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On filtration immersions and credit events*

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In credit risk models, it is usually assumed that the intensity processes contain all the necessary information about the default times. This is indeed the case when the appropriate immersion properties hold, so that one can compute the conditional law of the default times in terms of the intensity processes. In this paper, we characterize the immersion properties in terms of the terminal values of the compensators of the default processes. We also give an example of a model in which the immersion property does not hold, and the conditional law of the default times depend on the intensity and some other process.

1 Introduction

Modeling of default events is one of the interesting problems of modern finance. In the usual approach (see Lando [12]), the default times are constructed from given *intensity* processes adapted to some *reference* filtration, which contains all the information observable from the market. A more general approach was presented in Elliott, Jeanblanc and Yor [6]. The useful

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tool in that setting is the conditional law of the default times with respect to the given reference filtration. Starting with a family of given regular conditional laws (see Jiao [10]), it is usually possible to construct the default times (see Jeanblanc and Le Cam [8] and El Karoui, Jeanblanc and Jiao [5]). However, in the literature, there are only very few examples of such explicit conditional laws, except some trivial cases in which the *immersion* property holds. In those and only in those cases, the conditional laws of the default times can be computed in terms of the intensity processes (see El Karoui, Jeanblanc and Jiao [5]).

We recall that a filtration is said to be immersed in a larger one, whenever every martingale with respect to the former filtration keeps the martingale property with respect to the latter one (see, e.g. Mansuy and Yor [14; Chapter I]). For the first time, this situation was described in Brémaud and Yor [3] and referred to as the (H)-hypothesis. In the credit risk setting, as it was presented in Kusuoka [11], the given reference filtration is usually immersed in the filtration progressively enlarged with the default filtration. The assumption of filtration immersions under a particular equivalent martingale measure can be explained by the fact that the default times are unknown, and the addition of the default times into the model does not lead to the occurrence of arbitrage opportunities (see Jeanblanc and Le Cam [9]).

In reliability theory, where the reference filtration is trivial, Norros' lemma states the following assertion. If the failure times are finite, and neither two of them can occur at the same time, then the continuous compensator processes, evaluated at the failure times, are independent random variables, having standard exponential law (see, e.g. Norros [15]). We extend Norros' lemma for the case of credit risk models in which the reference filtration is no longer trivial. We prove that if the reference filtration is immersed into the filtrations progressively enlarged by any particular default time, then the terminal values of the compensator processes are independent of the observations. Moreover, we study the links between various immersion properties and (conditional) independence. When some additional condition also holds, the terminal values turn out to be conditionally independent with respect to the observations.

We introduce a model in which one asset is paying dividends with a rate, changing spontaneously from one fixed constant value to another one when some credit event occurs. Suppose that this change is hidden in the structure of the underlying asset values under an equivalent martingale measure. For simplicity of exposition, we consider a diffusion model for the asset value dynamics, and let the time of the credit event be exponentially distributed. We assume that the reference filtration is generated by the market prices of the risky asset, and that the time at which the credit event occurs is independent of the driving Brownian motion. In the resulting reduced-form model, the conditional probabilities of the occurrence of the credit event have the same form as the posterior probabilities of the occurrence of hidden changes in the drift rates of observed Wiener processes (see Shiryaev [21; Chapter IV]). We also generalize the model to a two-dimensional setting with constantly correlated driving Brownian motions.

The paper is organized as follows. In Section 2, we formulate a credit risk model with two default times and recall the notion of filtration immersions. In Section 3, we present an extension of Norros' lemma in the case of several defaults under non-trivial reference filtrations. In Section 4, we consider a diffusion model with random dividends and compute the conditional law of the default times in an explicit form.

2 The model

In this section, we introduce a credit risk model with two default times, and recall the notion of filtration immersions.

2.1 The setting

Let us suppose that on a probability space $(\Omega, \mathcal{G}, P)$ there exists a (nonnegative) finite random *default* time τ. Let $H = (H_t)_{t \ge 0}$ be the *default* process, associated with the default time τ and defined by $H_t = I(\tau \le t)$, where $I(\cdot)$ denotes the indicator function, and let $(\mathcal{H}_t)_{t \ge 0}$ be its natural filtration, so that $\mathcal{H}_t = \sigma(H_s \,|\, 0 \le s \le t)$, for all $t \ge 0$. Let us denote by $(\mathcal{F}_t)_{t \ge 0}$ the *reference* filtration, and define the filtrations $(\mathcal{G}_t)_{t \ge 0}$ by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, for $t \ge 0$. It is further assumed that all the considered filtrations are right-continuous and completed by all the sets of P-measure zero. Let $G = (G_t)_{t \ge 0}$ be the *conditional survival probability* process of the default time τ defined by $G_t = P[\tau > t \,|\, \mathcal{F}_t]$, for all $t \ge 0$.

Hypothesis 2.1. Assume that the process $G = (G_t)_{t \ge 0}$ is *continuous* and satisfies the condition $0 < G_t \le 1$, for all $t \ge 0$.

