

Topological regular variation. I. Slow variation.

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Abstract

Motivated by the Category Embedding Theorem, as applied to convergent automorphisms [BOst11], we unify and extend the multivariate regular variation literature by a reformulation in the language of topological dynamics. Here the natural setting are metric groups, seen as normed groups (mimicking normed vector spaces). We briefly study their properties as a preliminary to establishing that the Uniform Convergence Theorem (UCT) for Baire, group-valued slowly-varying functions has two natural metric generalizations linked by the natural duality between a homogenous space and its group of homeomorphisms. Each is derivable from the other by duality. One of these explicitly extends the (topological) group version of UCT due to Bajšanski and Karamata [BajKar] from groups to flows on a group. A multiplicative representation of the flow derived in [Ost-knit] demonstrates equivalence of the flow with the earlier group formulation. In

companion papers we extend the theory to regularly varying functions: we establish the calculus of regular variation in [BOst14] and we extend to locally compact, σ -compact groups the fundamental theorems on characterization and representation [BOst15]. In [BOst16], working with topological flows on homogeneous spaces, we identify an index of regular variation, which in a normed-vector space context may be specified using the Riesz representation theorem, and in a locally compact group setting may be connected with Haar measure.

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1 Introduction

Regular variation was described in 1987 as ‘essentially a chapter in classical real-variable theory’ by [BGT], since then a standard reference for the subject. That phrase precisely delimits the theory’s original scope: asymptotic behaviour of functions $h : \mathbb{R} \rightarrow \mathbb{R}$ which, as in classical analysis, are measurable, or alternatively have the property of Baire.

These dual foundations of the single-variable theory have recently been unified in two ways in [BOst4] and [BOst11]. In one they are unified structurally by their identical combinatorics. In the other, both are derived from a single source: Baire category. Both views translate immediately to \mathbb{R}^d . However, more recent developments in probability theory go beyond the domain \mathbb{R} to the domain \mathbb{R}^d (e.g. [dHOR], [Om], [BalEmb]) and further still (see e.g. [Res1], [Res2], [Res3], and [HLMS]). Our purpose here is to derive from general category considerations a foundation for a topological theory of regular variation, as a better fit to the current broader needs. We employ the *language* of topological dynamics, specifically two key terms: flow and cocycle (see [GoHe], [Be], or the more recent [Ell1], for the former, and [Ell2] for the latter). The first term refers to the action of a group T with identity element e_T on a space X . A continuous *flow* $\varphi : T \times X \rightarrow X$ has, as its defining characteristic, the flow equation:

$$\varphi(st, x) = \varphi(s, \varphi(t, x)) \text{ with } \varphi(e_T, x) = x.$$

The cocycle of $h : \mathbb{R} \rightarrow \mathbb{R}$, as it occurs in classical regular variation in the study of the asymptotic behaviour of h , is

$$\sigma_h(t, x) = h(tx)h(x)^{-1}.$$

In topological dynamics the defining characteristic of the *cocycle* is the cocycle equation:

$$\sigma(st, x) = \sigma(s, tx)\sigma(t, x).$$

In this language, whenever it exists, the limit function of a regularly varying h is just the limit cocycle, defined by:

$$\partial_X h(t) := \lim_{x \rightarrow \infty} \sigma_h(t, x).$$

The cocycle equation implies the multiplicative equation

$$\partial_X h(st) = \partial_X h(s) \cdot \partial_X h(t),$$

so that, for h Baire, $\partial_X h$ is a power function. (For an algebraic view, see [Mac] Section 3.8 where the cocycle is introduced to measure deviation from homomorphism and motivates the definition of an obstruction in the extension problem of group theory; compare also [As] Section 17.) The flexible notation of topological dynamics on the one hand denotes $\varphi(t, x)$ by $t(x)$ and so points to the group of homeomorphisms on X , with composition as multiplication:

$$st(x) = s(t(x)) \text{ and } e_T = id_X,$$

(t is a homeomorphism since its group inverse t^{-1} provides the continuous inverse function.) The alternative notation of $\varphi^t(x)$, suggesting an orbit, may be abbreviated to x^t and the latter makes intuitive the connection with powers (the focus of regular variation); here the flow equation reads:

$$x^{st} = (x^t)^s \text{ and } x^e = x \quad (\text{or even, } x^1 = x).$$

The first unification mentioned above – generic regular variation – works in a wider space of functions including both of the classical ones, and depends on the Kestelman-Borwein-Ditor theorem; see [BOst4]. The second unification, for which see [BOst11], derives the measure and category forms of Kestelman-Borwein-Ditor Theorem from a single topological result, the Category Embedding Theorem (reviewed in Section 4 below), by specialization to two topologies – the Euclidean topology and the density topology (for

which see [BOst11]). Both unifications have immediate generalizations to \mathbb{R}^d . The reason in the first case is that both the Steinhaus and Piccard theorems (that the distance set of a measurable/Baire set contains an interval) may be derived using subuniversality (for which see [BOst3]). The reason in the second case is that \mathbb{R}^d is a Baire space both in the usual and in the density topology.

It was realized by Bajšanski and Karamata [BajKar] in 1969 that some of the foundational work of regular variation can in fact be conducted in a group-theoretic framework. More specifically, the context may be functions $h : G \rightarrow H$ with G, H topological groups, provided category or measure assumptions are placed on a subgroup T of G , and, furthermore, either continuity or measurability in t for $t \in T$ is demanded of the map $t \rightarrow h(tx)$ for fixed $x \in G$. (There are additional technicalities, e.g. H needs to be second countable in the second case and hence a separable metric group.) We show below that the subgroup T should be interpreted as a group of actions on G .

In their framework, provided the usual limit procedure is followed (relative to a fixed countably generated filter \mathcal{F} on G), the Uniform Convergence Theorem (UCT) relativized to T holds in relation to the limit function $\partial_G h(t) = \mathcal{F}\text{-}\lim_{g \in G} \sigma_h(t, g)$ and the limit is taken over a filter \mathcal{F} on G (see Section 3 for a statement of UCT), as does the Continuous Homomorphism Theorem of the companion paper [BOst14] (which we derive from the Continuous Coboundary Theorem). With this apparatus, Bajšanski and Karamata were able to develop a representation theorem for the case $h : \mathbb{R}^d \rightarrow \mathbb{R}$, thus widening the scope of regular variation to Euclidean spaces, a new step in its time; however, they left untouched the issue of representation in a more general context. Furthermore, with the aim of demonstrating the strength of their approach, they deduced the Equicontinuity Principle from the UCT. This was one of their purposes in formulating the UCT as relativized to a subgroup T ; another was the implied need (not explicitly stated in [BajKar]) to capture the scaling tx of a vector in \mathbb{R}^d , as a product $te \cdot x$, (with $T = \{t\mathbf{1} : t \in \mathbb{R}\}$ and $\mathbf{1} = (1, 1, \dots, 1)$).

