k)$ constituents $M_\pi(k,y)$ for each π with $|\pi| \leq b$, where $M_\pi(k,y)$ is a matrix of size $e^*(b,\pi)$.

When we substitute $k = (x-1)(y-1)$ in $M(k,y)$ and apply Theorems 1 and 2, powers of $y-1$ arise. In fact, for each constituent $M_\pi(k,y)$ we can write

$$M_\pi((x-1)(y-1),y) \;=\; (y-1)^b N_\pi(x,y),$$

where $N_\pi(x,y)$ is a matrix of size $e^*(b,\pi)$ with entries in $\mathbb{Z}[x,y]$.

Theorem 4 Let $\{G_n\}$ be a family of bracelets with the base graph having b vertices. Then the Tutte polynomials of the graphs G_n can be written in the form of a sum over all integer-partitions π with $|\pi| \leq b$:

$$(x-1)T(G_n;x,y) \;=\; \sum_\pi m_\pi(x,y)\,\mathrm{tr}(N_\pi(x,y))^n.$$

Here $m_\pi(x,y)$ is a polynomial function of $k=(x-1)(y-1)$ and $N_\pi(x,y)$ is a matrix of size $e^*(b,\pi)$ with entries in the ring $\mathbb{Z}[x,y]$. □

For $b = 2,3$, the dimensions of the relevant matrices are

π	[0]	[1]	[20]	[11]	[300]	[210]	[111]
$d(\pi)$	1	1	1	1	1	2	1
$e^*(2,\pi)$	2	3	1	1	–	–	–
$e^*(3,\pi)$	5	10	6	6	1	2	1

4. The case $|V| = 2$

It follows from the theory given above that, for any family of bracelets in which the base graph B has vertex-set $V = \{1, 2\}$, the matrix $M = M(k, y)$ has

1 constituent of dimension 2,
$k-1$ isomorphic constituents of dimension 3,
$k(k-3)/2$ isomorphic constituents of dimension 1,
$(k-1)(k-2)/2$ isomorphic constituents of dimension 1.

The constituents are represented by matrices which depend on the linking set L, although their sizes 2,3,1,1, and their multiplicities do not.

We shall work out the details for the cases $B = K_2$, $L = \{11, 22\}$ and $L = \{11, 12, 22\}$, when the bracelets are the cubic and quartic plane ladders CPL_n and QPL_n respectively. The theory of Specht modules provides standard bases for the various constituents, but we can use any bases that arise naturally. For example, for the 2-dimensional constituent we define vectors $\mathbf{u}, \mathbf{v}$ as follows:

$$\mathbf{u}_\sigma = 1 \quad \text{for all } \sigma, \qquad \mathbf{v}_\sigma = \begin{cases} 1 & \text{if } \sigma_1 = \sigma_2; \\ 0 & \text{otherwise.} \end{cases} .$$

Lemma 4.1 For CPL_n the action of M on the vectors $\mathbf{u}, \mathbf{v}$ is given by

$$\begin{aligned} M\mathbf{u} &= (y+k-1)^2\mathbf{u} + (y+k-1)^2(y-1)\mathbf{v} \\ M\mathbf{v} &= (2y+k-2)\mathbf{u} + (y^2+y+k-2)(y-1)\mathbf{v}. \end{aligned}$$

Proof We have

$$(M\mathbf{u})_\sigma = \sum_\tau M_{\sigma\tau}\mathbf{u}_\tau = \sum_\tau M_{\sigma\tau}.$$

Given σ, there is one τ for which $\tau_1 = \sigma_1$ and $\tau_2 = \sigma_2$ both hold, $2(k-1)$ τ's for which exactly one of $\tau_1 = \sigma_1$, $\tau_2 = \sigma_2$ holds, and $(k-1)^2$ τ's for which neither equation holds. If $\sigma_1 = \sigma_2$, these three cases give $M_{\sigma\tau} = y^3, y^2, y$ respectively, while if $\sigma_1 \neq \sigma_2$, the values are $M_{\sigma\tau} = y^2, y, 1$ respectively. Hence

$$\begin{aligned} M\mathbf{u} &= y(y+k-1)^2\mathbf{v} + (y+k-1)^2(\mathbf{u}-\mathbf{v}) \\ &= (y+k-1)^2\mathbf{u} + (y+k-1)^2(y-1)\mathbf{v}. \end{aligned}$$