Note that the latter condition yields that τ is not an $(\mathcal{F}_t)_{t\geq 0}$-stopping time. Being a continuous $(\mathcal{F}_t)_{t\geq 0}$-supermartingale, the process G admits the continuous compensator $C = (C_t)_{t\geq 0}$ such that $C_0 = 1 - G_0 = 0$ and $G + C$ forms an $(\mathcal{F}_t)_{t\geq 0}$-martingale. In the same way, there exists a $(\mathcal{G}_t)_{t\geq 0}$-predictable increasing process $A = (A_t)_{t\geq 0}$ such that the process $M = (M_t)_{t\geq 0}$ defined by:

$$M_t = H_t - A_t \tag{2.1}$$

is a $(\mathcal{G}_t)_{t\geq 0}$-martingale. It is well known (see, e.g. [6]) that $A_t = A_{t\wedge\tau}$ and $A_t\, I(t \leq \tau) = \Lambda_t\, I(t \leq \tau)$, where $\Lambda = (\Lambda_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-predictable continuous increasing process given by:

$$\Lambda_t = \int_0^t \frac{dC_s}{G_s} \tag{2.2}$$

for all $t \geq 0$. It follows that the $(\mathcal{G}_t)_{t\geq 0}$-*compensator* process A of the default time τ is continuous (see, e.g. [1; Proposition 6.1.2]). Hence, the default time τ turns out to be a $(\mathcal{G}_t)_{t\geq 0}$-totally inaccessible stopping time (see, e.g. [18; Chapter VI, Section 13]). In the credit risk literature, A is called the $(\mathcal{G}_t)_{t\geq 0}$-*intensity* process, and Λ is called the $(\mathcal{F}_t)_{t\geq 0}$-*intensity* process of the default time τ.

2.2 Immersion properties

Let $(\mathcal{F}'_t)_{t\geq 0}$ and $(\mathcal{F}''_t)_{t\geq 0}$ be two filtrations such that $\mathcal{F}'_t \subseteq \mathcal{F}''_t$, for all $t \geq 0$. The filtration $(\mathcal{F}'_t)_{t\geq 0}$ is said to be *immersed* in the filtration $(\mathcal{F}''_t)_{t\geq 0}$ if any $(\mathcal{F}'_t)_{t\geq 0}$-martingale remains an $(\mathcal{F}''_t)_{t\geq 0}$-martingale. This notion is also known in the literature as (H)-*hypothesis* for the filtrations $(\mathcal{F}'_t)_{t\geq 0}$ and $(\mathcal{F}''_t)_{t\geq 0}$ (see, e.g. [3] or [14; Chapter V, Section 4]) and is equivalent to the conditional independence of $\mathcal{F}''_t$ and $\mathcal{F}'_\infty$ with respect to $\mathcal{F}'_t$. We recall that, in the particular case where $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, the filtration $(\mathcal{F}_t)_{t\geq 0}$ is immersed in the filtration $(\mathcal{G}_t)_{t\geq 0}$ if and only if:

$$P[\tau > t \,|\, \mathcal{F}_t] = P[\tau > t \,|\, \mathcal{F}_\infty] \tag{2.3}$$

holds true (see, e.g. [3] or [6]). Note that, in the case when $(\mathcal{F}_t)_{t\geq 0}$ is a trivial filtration (as it is assumed in the models of reliability theory), the (H)-hypothesis holds for $(\mathcal{F}_t)_{t\geq 0}$ and $(\mathcal{G}_t)_{t\geq 0}$ automatically. Observe that when (2.3) holds, the process G turns out to be decreasing, so that, because of the assumption of continuity of G, we have $C_t = 1 - G_t$, for all $t \geq 0$.

In the case of two default times τ_i, $i = 1, 2$, we denote by $H^i = (H^i_t)_{t \geq 0}$ the *default* process, associated with the default time τ_i, and by $(\mathcal{H}^i_t)_{t \geq 0}$ its natural filtration. We define the filtrations $(\mathcal{G}^i_t)_{t \geq 0}$ by $\mathcal{G}^i_t = \mathcal{F}_t \vee \mathcal{H}^i_t$, and $(\mathcal{G}_t)_{t \geq 0}$ by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}^1_t \vee \mathcal{H}^2_t$, for $t \geq 0$. For every $i = 1, 2$ fixed, let $G^i = (G^i_t)_{t \geq 0}$ be the $(\mathcal{F}_t)_{t \geq 0}$-*conditional survival probability* process of the default time τ_i, defined by $G^i_t = P[\tau_i > t \,|\, \mathcal{F}_t]$, for all $t \geq 0$. We also assume that Hypothesis 2.1 holds for $i = 1, 2$. We will further study the case where:

$$P[\tau_i > t \,|\, \mathcal{F}_t] = P[\tau_i > t \,|\, \mathcal{G}^{3-i}_t] \tag{2.4}$$

holds true, which is equivalent to $\mathcal{G}^i_t$ and $\mathcal{G}^{3-i}_t$ are conditionally independent with respect to $\mathcal{F}_t$, for $t \geq 0$ (see, e.g. [4]). Here, for $i = 1$ we have $3 - i = 2$, and for $i = 2$ we have $3 - i = 1$, obviously. We also see that $(\mathcal{G}^i_t)_{t \geq 0}$ is immersed in the filtration $(\mathcal{G}_t)_{t \geq 0}$ if and only if:

$$P[\tau_i > t \,|\, \mathcal{G}^{3-i}_t] = P[\tau_i > t \,|\, \mathcal{G}^{3-i}_\infty] \tag{2.5}$$

holds true for $t \geq 0$ and every $i = 1, 2$.