A flow approach necessarily includes the formulation of Bajšanski and Karamata when T is a subgroup of a topological group X ; here the multiplicative flow $\tau : (t, x) \rightarrow \tau^t(x) := tx$, i.e. a left-translation through t under the group multiplication in the space X , describes exactly their setting. Our reformulation sits well with modern applications of regular variation and answers the need for a richer setting of actions such as affine actions (cf. self-similarity, for which see [BGT] Section 8.5, or the work of Balkema and

Embrechts on high risk scenarios, [BalEmb]); we shortly mention a lead examples. We show in [Ost-knit] that any continuous flow on a topological group may in fact be represented as a multiplicative flow on an appropriately constructed group. (Although we have more structure here, this is similar in spirit to the semi-direct product of group theory, which describes a ‘split extension’ of a group G by a group A of automorphisms of G ; see eg [As] Sect. 10.) We thus have some flexibility as to which of the flow and group formulations to use. We generally use flow language, on grounds of directness, intuitiveness and convenience.

We illustrate how the current theory encompasses modern applications in probability. For X a normed vector space, consider the usual action in the function space $L^1(X, \mathcal{A}, \mu)$ defined on an element f by the formula $\tau_t(f)(x) := f(t^{-1}x)$, or just $(tf)(x) = f(t^{-1}x)$, with τ_t referring now to a left-translation of the domain. In the subspace of indicator functions $\mathbf{1}_B$ with B a measurable set in a measure space (X, \mathcal{A}, μ) , we observe that, for non-zero scalars t , $\mathbf{1}_{tB}(x) = 1$ iff $\mathbf{1}_B(t^{-1}x) = 1$; thus the translate $t\mathbf{1}_B$ corresponds to the scaled set tB . (The transformation group, T , here is the multiplicative group \mathbb{R}^* of strictly positive reals). More recent work in multivariate stochastic processes defines regular variation of the distribution of a random element X by reference to the limit as $t \rightarrow \infty$ of the ratios

$$\frac{\mathbb{P}(\|X\| > xt, X/\|X\| \in A)}{\mathbb{P}(\|X\| > t)} = \frac{\mathbb{P}((xt)^{-1}X \in B_1^c, X \in \bigcup_{r>0}(rA \cap S_r))}{\mathbb{P}(t^{-1}X \in B_1^c)},$$

for $x, t \in \mathbb{R}^*$ (again the strictly positive reals), with $A \subseteq S_1$, where S_r is the r -sphere, and B_1^c the complement of a unit ball (see [Lind] and [HLMS]). This expression takes the form $\mu(tx\mathbf{1}_B \cap C)/\mu(t\mathbf{1}_B)$ and so may be interpreted as

$$h(tx\mathbf{1}_B \mathbf{1}_C)h(t\mathbf{1}_B)^{-1}$$

(where $tx\mathbf{1}_B$ denotes the translate $\tau_{tx}\mathbf{1}_B$ of the function $\mathbf{1}_B$, not its multiple). Applications in harmonic analysis based on the above action are given in [Ost-knit].

There is a natural duality between the space X and the action of T on X : indeed on a formal level one may interpret X via its topological second dual as acting on T (see [Ost-knit], or [BOst12] for details). On an informal level it is already clear that there are two possible interpretations of cocycles for $h : X \rightarrow H$, with H a group, according as one holds the space variable

fixed (as in [BajKar]), or the action variable fixed:

$$h(tx)h(x)^{-1} \text{ or } h(tx)h(te_X)^{-1}, \text{ with } e_X \text{ a distinguished point of } X.$$

Specializing to X a group (with its identity as e_X) and T a subgroup, we obtain the two cocycles

$$h(tx)h(x)^{-1}, \text{ or } h(tx)h(te_X)^{-1},$$

offering a *strong* limit $\partial_X h(t)$ over X , or a *weak* limit $\partial_T h(x)$ over T , once a filter \mathcal{F} on X or T is given. It is the metric context which most easily supplies the notion of limit. (We restrict to the metric case for convenience only – the theory might readily be developed in the setting of uniform spaces.)

Corresponding to the space-action duality, we develop a primal and a dual UCT, both in fact being examples of a single Action UCT. In the primal UCT, uniformity of convergence to $\partial_T h(x)$ holds on compact subsets of the space X ; in the dual UCT, uniformity of convergence to $\partial_X h(t)$ holds on compact subsets of the action group T (the Bajšanski-Karamata case). We demonstrate that the Equicontinuity Principle also follows from the primal UCT.

In companion papers we extend the theory to regularly varying functions: we establish the calculus of regular variation in [BOst14] and we extend to locally compact, σ -compact groups the fundamental theorems on characterization and representation [BOst15]. In [BOst16], working with topological \mathbb{R} -flows on homogeneous spaces, we identify an index of regular variation, which in a normed-vector space context may be specified using the Riesz representation theorem, and in a locally compact group setting may be connected with Haar measure; this embraces the representation results of Meerschaert and Scheffler (cf. [MeSh]).

2 Strong local homogeneity in normed groups

In the UCT of Section 5 we are concerned with metric, topological homogeneous spaces (defined below). As [MZ] say “homogeneous spaces although topologically more general than group manifolds form a very restricted class of spaces”. So, with the exception of Section 5, we develop the theory of regular variation in the context of metric groups. Indeed, when one surveys the literature of applications of multivariate regular variation, the context is

most usually \mathbb{R}^d or a function space, most usually $\mathcal{C}[0, 1]$, $\mathcal{D}[0, 1]$ with various norms, and also $\mathcal{U}[0, 1]$ the space of upper semicontinuous functions considered by T. Norberg [Nor]. These easily fall under the scope of our theory, which embraces the multivariate theory of regular variation developed in \mathbb{R}^d . In a way these examples are canonical, since any metric space is isometrically embeddable as a closed linearly independent subset of a normed vector space and, furthermore, any separable complete metric space is algebraically and topologically isomorphic to a Hilbert space ℓ_2 (see [BePe] Ch. 2.1 and Ch. 6 Prop 7.10).

As a matter of convenience, but again without loss of generality (in view of the Birkhoff-Kakutani Theorem, for which see [Bir], [Kak], or [Kel, Ch. 6 Problem O] or [Ru-FA2, Th. 1.24]), we restrict ourselves to *normed groups*: groups equipped with a group norm defined as follows.

Definition. We say that $\|\cdot\| : X \rightarrow \mathbb{R}_+$ is a *group-norm* if the following properties hold:

- (i) Subadditivity (Triangle inequality): $\|xy\| \leq \|x\| + \|y\|$;
- (ii) Positivity: $\|x\| > 0$ for $x \neq e$;
- (iii) Inversion (Symmetry): $\|x^{-1}\| = \|x\|$.