The equation for $M\mathbf{v}$ is obtained in a similar way. □

Substituting $k = (x-1)(y-1)$ in the coefficients obtained above we find

$$y+k-1 = x(y-1), \quad 2y+k-2 = (x+1)(y-1), \quad y^2+y+k-2 = (x+y+1)(y-1).$$

Thus if $\mathbf{v}^* = (y-1)\mathbf{v}$ the action of M on the subspace $\langle \mathbf{u}, \mathbf{v}^* \rangle$ is given by the matrix $(y-1)^2 N_{[0]}$ where

$$N_{[0]} = \begin{pmatrix} x^2 & x^2 \\ x+1 & x+y+1 \end{pmatrix}.$$

Next, the 3-dimensional constituent, which corresponds to the partition [1]. For each $i \in \{1, 2, \ldots, k\}$ define vectors $\mathbf{r}^i, \mathbf{s}^i, \mathbf{t}^i$ as follows:

$$\mathbf{r}^i_\sigma = \begin{cases} 1 & \text{if } \sigma_1 = i \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbf{s}^i_\sigma = \begin{cases} 1 & \text{if } \sigma_2 = i \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathbf{t}^i_\sigma = \begin{cases} 1 & \text{if } \sigma_1 = \sigma_2 = i \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.2 For CPL_n the action of M on the vectors $\mathbf{r}^i, \mathbf{s}^i, \mathbf{t}^i$ is given by

$$M\mathbf{r}^i = (y-1)(y+k-1)\mathbf{r}^i + (y-1)^2(y+k-1)\mathbf{t}^i + (y+k-1)(\mathbf{u} + (y-1)\mathbf{v}).$$

$$M\mathbf{s}^i = (y-1)(y+k-1)\mathbf{s}^i + (y-1)^2(y+k-1)\mathbf{t}^i + (y+k-1)(\mathbf{u} + (y-1)\mathbf{v}).$$

$$M\mathbf{t}^i = (y-1)\mathbf{r}^i + (y-1)\mathbf{s}^i + (y-1)^2(y+2)\mathbf{t}^i + \mathbf{u} + (y-1)\mathbf{v}.$$

Proof For convenience, when $\tau_1 = a$ and $\tau_2 = b$ we write $\tau = (ab)$, and $M_{\sigma\tau} = M(\sigma, (ab))$. With this notation we have

$$(M\mathbf{r}^i)_\sigma = \sum_{j=1}^{k} M(\sigma, (ij)).$$

There are five cases for this sum of k terms.

(1) If $\sigma_1 = i$, $\sigma_2 \neq i$, one term is y^2 and the remaining $k-1$ terms are y.
(2) If $\sigma_1 \neq i$, $\sigma_2 = i$, one term is y and the rest are 1.
(3) If $\sigma_1 = \sigma_2 = i$, one term is y^3 and the rest are y^2.
(4) If $\sigma_1 = \sigma_2 \neq i$ one term is y^2 and the rest are y.
(5) If σ_1, σ_2 and i are all different, one term is y and the rest are 1.

Hence

$$M\mathbf{r}^i = y(y+k-1)(\mathbf{r}^i - \mathbf{t}^i) + (y+k-1)(\mathbf{s}^i - \mathbf{t}^i) + y^2(y+k-1)\mathbf{t}^i + y(y+k-1)(\mathbf{v} - \mathbf{t}^{\mathbf{i}})$$

$$+(y+k-1)(\mathbf{u}-\mathbf{v}-\mathbf{r}^i-\mathbf{s}^i+2\mathbf{t}^i).$$

Rearranging, we get the equation in the statement of the lemma.
The equation for $M\mathbf{s}^i$ follows by symmetry. For $M\mathbf{t}^i$ we have

$$(M\mathbf{t}^i)_\sigma = M(\sigma,(ii)).$$

The values of this term in the five cases listed above are, respectively: $y, y, y^3, y, 1$. Hence

$$M\mathbf{t}^i = y(\mathbf{r}^i-\mathbf{t}^i)+y(\mathbf{s}^i-\mathbf{t}^i)+y^3\mathbf{t}^i+y(\mathbf{v}-\mathbf{t}^i)+(\mathbf{u}-\mathbf{v}-\mathbf{r}^i-\mathbf{s}^i+2\mathbf{t}^i),$$

which reduces to the stated equation. □

Lemma 4.2 shows that, in the action of M on $\mathbf{r}^i$, $\mathbf{s}^i$ and $(\mathbf{t}^*)^i = (y-1)\mathbf{t}^i$, the coefficients of those vectors are given by the matrix

$$(y-1)\begin{pmatrix} y+k-1 & 0 & (y+k-1) \\ 0 & y+k-1 & (y+k-1) \\ y-1 & y-1 & (y-1)(y+2) \end{pmatrix}.$$