3 Extensions of Norros' lemma

In this section, we study the link between filtration immersions and properties of the terminal value of the compensator A. We begin with an assertion for a model with one default time. In the case of a trivial reference filtration, part (i) was obtained in [15; Theorem 2.1]. In the case of general reference filtration, the assertion of part (ii) and its inverse (see Remark 3.2 below) can be found as an exercise in [14; page 99, Example 38]. We keep a proof of this result for completeness.

Proposition 3.1. *Let the process $G = (G_t)_{t \geq 0}$ be continuous and such that $G_0 = 1$. Then the following conclusions hold:*

(i) the variable A_τ, defined in (2.2), has standard exponential law (with parameter 1);

(ii) if $(\mathcal{F}_t)_{t \geq 0}$ is immersed in $(\mathcal{G}_t)_{t \geq 0}$ (i.e., if (2.3) holds for all $t \geq 0$), then the variable A_τ is independent of $\mathcal{F}_\infty$.

Proof. (i) In this part, we reproduce the arguments from [15] for the reader convenience. Consider the process $L = (L_t)_{t \ge 0}$, defined by:

$$L_t = (1+z)^{H_t} \, e^{-zA_t} \tag{3.1}$$

for all $t \ge 0$ and any $z > 0$ fixed. Then, applying the integration by parts formula, we get:

$$dL_t = z \, e^{-zA_t} \, dM_t \tag{3.2}$$

where the process M, defined in (2.1), is a $(\mathcal{G}_t)_{t \ge 0}$-martingale. Hence, by virtue of the assumption that $z > 0$, it follows from (3.2) that L is also a $(\mathcal{G}_t)_{t \ge 0}$-martingale, so that:

$$E\big[(1+z)^{H_t} \, e^{-zA_t} \, \big| \, \mathcal{G}_s\big] = (1+z)^{H_s} \, e^{-zA_s} \tag{3.3}$$

holds true, for all $0 \le s \le t$. In view of the implied by $z > 0$ uniform integrability of L, we may let t go to infinity in (3.3). Therefore, setting s equal to zero in (3.3) and using the fact that $A_\infty = A_\tau$, we obtain:

$$E\big[(1+z) \, e^{-zA_\tau}\big] = 1 \, . \tag{3.4}$$

This means that the Laplace transform of A_τ is the same as one of a standard exponential variable and thus proves the claim. This property was also proved by Azéma.

(ii) Applying the change-of-variable formula, we get:

$$\begin{aligned} e^{-zA_t} &= 1 - z \int_0^t \exp\left(-z \int_0^s \frac{I(\tau > u)}{G_u} \, dC_u \right) \frac{I(\tau > s)}{G_s} \, dC_s \\ &= 1 - z \int_0^t \exp\left(-z \int_0^s \frac{dC_u}{G_u} \right) \frac{I(\tau > s)}{G_s} \, dC_s \\ &= 1 - z \int_0^t e^{-z\Lambda_s} \, \frac{I(\tau > s)}{G_s} \, dC_s \end{aligned} \tag{3.5}$$

for all $t \ge 0$ and any $z > 0$ fixed. Then, taking conditional expectations under $\mathcal{F}_t$ from both parts of expression (3.5) and applying Fubini's theorem, we obtain from the immersion of $(\mathcal{F}_t)_{t \ge 0}$ in $(\mathcal{G}_t)_{t \ge 0}$ that:

$$\begin{aligned} E[e^{-zA_t} \, | \, \mathcal{F}_t] &= 1 - z \int_0^t E\left[e^{-z\Lambda_s} \, \frac{I(\tau > s)}{G_s} \, \Big| \, \mathcal{F}_t \right] dC_s \\ &= 1 - z \int_0^t e^{-z\Lambda_s} \, \frac{P[\tau > s \, | \, \mathcal{F}_t]}{G_s} \, dC_s \\ &= 1 - z \int_0^t e^{-z\Lambda_s} \, dC_s \end{aligned} \tag{3.6}$$

for all $t \geq 0$. Hence, using the fact that the immersion of $(\mathcal{F}_t)_{t\geq 0}$ in $(\mathcal{G}_t)_{t\geq 0}$ implies that the process G is decreasing, and thus $C_t = 1 - G_t$ and $\Lambda_t = -\ln G_t$, we see from (3.6) that:

$$E\big[e^{-zA_t}\,|\,\mathcal{F}_t\big] = 1 + \frac{z}{1+z}\big((G_t)^{1+z} - (G_0)^{1+z}\big) \tag{3.7}$$

holds true, for all $t \geq 0$. Therefore, letting t go to infinity and using the assumption $G_0 = 1$, as well as the fact that $G_\infty = 0$ (P-a.s.), we have:

$$E\big[e^{-zA_\tau}\,|\,\mathcal{F}_\infty\big] = \frac{1}{1+z} \tag{3.8}$$

that signifies the desired assertion. Note that a similar result was obtained by N. El Karoui (private communication), by means of the time-change technique and under the assumption of the strict decrease of the process G. □