We say that a group-norm, is *abelian*, or more precisely *cyclically permutable* if

- (iv) Abelian norm (strong norm): $\|xy\| = \|yx\|$ for all x, y .

A normed group gives rise to a right-invariant metric $d_X(x, y) = \|xy^{-1}\|$ and a right-invariant metric gives rise to the norm $\|x\| := d_X(e, x)$ where e_X is the group identity. The group norm is abelian iff

$$\|xa(yb)^{-1}\| \leq \|xy^{-1}\| + \|ab^{-1}\|,$$

cf. [Klee]; for details and a wider discussion see [BOst12], in particular we note that when the group is metric, the Birkhoff-Kakutani Theorem Metrization (in fact Normability) Theorem assures the existence of a group norm which in the case of a non-compact group is unbounded (but of course bounded on compact subsets). So in what follows we may assume without loss that all metric groups are normed.

We denote by $Auth(X)$ the group of self-homeomorphisms of X under composition. $\mathcal{H}(X)$ denotes the subgroup

$$\{h \in Auth(X) : \|h\| < \infty\},$$

where, in turn,

$$\|h\| := d_X^*(h, e_{\mathcal{H}(X)}) = \sup d_X(h(x), x)$$

denotes the group-norm on $\mathcal{H}(X)$, which metrizes it by the right-invariant metric $d(g, h) = \|gh^{-1}\|$. This is for us the canonical example of a normed group.

A metric space (X, d) with *distinguished point* z_0 is said to be algebraically \mathcal{H} -homogeneous, if \mathcal{H} acts *transitively* on X , i.e. for any pair of points a, b there is a (bounded) homeomorphism $h \in \mathcal{H}(X)$ with $b = h(a)$ ([Kur-I] Ch. I. 13. XI). Of interest here is the theorem which goes back to work of van Dantzig and van der Waerden that for X a connected, locally compact metric space the isometries under the pointwise convergence topology form a locally compact group (acting properly), for which see [KoNo]. The space X is said to be a *topological \mathcal{H} -homogeneous space*, or a \mathcal{H} -coset space, if X is homeomorphic to $\mathcal{H}/\mathcal{H}_{z_0}$, where $\mathcal{H}_{z_0} = \{h \in \mathcal{H} : h(z_0) = z_0\}$ is the *stabilizer* of z_0 (see [Na] Ch. III. 3). Ford's Theorem ([For]) below identifies usefully for us when a homogeneous space is a coset space; we note in this connection Arens' Theorem referring in the locally compact case to the component of the identity ([Ar], or [MZ] Ch. II Th. 2.13) and results due to Freudenthal [Fr1],[Fr2], [Fr3]. Evidently, if \mathcal{H}_{z_0} is a normal subgroup of \mathcal{H} , then X is a topological group; thus, for \mathcal{H}_{z_0} trivial, X is a group, then referred to as a *principal* homogeneous space (cf. e.g. [Na] Ch. III. 3 p. 128). When this circumstance does not obtain, it may be possible to 'cut down' \mathcal{H} to the principal case. See [MZ] for the following scenario: for \mathcal{H} locally compact and first countable, \mathcal{H} contains a normal subgroup \mathcal{H}' such that \mathcal{H}/\mathcal{H}' is metrizable, and so if \mathcal{H} is effective (that is, the equation $h(x) = x$ holds for all $x \in X$ iff $h = e_{\mathcal{H}}$, meaning 'there is no identity but the Identity'), then $\mathcal{H}' = \{e\}$. In all these circumstances the intuitive picture of the group of actions \mathcal{H} is that of a group of topological shifts (i.e. continuous deformations by, say, left-translations).

We are usually interested in the case when \mathcal{H} acts continuously on X , i.e. X is given an \mathcal{H} -flow, so that \mathcal{H}_{z_0} is closed.

For the purposes of the UCT we need to consider a strengthening of algebraic \mathcal{H} -homogeneity which automatically leads, by Ford's Theorem below, to topological \mathcal{H} -homogeneity. Recall that the action of \mathcal{H} on X is *weakly transitive* if $\{h(e_X) : h \in \mathcal{H}\}$ is dense in X (cf. [Se] and [Itz]); see [RaoRao] who examine its relation to a category version of the Hewitt-Savage zero-one

law, a result closely connected to Kuratowski's zero-one law ([Kur-I] Ch. I. 13. XII). In the metric setting this demands that, for any $x_0 \in X$, there is a sequence $x_n = h_n(e_X)$ with $h_n \in \mathcal{H}$ and $x_n \rightarrow x_0$. This property is assumed in [Itz], where X is a uniform space, and is instrumental in making a locally compact uniform space X topologically homogeneous. In the UCT we will need the following strengthening.

Definition. For any subgroup \mathcal{H} of $\mathcal{H}(X)$, say that X has the \mathcal{H} -*crimping property* at x_0 if, for any sequence $x_n \rightarrow x_0$, there is a sequence of homeomorphisms ψ_n in \mathcal{H} converging to the identity, so necessarily in $\mathcal{H}(X)$, with $\psi_n(z_0) = z_n$. We refer to the ψ_n as a *crimping sequence* at x_0 . Say that X has the crimping property *globally* if it has the \mathcal{H} -crimping property at all points.

Note that weak transitivity would yield only a sequence $h_n(z_0)$ with $d(h_n(z_0), z_n) \rightarrow 0$ without even having the $\mathcal{H}(X)$ -norms converge to zero. Normed groups are a natural domain for regular variation theory, since left-translations provide the crimping property when the metric is right-invariant.

Proposition 1. *Let X be a normed group with the identity e_X as its distinguished point z_0 . Then:*

- (i) *for $z_n \rightarrow z_0$, the sequence $\psi_n(x) = z_n z_0^{-1} x$ is a $\mathcal{H}(X)$ -crimping sequence at z_0 ;*
- (ii) *$\sigma(x) := x_0 x$ is a bounded homeomorphism with $\sigma(z_0) := x_0$, and further:*
- (iii) *for $x_n \rightarrow x_0$, $\tilde{\psi}_n(x) := x_n x_0^{-1} x$ is a $\mathcal{H}(X)$ -crimping sequence at x_0 , such that, for the null sequence $z_n := x_0^{-1} x_n$, $\tilde{\psi}_n = \sigma \psi_n \sigma^{-1}$, i.e. $\tilde{\psi}_n$ is a conjugate of ψ_n .*

Thus X has the $\mathcal{H}(X)$ -crimping property globally.