Changing the basis by adding suitable multiples of $\mathbf{u}$ and $\mathbf{v}^*$ we obtain a subspace on which the action of M is represented by the matrix given above. On substituting $k=(x-1)(y-1)$ we obtain $(y-1)^2 N_{[1]}$, where

$$N_{[1]} = \begin{pmatrix} x & 0 & x \\ 0 & x & x \\ 1 & 1 & y+2 \end{pmatrix}.$$

Note that there are k such representations, one for each value of i. However, their 'sum' is zero, and so there are only $k-1$ linearly independent ones, in accordance with the fact that $m_{[1]}(k) = k-1$.
The 1-dimensional representations are derived from the action of M on the vectors $\mathbf{e}^{ij}$, where

$$(\mathbf{e}^{ij})_\sigma = \begin{cases} 1 & \text{if } \sigma_1 = i,\ \sigma_2 = j; \\ 0 & \text{otherwise;} \end{cases} \qquad (i \neq j).$$

A simple calculation gives the equation

$$M\mathbf{e}^{ij} = (y-1)^2\mathbf{e}^{ij} + \text{ terms in the vectors defined above.}$$

This leads to $k(k-1)$ representations with matrix $[(y-1)^2]$. The representations corresponding to [20] and [11] are the same, and taking into account the linear dependencies we obtain a total of

$$m_{[20]}(k) + m_{[11]}(k) = k^2 - 3k + 1$$

independent representations. Since $N_{[20]}$ and $N_{[11]}$ are both equal to the matrix [1], there is a 'constant' term $\delta(x,y)$, obtained by substituting $k = (x-1)(y-1)$ in k^2-3k+1.

The Tutte polynomial for CPL_n is therefore

$$(x-1)T(CPL_n;x,y) = \mathrm{tr}\,(N_{[0]})^n + (xy-x-y)\mathrm{tr}\,(N_{[1]})^n + \delta(x,y),$$

where $N_{[0]}$, $N_{[1]}$ are the matrices displayed above, and

$$\delta(x,y) = x^2y^2 - 2x^2y - 2xy^2 + x^2 + xy + y^2 + x + y - 1.$$

For any bracelet with $b=2$, the decomposition of $M(k,y)$ takes the same form as for CPL_n, and we can use the bases given above. For example, when $L = \{11, 12, 22\}$ we get the quartic plane ladders QPL_n.

Lemma 4.3 For QPL_n, the action of M on the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{r}^i$, $\mathbf{s}^i$, $\mathbf{t^i}$ is given by

$$M\mathbf{u} = (2y^2 + (3k-4)y + k^2 - 3k + 2)\mathbf{u} + (y-1)(y+k-1)(y^2+y+k-2)\mathbf{v},$$
$$M\mathbf{v} = (y^2+y+k-2)\mathbf{u} + (y-1)(y^3+y^2+k-2)\mathbf{v};$$

$$\begin{aligned} M\mathbf{r}^i &= (y-1)(2y+k-2)\mathbf{r}^i + (y-1)^2(y^2+y+k-2)\mathbf{t}^i \\ &\quad +(2y+k-2)\mathbf{u} + (y-1)(y^2+y+k-2)\mathbf{v}, \\ M\mathbf{s}^i &= (y-1)(y+k-1)\mathbf{r}^i + (y-1)(y+k-1)\mathbf{s}^i + (y-1)^2(y+2)(y+k-1)\mathbf{t}^i \\ &\quad +(y+k-1)(\mathbf{u} + (y-1)(y+k-1)\mathbf{v}), \\ M\mathbf{t}^i &= (y^2-1)\mathbf{r}^i + (y-1)\mathbf{s}^i + (y-1)^2(y^2-y+2)\mathbf{t}^i + \mathbf{u} + (y-1)\mathbf{v}. \end{aligned}$$ □

Applying the same reductions as for CPL_n, the result is as follows.

Theorem 5 The Tutte polynomials for the graphs QPL_n are given by

$$(x-1)T(QPL_n;x,y) = \mathrm{tr}(N_{[0]})^n + (xy-x-y)\,\mathrm{tr}(N_{[1]})^n + \delta,$$

where $N_{[0]}$, $N_{[1]}$, and δ are

$$N_{[0]} = \begin{pmatrix} x+x^2 & x(1+x+y) \\ 1+x+y & 1+x+2y+y^2 \end{pmatrix}, \quad N_{[1]} = \begin{pmatrix} 1+x & 0 & 1+x+y \\ x & x & 2x+xy \\ 1+y & 1 & 2+2y+y^2 \end{pmatrix},$$

$$\delta = x^2y^2 - 2x^2y - 2xy^2 + x^2 + xy + y^2 + x + y - 1.$$

□

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