Remark 3.2. To show that an assertion inverse to part (ii) of Proposition 3.1 holds, we observe the assumption of continuity of G^i yields that the process A^i is continuous too. Then, the default time τ_i can be obviously represented in the form:

$$\tau = \inf\{t \geq 0\,|\,A_t \geq A^i_{\tau_i}\}\,. \tag{3.9}$$

Hence, if A_τ is independent of $\mathcal{F}_\infty$, we obtain:

$$P[\tau > t\,|\,\mathcal{F}_t] = P[A_\tau > A_t\,|\,\mathcal{F}_t] = P[A_\tau > A_t\,|\,\mathcal{F}_\infty] = P[\tau > t\,|\,\mathcal{F}_\infty] \tag{3.10}$$

for all $t \geq 0$, so that, condition (2.3) holds, signifying that $(\mathcal{F}_t)_{t\geq 0}$ is immersed in $(\mathcal{G}_t)_{t\geq 0}$ (see also [14; page 99, Example 38]).

Let us now formulate and prove the appropriate result for the two defaults setting.

Proposition 3.3. *Let the processes $G^i = (G^i_t)_{t\geq 0}$, $i = 1, 2$, be continuous and such that $G^i_0 = 1$, and assume that $P[\tau_1 = \tau_2] = 0$ is satisfied. Then the following conclusions hold:*

(i) if $(\mathcal{G}^i_t)_{t\geq 0}$, $i = 1, 2$, are immersed in $(\mathcal{G}_t)_{t\geq 0}$ (i.e., if (2.5) holds for all $t \geq 0$), then the variables $A^i_{\tau_i}$, $i = 1, 2$, are independent;

(ii) if (2.3) and (2.4) hold for all $t \geq 0$, then the variables $A^i_{\tau_i}$, $i = 1, 2$, are conditionally independent with respect to $\mathcal{F}_\infty$.

Proof. (i) Observe that condition (2.5) yields that the process L^i from (3.1) is also a $(\mathcal{G}_t)_{t\geq 0}$-martingale. Then, following the arguments from [15] and applying the implied by

$P[\tau_1 = \tau_2] = 0$ orthogonality of the pure jump processes L^i, $i = 1, 2$, we obtain:

$$E\big[(1+z_1)^{H^1_t}\, e^{-z_1 A^1_t}\, (1+z_2)^{H^2_t}\, e^{-z_2 A^2_t} \,\big|\, \mathcal{G}_s\big] = (1+z_1)^{H^1_s}\, e^{-z_1 A^1_s}\, (1+z_2)^{H^2_s}\, e^{-z_2 A^2_s} \tag{3.11}$$

for all $0 \le s \le t$. Hence, letting t go to infinity and setting s equal to zero in (3.11), we get:

$$E\big[(1+z_1)\, e^{-z_1 A^1_{\tau_1}}\, (1+z_2)\, e^{-z_2 A^2_{\tau_2}}\big] = 1 \tag{3.12}$$

from where, upon recalling (3.4), we see that:

$$E\big[e^{-z_1 A^1_{\tau_1}}\, e^{-z_2 A^2_{\tau_2}}\big] = E\big[e^{-z_1 A^1_{\tau_1}}\big]\, E\big[e^{-z_2 A^2_{\tau_2}}\big] \tag{3.13}$$

thus proving the claim.

(ii) Using the arguments from the part (ii) of Proposition 3.1 above, we see that (3.5) implies:

$$\begin{aligned} e^{-z_1 A^1_t}\, e^{-z_2 A^2_t} = 1 &- z_1 \int_0^t e^{-z_1 \Lambda^1_u}\, \frac{I(\tau_1 > u)}{G^1_u}\, dC^1_u - z_2 \int_0^t e^{-z_2 \Lambda^2_v}\, \frac{I(\tau_2 > v)}{G^2_v}\, dC^2_v \\ &+ z_1 z_2 \int_0^t \int_0^t e^{-z_1 \Lambda^1_u}\, e^{-z_2 \Lambda^2_v}\, \frac{I(\tau_1 > u, \tau_2 > v)}{G^1_u G^2_v}\, dC^1_u\, dC^2_v \end{aligned} \tag{3.14}$$

for all $t \ge 0$. Then, taking conditional expectations under $\mathcal{F}_t$ from both parts of the expression in (3.14) and applying Fubini's theorem, we have:

$$\begin{aligned} E\big[e^{-z_1 A^1_t}\, e^{-z_2 A^2_t} \,|\, \mathcal{F}_t\big] = 1 &- z_1 \int_0^t e^{-z_1 \Lambda^1_u}\, dC^1_u - z_2 \int_0^t e^{-z_2 \Lambda^2_v}\, dC^2_v \\ &+ z_1 z_2 \int_0^t \int_0^t e^{-z_1 \Lambda^1_u}\, e^{-z_2 \Lambda^2_v}\, \frac{P[\tau_1 > u, \tau_2 > v \,|\, \mathcal{F}_t]}{G^1_u G^2_v}\, dC^1_u\, dC^2_v \end{aligned} \tag{3.15}$$