Proof. Let d_X be a right-invariant metric. For (i),

$$\|\psi_n\| = \sup_x d_X(z_n z_0^{-1} x, x) = d_X(z_n z_0^{-1}, e_X) = d(z_n, z_0) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

For (ii), evidently $\|\sigma\| := d(x_0 x, x) = \|x_0\|$ is bounded. Then (iii) follows similarly. \square

In the absence of a right-invariant metric, i.e. in a general topological homogeneous space, we will demand the crimping property at all x_0 . However, one may sometimes pass from local crimping to global crimping, as the following result shows.

Proposition 2. *If X is $\mathcal{H}_u(X)$ -homogeneous and has the crimping property at z_0 , then X has the $\mathcal{H}_u(X)$ -crimping property globally. In particular, a Klee group is $\mathcal{H}_u(X)$ -homogeneous and so has the $\mathcal{H}_u(X)$ -crimping property globally.*

Proof. Suppose X has the crimping property at z_0 . Suppose $x_n \rightarrow x_0$ with x_0 any point in X . As X is $\mathcal{H}_u(X)$ -homogeneous, there is $\sigma \in \mathcal{H}_u(X)$ with $\sigma(z_0) = x_0$; so $z_n := \sigma^{-1}(x_n) \rightarrow \sigma^{-1}(x_0) = z_0$. Suppose ψ_n is a sequence converging to the identity with $\psi_n(z_0) = z_n$; then, by Lemma 2 of the previous section, $\tilde{\psi}_n = \sigma\psi_n\sigma^{-1}$ is also convergent to the identity and so verifies the crimping property at x_0 .

In a Klee group, for $\sigma_a(x) := a^{-1}x$, we have $\|\sigma_a\| := \|a^{-1}\| = \|a\|$ so $\sigma \in \mathcal{H}(X)$. Furthermore, by Proposition 7,

$$d(\sigma_a(x), \sigma_a(y)) = \|a^{-1}xy^{-1}a\| = \|xy^{-1}\| = d(x, y),$$

so $\sigma \in \mathcal{H}_u(X)$. \square

We recall here the algebraic treatment of homeomorphisms of X in the Weil approach to homogeneity – for which we follow [Bour] Part I Sections 3.4 and 3.5 for group actions on a topological space X (cf. [Na] Ch. 3 Sect. 3) There the focal point is a *topological group* G which acts *transitively*, i.e., for each x, y in X , there is g with $gy = x$. For an arbitrary fixed z in X , denote by H_z the *stabilizer* subgroup of elements h fixing z , i.e. with $hz = z$. The coset mapping taking gH_z to the point gz in X is a continuous bijective mapping onto X (noting that $gz = kz$ implies $g^{-1}k \in H_z$). If this mapping happens to be a homeomorphism, then X is called a *coset space* (cf. e.g. [vM]), or according to [Bour] a *topologically homogeneous space*; X is then represented by the algebraic quotient G/H_z . (Note that if h is in H_z and $gy = z$, then $hgy = gy$, i.e. $g^{-1}hgy = y$, so $g^{-1}H_zg$ is the subgroup fixing y ; thus the conclusions do not depend on the choice of z .) Of interest here is the necessary and sufficient condition for the bijection from G/H_z to X to be a homeomorphism, that for each fixed x , the mapping $g \rightarrow gx$ be open.

We compare the crimping property to a related notion: a homogenous space X is *strongly locally homogeneous* if, for each point x and each sufficiently small ε -ball around x , there is, for each pair of points a, b in the ball, a homeomorphism h_0 of the closed ball taking a to b and fixing its boundary; that is, $h_0(a) = b$ and $h_0(z) = z$ for z with $d(x, z) = \varepsilon$ (cf. strong local homogeneity as defined in [For]). By fixing the entire exterior of the ball we

may extend h_0 to a homeomorphism h of X with $\|h\| \leq \varepsilon$, thereby achieving $h \in \mathcal{H}(X)$.

This latter consideration leads to a further definition. Say that (X, d) is *locally crimping*, if, for any $a \in X$ and any sufficiently small $\varepsilon > 0$, there is $\delta > 0$ such that for all b with $d(a, b) < \delta$ there exists $h \in \mathcal{H}(X)$ with $\|h\| < \varepsilon$ and $b = h(a)$. (For a similar notion demanding also connectedness see [Mon2].) The argument of [For] (compare [AlpPras] Lemma 2.2 p. 10) referring to rotation around the midpoint of a, b shows that Euclidean manifolds, and more generally locally convex spaces, are strongly locally homogeneous. The local crimping property holds in any topological group with a right-invariant metric, since again the mapping $h(x) = ba^{-1}x$ satisfies $\|h\| = \sup_x d(ba^{-1}x, x) = d(a, b)$.

The crimping property is critical to the UCT. We include the proof of the key result here, Ford's theorem, as it is short.

Proposition 3 (Ford's Theorem, [For], [vM]). *Suppose that X is strongly locally homogeneous, or more generally has the local crimping property. Then, for any fixed x , the mapping $g \rightarrow g(x)$ is open, and so X is homogeneous in the sense of Weil with $X = \mathcal{H}(X)/\mathcal{H}_0(X)$, where $\mathcal{H}_0(X)$ is the stabilizer subgroup.*

Proof. Let U be open and $h_0 \in U$. There is $\varepsilon_0 > 0$ such that for all h with $d(h_0, h) < \varepsilon_0$, $h \in U$. Fix x . Put $z = h_0(x)$. Let $\varepsilon \leq \varepsilon_0$ be small enough and positive so that pairs of points of the ε -ball about z may be mapped to each other by a homeomorphism of norm less than ε . Then $\{y : d(y, z) < \varepsilon\} \subset \{g(x) : g \in U\}$. Indeed, for some $h \in \mathcal{H}(X)$ with $\|h\| \leq \varepsilon$ we have $y = h(z)$. But $d(h_0, hh_0) = d(id, h) = \|h\| < \varepsilon$, so $hh_0 \in U$ and $y = hh_0(x)$. \square

3 Unconditional Divergence

Definition. Let $\psi_n : X \rightarrow X$ be auto-homeomorphisms.

We say that a sequence ψ_n in $\mathcal{H}(X)$ *converges to the identity* if

$$\|\psi_n\| = d^*(\psi_n, id) := \sup_{t \in X} d(\psi_n(t), t) \rightarrow 0.$$

Thus, for all t , we have $z_n(t) := d(\psi_n(t), t) \leq \|\psi_n\|$ and $z_n(t) \rightarrow 0$. Thus the sequence $\|\psi_n\|$ is bounded.

Examples. In \mathbb{R} we may consider $\psi_n(t) = t + z_n$ with $z_n \rightarrow 0$. In a more general context, we note that a natural example of a convergent sequence of homeomorphisms is provided by a flow parametrized by *discrete time* (thus also termed a ‘chain’) towards a sink. If $\psi : \mathbb{N} \times X \rightarrow X$ is a flow and $\psi_n(x) = \psi(n, x)$, then, for each t , the orbit $\{\psi_n(t) : n = 1, 2, \dots\}$ is the image of the real null sequence $\{z_n(t) : n = 1, 2, \dots\}$.

We note, in the context of normalizations of a sequence of random variables by affine transformations, two normalizations are said to be equivalent if their transformations α_n, β_n are asymptotic in the sense that $\psi_n = \alpha_n^{-1}\beta_n$ converges to the identity ([BalEmb] Example 0.2).

Proposition. (i) For a sequence ψ_n in $\mathcal{H}(X)$, ψ_n converges to the identity iff ψ_n^{-1} converges to the identity.

(ii) Suppose X has abelian norm. For $h \in \mathcal{H}(X)$, if ψ_n converges to the identity then so does $h^{-1}\psi_n h$.

Proof. For (i), note that $\|\psi_n\| = \|\psi_n^{-1}\|$, the symmetry property of the norm, verified by

$$\|\psi_n\| := \sup_t d(\psi_n(t), t) = \sup_u d(u, \psi_n^{-1}(u)) = \|\psi_n^{-1}\|.$$

For (ii), note that $\|h^{-1}\psi_n h\| = \|h h^{-1}\psi_n\| = \|\psi_n\|$, by the assumed cyclic property. \square

Definitions.

1. For $\varphi_n : X \rightarrow X$ auto-homeomorphisms, we say that the sequence φ_n in \mathcal{G} *diverges uniformly* if for any $M > 0$ we have, for ultimately all n , that

$$d(\varphi_n(t), t) \geq M, \text{ for all } t.$$

Equivalently, putting

$$d_*(h, h') = \inf_{x \in X} d(h(x), h'(x)),$$

$$d_*(\varphi_n, id) \rightarrow \infty.$$

2. More generally, let $\mathcal{A} \subseteq \mathcal{H}(S)$ with \mathcal{A} a metrizable topological group. We say that α_n is a *pointwise divergent sequence in \mathcal{A}* if, for each $s \in S$,

$$d_S(\alpha_n(s), s) \rightarrow \infty,$$

equivalently, $\alpha_n(s)$ does not contain a bounded subsequence.

3. We say that α_n is a *uniformly divergent sequence in \mathcal{A}* if

$$\|\alpha_n\|_{\mathcal{A}} := d_{\mathcal{A}}(e_{\mathcal{A}}, \alpha_n) \rightarrow \infty,$$

equivalently, α_n does not contain a bounded subsequence.

Examples. In \mathbb{R} we may consider $\varphi_n(t) = t + x_n$ where $x_n \rightarrow \infty$. In a more general context, a natural example of a uniformly divergent sequence of homeomorphisms is again provided by a flow parametrized by discrete time from a source to infinity. If $\varphi : \mathbb{N} \times X \rightarrow X$ is a flow and $\varphi_n(x) = \varphi(n, x)$, then, for each x , the orbit $\{\varphi_n(x) : n = 1, 2, \dots\}$ is the image of the divergent real sequence $\{y_n(x) : n = 1, 2, \dots\}$, where $y_n(x) := d(\varphi_n(x), x) \geq d_*(\varphi_n, id)$.

Remark. Our aim is to offer analogues of the topological vector space characterization of boundedness: for a bounded sequence of vectors $\{x_n\}$ and scalars $\alpha_n \rightarrow 0$ ([Ru-FA2] cf. Th. 1.30) $\alpha_n x_n \rightarrow 0$. However $\alpha_n x_n$ is interpreted in the spirit of duality as $\alpha_n(x_n)$ with the homeomorphisms α_n converging to the identity.

Theoretical examples motivated by duality

1. Evidently, if $S = X$, the pointwise definition reduces to functional divergence in $\mathcal{H}(X)$ defined pointwise:

$$d_X(\alpha_n(x), x) \rightarrow \infty.$$

The uniform version corresponds to divergence in the supremum metric in $\mathcal{H}(X)$.

2. If $S = T$ and $\mathcal{A} = X = \Xi$, we have, by the Quasi-Isometric Duality Theorem [BOst12], that

$$d_T(\xi_{x(n)}(t), \xi_e(t)) \rightarrow \infty \text{ iff } d_X(x_n, e_X) \rightarrow \infty,$$

and the assertion reduces to ordinary divergence in X . Since

$$d_{\Xi}(\xi_{x(n)}, \xi_e) = d_X(x_n, e_X),$$

the uniform version also asserts that

$$d_X(x_n, e_X) \rightarrow \infty.$$

Recall that $\tau^x(s)(z) = s(x^{-1}z)$, so Ξ defines an action φ on T according to the formula

$$\varphi(\xi, t) = \xi_{x^{-1}(t)}(e) = t(x),$$

where $\xi = \xi_x$. In lieu of $\varphi^\xi(t)$, with $\xi = \xi_{x(n)}$, one may write $\xi_{x(n)}(t)$ and then

$$\xi_{x(n)}(t) = t(x_n).$$

When interpreting $\xi_{x(n)}$ in Ξ as x_n in X acting on t , note that

$$d_X(x_n, e_X) \leq d_X(x_n, t(x_n)) + d_X(t(x_n), e_X) \leq \|t\| + d_X(t(x_n), e_X),$$

so, as expected, the divergence of x_n implies the divergence of $t(x_n)$.

Definition. We say that pointwise (resp. uniform) divergence is *unconditional* in \mathcal{A} if, for any (pointwise/uniform) divergent sequence α_n ,
(i) for any bounded σ , the sequence $\sigma\alpha_n$ is (pointwise/uniform) divergent;
and,
(ii) for any ψ_n convergent to the identity, $\psi_n\alpha_n$ is (pointwise/uniform) divergent.

Remarks. In clause (ii) each of the functions ψ_n has a bound depending on n . The two clauses could be combined into one by requiring that if the bounded functions ψ_n converge to ψ_0 in the supremum norm, then $\psi_n\alpha_n$ is (pointwise/uniform) divergent.

By Lemma 3 of [BOst12] Section 4 uniform divergence in $\mathcal{H}(X)$ is unconditional. We move to other forms of this result.