for all $t \ge 0$. Observe that from assumptions (2.3) and (2.4) it follows that:

$$\begin{aligned} P[\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{F}_t] &= P[\tau_i > u \,|\, \mathcal{F}_t]\, P[\tau_{3-i} > v \,|\, \mathcal{F}_t] \\ &= P[\tau_i > u \,|\, \mathcal{F}_u]\, P[\tau_{3-i} > v \,|\, \mathcal{F}_v] = G^i_u G^{3-i}_v \end{aligned} \tag{3.16}$$

holds true, for all $0 \le u, v \le t$ and every $i = 1, 2$. Hence, using (3.7) and the fact that assumption (2.3) yields that the process G^i is decreasing, we get from (3.15)-(3.16) that:

$$\begin{aligned} E\big[e^{-z_1 A^1_t}\, e^{-z_2 A^2_t} \,|\, \mathcal{F}_t\big] = &\left(1 + \frac{z_1}{1+z_1}\, \big((G^1_t)^{1+z_1} - (G^1_0)^{1+z_1}\big)\right) \\ &\times \left(1 + \frac{z_2}{1+z_2}\, \big((G^2_t)^{1+z_2} - (G^2_0)^{1+z_2}\big)\right) \end{aligned} \tag{3.17}$$

for all $t \geq 0$. Therefore, letting t go to infinity and using the assumption $G_0^i = 1$ as well as the fact that $G_\infty^i = 0$ (P-a.s.), we obtain from uniform integrability of L^i, $i = 1, 2$, that:

$$E\big[e^{-z_1 A^1_{\tau_1}} \, e^{-z_2 A^2_{\tau_2}} \,|\, \mathcal{F}_\infty\big] = \frac{1}{1+z_1} \, \frac{1}{1+z_2} \tag{3.18}$$

from where, upon recalling (3.8), we conclude that:

$$E\big[e^{-z_1 A^1_{\tau_1}} \, e^{-z_2 A^2_{\tau_2}} \,|\, \mathcal{F}_\infty\big] = E\big[e^{-z_1 A^1_{\tau_1}} \,|\, \mathcal{F}_\infty\big] \, E\big[e^{-z_2 A^2_{\tau_2}} \,|\, \mathcal{F}_\infty\big] \tag{3.19}$$

that signifies the desired assertion. □

4 Some examples

In this section, we introduce a model of a financial market with random dividends, in which the dividend rates change their constant values when some unpredictable credit events occur. This model is a two-dimensional extension of the one considered in [19] (see also [21; Chapter IV, Section 4] or [16; Chapter VI, Section 22]) with the aim of solving the problem of detecting a change in the drift rate of an observed Wiener process.

4.1 A one-dimensional model with random dividends

Let $W = (W_t)_{t \geq 0}$ be a Brownian motion defined on the probability space $(\Omega, \mathcal{G}, P)$, and τ a random time, independent of W and such that $P[\tau > t] = e^{-\lambda t}$, for all $t \geq 0$ and some $\lambda > 0$ fixed. We define $X = (X_t)_{t \geq 0}$ as the process:

$$X_t = x \, \exp\left(\left(r - \frac{\sigma^2}{2} - \delta_0 \right) t - (\delta_1 - \delta_0)(t - \tau)^+ + \sigma \, W_t \right) \tag{4.1}$$

where $r \geq 0$, and x, σ, δ_j are some given strictly positive constants, for every $j = 0, 1$. It is easily shown that the process X solves the stochastic differential equation:

$$dX_t = X_t \left(r - \delta_1 + (\delta_1 - \delta_0) \, I(\tau > t) \right) dt + X_t \, \sigma \, dW_t \tag{4.2}$$

Let us assume that the process X describes the risk-neutral dynamics of the value of a dividend paying asset, and τ is a random time at which some credit event occurs, leading to a change of the dividend rate. In more details, the asset pays dividends at the rate δ_0 until the time τ

at which the credit event occurs and the dividend rate is changed to δ_1. Here, r is the interest rate of a riskless banking account, and σ is the volatility coefficient.

Let us denote by $(\mathcal{F}_t)_{t \geq 0}$ the natural filtration of the process X, that is $\mathcal{F}_t = \sigma(X_s \,|\, 0 \leq s \leq t)$, for all $t \geq 0$. By means of standard arguments (see, e.g. [21; Chapter IV, Section 4]), it is shown that the process X admits the following representation on its own filtration:

$$dX_t = X_t \big(r - \delta_1 + (\delta_1 - \delta_0) \, G_t \big) \, dt + X_t \, \sigma \, d\overline{W}_t \tag{4.3}$$

(see also [13; Chapter IX]). Here, $G = (G_t)_{t \geq 0}$ is the survival probability process given by $G_t = P[\tau > t \,|\, \mathcal{F}_t]$, for all $t \geq 0$, and the innovation process $\overline{W} = (\overline{W}_t)_{t \geq 0}$ defined by:

$$\overline{W}_t = W_t + \frac{\delta_1 - \delta_0}{\sigma} \int_0^t \big(I(\tau > s) - G_s \big) \, ds \tag{4.4}$$

is a standard Brownian motion, according to P. Lévy's characterization theorem (see, e.g. [17; Chapter IV, Theorem 3.6]). It is easily shown using the arguments based on the notion of strong solutions of stochastic differential equations (see, e.g. [13; Chapter IV]) that the natural filtration of $\overline{W}$ coincides with $(\mathcal{F}_t)_{t \geq 0}$. It follows from [13; Chapter IX] (see also [21; Chapter IV, Section 4]) that the process G solves the stochastic differential equation:

$$dG_t = -\lambda G_t \, dt + \frac{\delta_1 - \delta_0}{\sigma} \, G_t (1 - G_t) \, d\overline{W}_t \tag{4.5}$$

with $G_0 = 1$.