Proposition. *If the metric on \mathcal{A} is left- or right-invariant, then uniform divergence is unconditional in \mathcal{A} .*

Proof. If the metric $d = d_{\mathcal{A}}$ is left-invariant, then observe that if β_n is a bounded sequence, then so is $\sigma\beta_n$, since

$$d(e, \sigma\beta_n) = d(\sigma^{-1}, \beta_n) \leq d(\sigma^{-1}, e) + d(e, \beta_n).$$

Since $\|\beta_n^{-1}\| = \|\beta_n\|$, the same is true for right-invariance. Further, if ψ_n is convergent to the identity, then also $\psi_n\beta_n$ is a bounded sequence, since

$$d(e, \psi_n\beta_n) = d(\psi_n^{-1}, \beta_n) \leq d(\psi_n^{-1}, e) + d(e, \beta_n).$$

Here we note that, if ψ_n is convergent to the identity, then, so is ψ_n^{-1} by continuity of inversion (or by metric invariance). The same is again true for right-invariance. \square

The case where the subgroup \mathcal{A} of autohomeomorphisms is the translations Ξ , though immediate, is worth noting.

Theorem 1. (The case $\mathcal{A} = \Xi$.) *If the metric on the group X is left- or right-invariant, then uniform divergence is unconditional in $\mathcal{A} = \Xi$.*

Proof. We have already noted that Ξ is isometrically isomorphic to X . \square

Remarks.

1. If the metric is bounded, there may not be any divergent sequences.
2. We already know from Lemma 3 that uniform divergence in $\mathcal{A} = \mathcal{H}(X)$ is unconditional.
3. The unconditionality condition (i) corresponds directly to the technical condition placed in [BajKar] on their filter \mathcal{F} . In our metric setting, we thus employ a stronger notion of limit to infinity than they do. The filter implied by the pointwise setting is generated by sets of the form

$$\bigcap_{i \in I} \{ \alpha : d_X(\alpha_n(x_i), x_i) > M \text{ ultimately} \} \text{ with } I \text{ finite.}$$

However, whilst this is not a countably generated filter, its projection on the x -coordinate:

$$\{ \alpha : d_X(\alpha_n(x), x) > M \text{ ultimately} \},$$

is.

4. When the group is locally compact, ‘bounded’ may defined as ‘pre-compact’, and so ‘divergent’ becomes ‘unbounded’. Here divergence is unconditional (because continuity preserves compactness).

The supremum metric need not be left-invariant; nevertheless we still do have unconditional divergence.

Theorem 2. *For $\mathcal{A} \subseteq \mathcal{H}(S)$, pointwise divergence in \mathcal{A} is unconditional.*

Proof. For fixed $s \in S$, $\sigma \in \mathcal{H}(S)$ and $d_X(\alpha_n(s), s)$ unbounded, suppose that $d_X(\sigma\alpha_n(s), s)$ is bounded by K . Then

$$\begin{aligned} d_S(\alpha_n(s), s) &\leq d_S(\alpha_n(s), \sigma(\alpha_n(s))) + d_S(\sigma(\alpha_n(s)), s) \\ &\leq \|\sigma\|_{\mathcal{H}(S)} + K, \end{aligned}$$

contradicting that $d_S(\alpha_n(s), s)$ is unbounded. Similarly, for ψ_n converging to the identity, if $d_S(\psi_n(\alpha_n(x)), x)$ is bounded by L , then

$$\begin{aligned} d_S(\alpha_n(s), s) &\leq d_S(\alpha_n(s), \psi_n(\alpha_n(s))) + d_S(\psi_n(\alpha_n(s)), s) \\ &\leq \|\psi_n\|_{\mathcal{H}(S)} + L, \end{aligned}$$

contradicting that $d_S(\alpha_n(s), s)$ is unbounded. \square

Corollary 1. *Pointwise divergence in $\mathcal{A} \subseteq \mathcal{H}(X)$ is unconditional.*

Corollary 2. *Pointwise divergence in $\mathcal{A} = \Xi$ is unconditional.*

Proof. In Theorem 2, take $\alpha_n = \xi_{x(n)}$. Then unboundedness of $d_T(\xi_{x(n)}(t), t)$ implies unboundedness of $d_T(\sigma\xi_{x(n)}(t), t)$ and of $d_T(\psi_n\xi_{x(n)}(t), t)$. \square

4 Category Embedding Theorem

If ψ_n converges to the identity, then, for large n , each ψ_n is almost an isometry. Indeed by the Proposition on Permutation metrics [BOst12], we have

$$d(x, y) - 2\|\psi_n\| \leq d(\psi_n(x), \psi_n(y)) \leq d(x, y) + 2\|\psi_n\|.$$

This motivates our next result; we need to recall a definition and the Category Embedding Theorem from [BOst11]. In what follows, the words *quasi everywhere (q.e.)*, or *for quasi all points* mean *for all points off a meagre set*.

Definition (weak category convergence). A sequence of homeomorphisms ψ_n satisfies the *weak category convergence* condition (wcc) if:

For any non-empty open set U , there is a non-empty open set $V \subseteq U$ such that, for each $k \in \omega$,

$$\bigcap_{n \geq k} V \setminus \psi_n^{-1}(V) \text{ is meagre.} \quad (\text{wcc})$$

Equivalently, for each $k \in \omega$, there is a meagre set M such that, for $t \notin M$,

$$t \in V \implies (\exists n \geq k) \psi_n(t) \in V.$$

Category Embedding Theorem. *Let X be a Baire space. Suppose given homeomorphisms $\psi_n : X \rightarrow X$ for which the weak category convergence condition (wcc) is met. Then, for any non-meagre Baire set T , for locally quasi all $t \in T$, there is an infinite set \mathbb{M}_t such that*

$$\{\psi_m(t) : m \in \mathbb{M}_t\} \subseteq T.$$

Examples. In \mathbb{R} we may consider $\psi_n(t) = t + z_n$ with $z_n \rightarrow z_0 := 0$. It is shown in [BOst11] that for this sequence the condition (wcc) is satisfied in

both the usual topology and the density topology on \mathbb{R} . This remains true in \mathbb{R}^d (where the specific instance of the theorem is referred to as the Kestelman-Borwein-Ditor Theorem, see [Kes], [BoDi]). In fact in any metrizable group X with right-invariant metric d_X , for a null sequence tending to the identity $z_n \rightarrow z_0 := e_X$, the mapping defined by $\psi_n(x) = z_n x$ converges to the identity (see the Corollary to Ford's Theorem); here too (wcc) holds. This follows from the next result, which extends the proof of [BOst11].

Proposition *If ψ_n converges to the identity, then ψ_n satisfies the weak category convergence condition (wcc).*

Proof. Since ψ_n converges to the identity iff ψ_n^{-1} converges to the identity (see previous section), it is more convenient to prove the equivalent statement that ψ_n^{-1} satisfies the category convergence condition.

Put $z_n = \psi_n(z_0)$, so that $z_n \rightarrow z_0$. Let k be given.

Suppose that $y \in B_\varepsilon(z_0)$, i.e. $r = d(y, z_0) < \varepsilon$. For some $N > k$, we have $\varepsilon_n = d(\psi_n, id) < \frac{1}{3}(\varepsilon - r)$, for all $n \geq N$. Now

$$\begin{aligned} d(y, z_n) &\leq d(y, z_0) + d(z_0, z_n) \\ &= d(y, z_0) + d(z_0, \psi_n(z_0)) \leq r + \varepsilon_n. \end{aligned}$$

For $y = \psi_n(x)$ and $n \geq N$,

$$\begin{aligned} d(z_0, x) &\leq d(z_0, z_n) + d(z_n, y) + d(y, x) \\ &= d(z_0, z_n) + d(z_n, y) + d(x, \psi_n(x)) \\ &\leq \varepsilon_n + (r + \varepsilon_n) + \varepsilon_n < \varepsilon. \end{aligned}$$

So $x \in B_\varepsilon(z_0)$, giving $y \in \psi_n(B_\varepsilon(z_0))$. Thus

$$y \notin \bigcap_{n \geq N} B_\varepsilon(z_0) \setminus \psi_n(B_\varepsilon(z_0)) \supseteq \bigcap_{n \geq k} B_\varepsilon(z_0) \setminus \psi_n(B_\varepsilon(z_0)).$$

It now follows that

$$\bigcap_{n \geq k} B_\varepsilon(z_0) \setminus \psi_n(B_\varepsilon(z_0)) = \emptyset. \quad \square$$

We refer the reader to [BOst12] for applications which include generalizations to normed groups of the Steinhaus Theorem. In the real line the latter asserts that $A + A$ and $A - A$, for A Baire and non-meagre, contains an interval (see [Kom] for the topological vector space setting and earlier literature including the work of Piccard [P] and Pettis [Pet1]).

5 The Uniform Convergence Theorems

Definition. Let X be a homogeneous metric space with distinguished point z_0 . We say that $h : X \rightarrow H$ is *unconditionally α -slowly varying on S* if α is divergent and

$$h(\alpha'_n s)h(\alpha'_n z_0)^{-1} \rightarrow 0, \text{ as } n \rightarrow \infty \text{ (for each } s \in S), \quad (1)$$

for any sequence $\alpha' = \{\alpha'_n\}$ with $\alpha'_n = \psi_n \sigma \alpha_n$, for any fixed bounded σ and any sequence ψ_n convergent to the identity.

Example. Take $X = \mathcal{C}[0, 1]$ and for $\alpha(t)$ continuous, let f be defined by

$$f(x) = \int_0^1 \alpha(t) \log(x(t)) dt.$$

Then

$$f(u+x) - f(x) = \int_0^1 \alpha(t) \log\left(1 + \frac{u(t)}{x(t)}\right) dt \rightarrow 0,$$

as $\|x\| \rightarrow \infty$. More particularly, suppose that $\{x_n\}$ is a sequence in X with $\|x_n\| \rightarrow \infty$. Put $\varphi_n(u) := u + x_n$; then we have

$$f(\varphi_n(u)) - f(\varphi_n(z_0)) = \int_0^1 \alpha(t) \log\left(\frac{\varphi_n(u(t))}{\varphi_n(z_0(t))}\right) dt \rightarrow 0.$$

Thus f is Φ -slowly varying for Φ the group of all shifts $\varphi_x(u) := u + x$.

To state and prove our first main theorem, we recall the notion of crimping from Section 4.

The General Uniform Convergence Theorem for Actions (General UCT).

Suppose the following:

(i) $\mathcal{A} \subseteq \mathcal{H}(S)$ is a (topological) group of homeomorphisms acting on the metric space S such that S is of second category in itself;

(ii) for $z_n \rightarrow z_0$ in S , there is a crimping sequence of maps with $\psi_n z_0 = z_n$;

(iii) for each $z \in S$ there is a bounded shift σ_z with $\sigma_z z_0 = z$, such that $\tilde{\psi}_n = \sigma_z \psi_n \sigma_z^{-1}$ is a crimping sequence of maps with $\psi_n z_0 = z_n$;

- (iv) divergence in \mathcal{A} is unconditional;
- (v) $h : S \rightarrow H$ is Baire unconditionally α -slowly varying on S , so that

$$h(\alpha_n s)h(\alpha_n z_0)^{-1} \rightarrow e_H, \text{ as } n \rightarrow \infty \text{ (for each } s \in S).$$

Then the convergence is uniform for s on compact subsets K of the space S .

Proof. If not, then, for some Baire function h , some $\varepsilon > 0$, some divergent sequence α_n in \mathcal{A} , and some convergent sequence of points $u_n \rightarrow u$ in S ,

$$|h(\alpha_n u_n)h(\alpha_n)^{-1}| \geq 2\varepsilon. \quad (2)$$

Now, by (iii) there exists a bounded homeomorphism $\sigma = \sigma_u$ of \mathcal{A} with $\sigma(z_0) = u$.

Put $z_n = \sigma^{-1}(u_n)$, so that, by continuity, $u_n \rightarrow u$ implies that $z_n \rightarrow z_0$. Evidently we have $u_n = \sigma(z_n) \rightarrow \sigma(z_0) = u$.

Let ψ_n converging to the identity be selected with $z_n = \psi_n(z_0)$. Thus $\sigma(z_n) = \sigma(\psi_n(z_0)) = u_n$. Moreover, note that

$$\psi_n(u_n) \rightarrow u,$$

as

$$d_S(\psi_n(u_n), u) \leq d_S(\psi_n(u_n), u_n) + d_S(u_n, u) \leq \varepsilon_n + d_S(u_n, u).$$

Define Baire sets S_k by

$$S_k := \bigcap_{n \geq k} \{s \in S : |h(\alpha_n(\sigma(s)))h(\alpha_n(z_0))^{-1}| < \varepsilon\}.$$

Then, by definition of slowly varying, we have

$$S = \bigcup_k S_k.$$

So, by (i), for some k the set S_k is non-meagre. Hence, by the Category Embedding Theorem, there are $t \in S_k$ and an infinite \mathbb{M}_t such that

$$\{\psi_n(t) : n \in \mathbb{M}_t\} \subseteq S_k.$$

Thus, for $n \in \mathbb{M}_t$, we have

$$|h(\alpha_n(\sigma(\psi_n(t))))h(\alpha_n(z_0))^{-1}| < \varepsilon.$$

In the case when $u = z_0$ we have, as $\psi_n(z_0) = z_n$, that

$$\begin{aligned} |h(\alpha_n z_n)h(\alpha_n(z_0))^{-1}| &= |h(\alpha_n \psi_n(t))h(\alpha_n \psi_n(z_0))^{-1}h(\alpha_n \psi_n(a))h(\alpha_n(z_0))^{-1}| \\ &\leq |h(\alpha_n \psi_n(t))h(\alpha_n \psi_n(z_0))^{-1}| \\ &\quad + |h(\alpha_n \psi_n(a))h(\alpha_n(z_0))^{-1}|. \end{aligned}$$