Observe that the process $(e^{\lambda t} \, G_t)_{t \geq 0}$ admits the representation:

$$d(e^{\lambda t} \, G_t) = \frac{\delta_1 - \delta_0}{\sigma} \, e^{\lambda t} \, G_t (1 - G_t) \, d\overline{W}_t \tag{4.6}$$

and thus, it is an $(\mathcal{F}_t)_{t \geq 0}$-martingale (to establish the true martingale property, note that the process $(G_t (1 - G_t))_{t \geq 0}$ is bounded). The equality (4.5) provides the Doob-Meyer decomposition of the super-martingale G, while $G_t = (G_t \, e^{\lambda t}) \, e^{-\lambda t}$ gives its multiplicative decomposition. From these decompositions, it follows that the $(\mathcal{F}_t)_{t \geq 0}$-intensity process of τ is λt.

Remark 4.1. We observe from (4.5) that the conditional survival probability G is not an increasing process, and we thus conclude that the filtration $(\mathcal{F}_t)_{t \geq 0}$ is not immersed in $(\mathcal{G}_t)_{t \geq 0}$. In particular, $\overline{W}$ is not a Brownian motion in the filtration $(\mathcal{G}_t)_{t \geq 0}$, which is equal to the filtration generated by W and H. Indeed, observing the process X after the random time τ provides information on the conditional law of τ.

4.2 The conditional laws in the one-dimensional model

From the definition of the conditional survival probability $G_t = P[\tau > t \,|\, \mathcal{F}_t]$ and the fact that $(G_t\, e^{\lambda t})_{t \ge 0}$ is a martingale, it follows that the conditional probability process can be expressed as:

$$P[\tau > u \,|\, \mathcal{F}_t] = E[P[\tau > u \,|\, \mathcal{F}_u] \,|\, \mathcal{F}_t] = E[G_u\, e^{\lambda u} \,|\, \mathcal{F}_t]\, e^{-\lambda u} = G_t\, e^{\lambda t}\, e^{-\lambda u} \tag{4.7}$$

for $0 \le t < u$.

From the standard arguments in [20; Chapter IV, Section 4] (which are compressed in [21; Chapter IV, Section 4]), resulting from the application of Bayes' formula, it follows that the conditional probability process can be expressed as:

$$P[\tau > u \,|\, \mathcal{F}_t] = 1 - \frac{Y_{u \wedge t}}{Y_t} + \frac{Z_{u \wedge t}}{Y_t}\, e^{-\lambda u} \tag{4.8}$$

for $t, u \ge 0$. Here, the process $Y = (Y_t)_{t \ge 0}$ is defined by:

$$Y_t = \lambda \int_0^t Z_s\, e^{-\lambda s}\, ds + Z_t\, e^{-\lambda t} \tag{4.9}$$

and the process $Z = (Z_t)_{t \ge 0}$ is given by:

$$Z_t = \exp\left(\frac{\delta_1 - \delta_0}{\sigma^2} \left(\ln \frac{X_t}{x} - \frac{2r - \delta_1 - \delta_0 - \sigma^2}{2}\, t \right) \right) \tag{4.10}$$

for all $t \ge 0$. In particular, from (4.9), we deduce that the process $(Z_t/Y_t)_{t \ge 0}$, is equal to $(e^{\lambda t}\, G_t)_{t \ge 0}$, hence is an $(\mathcal{F}_t)_{t \ge 0}$-martingale. Moreover, by tedious computations, we see that the process $(1/Y_t)_{t \ge 0}$, or its equivalent $(e^{\lambda t}\, G_t/Z_t)_{t \ge 0}$, admits the representation:

$$d\Big(\frac{1}{Y_t} \Big) = d\Big(\frac{e^{\lambda t}\, G_t}{Z_t} \Big) = -\frac{\delta_1 - \delta_0}{\sigma}\, \frac{e^{\lambda t}\, G_t}{Y_t}\, d\overline{W}_t \tag{4.11}$$

and thus, it is also an $(\mathcal{F}_t)_{t \ge 0}$-martingale. Hence, one checks that for fixed u, the expression in (4.9) defines an $(\mathcal{F}_t)_{t \ge 0}$-martingale.

We also get the representation:

$$P[\tau > u \,|\, \mathcal{F}_t] = \int_u^\infty \alpha_t(s)\, ds \tag{4.12}$$

where the density $\alpha_t(s)$ is given by:

$$\alpha_t(s) = \lambda \frac{Z_{s \wedge t}}{Y_t} e^{-\lambda s} \tag{4.13}$$

and hence, by the definition of the process Y in (4.9), we have:

$$\int_0^\infty \alpha_t(s)\, ds = 1 \tag{4.14}$$

as expected. Thus, we get an example of model with an explicit form of the density $\alpha_t(s)$ (see [8] and [5]).

Remark 4.2. This model shows that we can construct two random times with constant intensity λ, associated with the given Brownian motion $\overline{W}$. One by using the usual construction as $\tau = \theta$, where θ is an exponential random variable with parameter λ, independent of $\overline{W}$, another one as a random time with density $\alpha_t(s)$ given by (4.13) with respect to $\overline{W}$ (see [8]). Note that, in this case, the process X satisfying (4.3) admits a (quite complicated) closed form representation by virtue of the expressions in (4.8)-(4.10) as:

$$dX_t = X_t \left(a + \frac{bX_t^\nu}{\lambda e^{(\alpha+\lambda)t} \int_0^t X_s^\nu e^{-(\alpha+\lambda)s} ds + X_t^\nu} \right) dt + \sigma X_t \, d\overline{W}_t \tag{4.15}$$

in which coefficients a, b, ν and α are given in terms of δ_i, $i = 0, 1$. It is quite remarkable that, in both models, the prices of defaultable zero-coupon bonds turn out to be the same. Indeed, in the second construction the price of a defaultable zero-coupon bond is:

$$P[\tau > T \,|\, \mathcal{G}_t] = \frac{I(t<\tau)}{G_t} P[\tau > T \,|\, \mathcal{F}_t] = \frac{I(t<\tau)}{G_t} G_t e^{-\lambda T} e^{\lambda t} = I(t<\tau) e^{-\lambda(T-t)} \tag{4.16}$$

for $0 \le t \le T$, which is exactly the price obtained in the first construction. The same remark can be done for payoffs of the form $h(\tau)$ where h is a deterministic function

$$\begin{aligned} E[h(\tau)I(t<\tau\le T) \,|\, \mathcal{G}_t] &= \frac{I(t<\tau)}{G_t} E[h(\tau)I(t<\tau\le T) \,|\, \mathcal{F}_t] \\ &= \frac{I(t<\tau)}{G_t} \int_t^T h(s)\alpha_t(s)\, ds = I(t<\tau) \int_t^T h(s) e^{-\lambda(s-t)}\, ds \end{aligned} \tag{4.17}$$

for $0 \le t \le T$ (see [2; Corollary 2.2] for the computation of the price of the same payoff in the first construction). However, for payoffs of the form $F(X_T)I(T<\tau)$, we obtain:

$$E[F(X_T)I(T<\tau) \,|\, \mathcal{G}_t] = \frac{I(t<\tau)}{G_t} E[F(X_T)I(T<\tau) \,|\, \mathcal{F}_t] = \frac{I(t<\tau)}{G_t} E[F(X_T)G_T \,|\, \mathcal{F}_t] \tag{4.18}$$

and to complete these calculations, we shall need the joint marginal density of (Y, Z). This can be derived by means of the arguments from [7; Section 4] (see also [16; Chapter VI, Section 22]) using the given law of a geometric Brownian motion and its integral.

4.3 A two-dimensional model with random dividends

Suppose that on the initial probability space $(\Omega, \mathcal{G}, P)$ the random times τ_i, $i = 1, 2$, are independent, and $P[\tau_i > t] = e^{-\lambda_i t}$, for all $t \geq 0$ and some $\lambda_i > 0$ fixed. Assume that there exist two (constantly correlated) standard Brownian motions $W^i = (W^i_t)_{t \geq 0}$, $i = 1, 2$, such that $\langle W^1, W^2 \rangle_t = \rho t$, for all $t \geq 0$ and some $-1 < \rho < 1$ fixed. Let $X^i = (X^i_t)_{t \geq 0}$, be given by:

$$X^i_t = x_i \exp\left(\left(r - \frac{\sigma_i^2}{2} - \delta_{i,0}\right) t - (\delta_{i,1} - \delta_{i,0})(t - \tau_i)^+ + \sigma_i W^i_t\right) \tag{4.19}$$

where $r \geq 0$, and x_i, σ_i, $\delta_{i,j}$ are some given strictly positive constants, for every $i = 1, 2$ and $j = 0, 1$. Assume that the random times τ_i, $i = 1, 2$, are independent of the Brownian motions W^i, $i = 1, 2$, that implies the existence of such a pair of processes (X^1, X^2), by means of standard change-of-measure arguments. Let us assume that the processes X^i, $i = 1, 2$, describe the risk-neutral dynamics of the values of some dividend paying assets, and τ_i, $i = 1, 2$, are random times at which some credit events occur, leading to the changes of the dividend rates. In more details, for every $i = 1, 2$ fixed, the i-th asset pays dividends at the rate $\delta_{i,0}$ until the time τ_i at which the i-th credit event occurs and the dividend rate is changed to $\delta_{i,1}$. Here, r is the interest rate of a riskless banking account, and σ_i is the volatility coefficient.