However, by (iv), $\alpha_n \psi_n$ is divergent, so the first term is ultimately smaller than ε (h slowly varying at t), and so is the second, by the Category Embedding Theorem. This contradicts the assumptions (2). For a general location u , the argument is similar. Here, as $\sigma(\psi_n(z_0)) = u_n$ (again by construction), we have

$$\begin{aligned} |h(\alpha_n u_n)h(\alpha_n(z_0))^{-1}| &\leq |h(\alpha_n \sigma \psi_n(t))h(\alpha_n \sigma \psi_n(z_0))^{-1}| \\ &\quad + |h(\alpha_n \sigma \psi_n(t))h(\alpha_n(z_0))^{-1}|, \end{aligned}$$

and this time $\alpha_n \sigma \psi_n$ is divergent (by (iv)), since σ is a bounded homeomorphism. \square

Since groups with invariant metrics have crimping sequences, and bounded shifts, we conclude as follows.

The Primal Uniform Convergence Theorem (State UCT). *Suppose the following:*

- (i) X is a Baire space,
- (ii) X is homogeneous under $\mathcal{H}(X)$, i.e., for any pair of points z, u , there is a bounded homeomorphism σ such that $\sigma(z) = u$;
- (iii) The crimping property holds: for any null sequence $z_n \rightarrow z_0$ in X , there is a sequence of homeomorphisms converging to the identity with $\psi_n(z_0) = z_n$.
- (iv) for each $z \in X$ there is a bounded shift σ_z with $\sigma_z z_0 = z$, such that $\tilde{\psi}_n = \sigma_z \psi_n \sigma_z^{-1}$ is a crimping sequence of maps with $\psi_n z_0 = z_n$;
- (v) h is Baire unconditionally φ -slowly varying for divergent φ in \mathcal{T} .

Then, for x in any compact set $K \subseteq X$, we have uniformly for $x \in K$ the convergence

$$h(\varphi_n(x))h(\varphi_n(z_0))^{-1} \rightarrow e_H,$$

i.e. the convergence in (1) is uniform on compact subsets K of the space X .

Proof: Take $S = X$ and $\mathcal{A} = \mathcal{T} \subseteq \mathcal{H}(X)$. Here $e_{\mathcal{A}} = id_X$, i.e. $e_{\mathcal{A}}(x) = x$ for $x \in X$ and $e_S = e_X = z_0$.

As $\varphi = \{\varphi_n\}$ diverges, for each $x \in X$, we have

$$d(\varphi_n(x), x) \rightarrow \infty.$$

This notion is unconditional by Corollary 1. This completes our check of the hypotheses of the General UCT. \square

A brief inspection of the proof leads to the conclusion that the uniform convergence theorem holds in a form where all the convergence assertions are relativized to a subspace of X .

Corollary (Relativized Primal UCT) *Suppose that (i)-(iv) of the Primal UCT hold and also that*

(v) *Y is a Baire subspace of X , $h : Y \rightarrow H$ is Baire unconditionally φ -slowly varying for divergent φ in \mathcal{T} on Y , i.e., for $x \in Y$,*

$$h(\varphi_n(x))h(\varphi_n(z_0))^{-1} \rightarrow e_H.$$

Then, for x in any compact set $K \subseteq Y$, we have uniformly for $x \in K$ the convergence

$$h(\varphi_n(x))h(\varphi_n(z_0))^{-1} \rightarrow e_H,$$

i.e. the convergence in (1) is uniform on compact subsets K of the subspace Y .

Application: Equicontinuity from the Relativized Primal UCT (State UCT).

Consider Y a Baire topological group and form the direct product group $X = Y \times \mathbb{N}$. Here

$$(t, m) \cdot (s, n) = (ts, mn).$$

Identify Y with $Y \times \{1\}$ and let $T = X$ so that X acts on itself. Suppose H is also a topological group and that, for $n \in \mathbb{N}$, $f_n : Y \rightarrow H$ are *homomorphisms* with $f_n(y) \rightarrow f_\infty(y)$ pointwise. For $x = (s, n) \in X$, put

$$F(x) = f_n(s).$$

Then, for $y = (t, 1) \in Y \times \{1\}$ and $x = (s, n) \in X$, we have, since f_n is a homomorphism, that

$$\sigma_F(x, y) = F(x)^{-1}F(xy) = f_n(s)^{-1}f_n(st) = f_n(t),$$

i.e. the cocycle is independent of x . Hence, with \mathcal{F} generated from $F_m = \{(s, n) : n \geq m\}$, we have, for y fixed,

$$\mathcal{F}\text{-}\lim_x F(x)^{-1}F(xy) = f_\infty(y).$$

Thus the limit cocycle on Y over X is $k(y) = f_\infty(y)$. Then UCT implies equicontinuity, i.e. $f_n(y) \rightarrow f_\infty(y)$ on compact subsets of Y . \square

The above is a modification of the argument presented by Bajšanski and Karamata in [BajKar]; however, we proceed dually, as here the limit taken is with the ‘space’ variable y held fixed (the action variable x goes to infinity). We now turn to the dual version of the UCT in which the space variable goes to infinity.

The Dual Uniform Convergence Theorem (Costate UCT).

Suppose the following:

- (i) $T \subseteq \mathcal{H}(X)$ is a (topological) group of homeomorphisms acting on the metric space X such that T is of second category in itself;
- (ii) for $z_n \rightarrow e$ in T , the sequence of maps $\psi_n : t \rightarrow tz_n$ satisfies the condition (wcc);
- (iii) the map $\theta_z : t \rightarrow tz$ is bounded, for each $z \in T$;
- (iv) $h : X \rightarrow H$ is Baire slowly varying on T , i.e. h is Baire and, for each divergent sequence x_n in X ,

$$h(tx_n)h(x_n)^{-1} \rightarrow e_H, \text{ as } n \rightarrow \infty \text{ (for each } t \in T\text{)}.$$

Then the convergence is uniform on compact subsets of the transformation group T .

Proof. Take $S = T$ and $\mathcal{A} = \Xi = \{\xi_x : x \in X\}$, which we identify with X . Here $e_{\mathcal{A}} = e_{\Xi}$ and $e_{\Xi} = \xi_e$ with $e = e_X$. That is, $e_{\mathcal{A}}(t) = t$, for $t \in T$. To obtain the notion of divergence, write $x(n)$ for x_n and demand for fixed t that

$$d_T(\xi_{x(n)}(t), t) \rightarrow \infty.$$

Then by the Lemma this is equivalent to $d_X(x_n, e_X) \rightarrow \infty$.

By Corollary 2 this notion is unconditional. This completes our check of the hypotheses of the General UCT. \square

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