Let us denote by $(\mathcal{F}_t)_{t \geq 0}$ the natural filtration of the processes X^i, $i = 1, 2$, that is $\mathcal{F}_t = \sigma(X^1_s, X^2_s \,|\, 0 \leq s \leq t)$, for all $t \geq 0$. By means of standard arguments (see, e.g. [21; Chapter IV, Section 4]), it is shown that the process X^i solves the stochastic differential equation:

$$dX^i_t = X^i_t \big(r - \delta_{i,1} + (\delta_{i,1} - \delta_{i,0})\, I(\tau_i > t)\big)\, dt + X^i_t \, \sigma_i \, dW^i_t \tag{4.20}$$

and admits the following representation on its own filtration:

$$dX^i_t = X^i_t \big(r - \delta_{i,1} + (\delta_{i,1} - \delta_{i,0})\, G^i_t\big)\, dt + X^i_t \, \sigma_i \, d\overline{W}^i_t \tag{4.21}$$

for every $i = 1, 2$. Here, $G^i = (G^i_t)_{t \geq 0}$ is the survival probability process given by $G^i_t = P[\tau_i > t \,|\, \mathcal{F}_t]$, for all $t \geq 0$, and the innovation process $\overline{W}^i = (\overline{W}^i_t)_{t \geq 0}$, defined by:

$$\overline{W}^i_t = W^i_t + \frac{\delta_{i,1} - \delta_{i,0}}{\sigma_i} \int_0^t \big(I(\tau_i > s) - G^i_s\big)\, ds \tag{4.22}$$

is a standard Brownian motion, according to P. Lévy's characterization theorem. It is easily shown that the natural filtration of $(\overline{W}^1, \overline{W}^2)$ coincides with $(\mathcal{F}_t)_{t \geq 0}$, and $\langle \overline{W}^1, \overline{W}^2 \rangle_t = \rho t$ for all $t \geq 0$.

4.4 The conditional laws in the two-dimensional model

Let us define the process $V = (V_t)_{t \geq 0}$ by:

$$V_t = \Psi_t^1 + \Psi_t^2 + \Phi_t^1 e^{-\lambda_2 t} + \Phi_t^2 e^{-\lambda_1 t} + e^{-\lambda_1 t} e^{-\lambda_2 t} \tag{4.23}$$

where the processes $\Psi^i = (\Psi_t^i)_{t \geq 0}$, $i = 1, 2$, are given by:

$$\Psi_t^i = \lambda_{3-i} \int_0^t \Phi_s^i \frac{Z_s^{i,0} Z_s^{3-i,1}}{Z_t^{i,0} Z_t^{3-i,1}} e^{-\lambda_{3-i} s} \, ds \tag{4.24}$$

with the processes $\Phi^i = (\Phi_t^i)_{t \geq 0}$, $i = 1, 2$, defined as:

$$\Phi_t^i = \lambda_i \int_0^t \frac{Z_w^{i,0}}{Z_t^{i,0}} e^{-\lambda_i w} \, dw \tag{4.25}$$

and the processes $Z^{i,j} = (Z_t^{i,j})_{t \geq 0}$ are obtained in terms of X^i by:

$$Z_t^{i,j} = \exp\left(\frac{\delta_{i,1} - \delta_{i,0}}{\sigma_i^2 (1 - \rho^2)} \Big(\ln \frac{X_t^i}{x_i} - \frac{\sigma_i \rho}{\sigma_{3-i}} \Big(\ln \frac{X_t^{3-i}}{x_{3-i}} - \frac{2r - 2\delta_{3-i,j} - \sigma_{3-i}^2}{2} \, t \Big) - \frac{2r - \delta_{i,0} - \delta_{i,1} - \sigma_i^2}{2} \, t \Big) \right) \tag{4.26}$$

for all $t \geq 0$ and every $i = 1, 2$ and $j = 0, 1$.

Using tedious computations, from the formulas (4.23)-(4.26), we get the representation which is the starting point of the study in [5]:

$$P[\tau_1 > u, \tau_2 > v \,|\, \mathcal{F}_t] = \int_u^\infty \int_v^\infty \alpha_t(s, w) \, ds dw \tag{4.27}$$

where the density $\alpha_t(s, w)$ is given by:

$$\alpha_t(s, w) = \frac{1}{V_t} \left(\lambda_1 \frac{Z_{s \wedge t}^{1,0}}{Z_t^{1,0}} e^{-\lambda_1 s} \, \lambda_2 \frac{Z_{w \wedge t}^{2,1}}{Z_t^{2,1}} e^{-\lambda_2 w} I(s < w) + \lambda_1 \frac{Z_{s \wedge t}^{1,1}}{Z_t^{1,1}} e^{-\lambda_1 s} \, \lambda_2 \frac{Z_{w \wedge t}^{2,0}}{Z_t^{2,0}} e^{-\lambda_2 w} I(s > w) \right) \tag{4.28}$$

for all $s, t, w \geq 0$. By the definition of the process V in (4.23), we have:

$$\int_0^\infty \int_0^\infty \alpha_t(s, w) \, ds dw = 1 \tag{4.29}$$

as expected. We also observe that, letting λ_2 tend to zero, we get that the conditional law in (4.16) with the density in (4.17) takes the form of one in (4.12) with the density in (4.13).

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