Normed groups: dichotomy and duality

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Abstract

The key vehicle of the recent development of a topological theory of regular variation based on topological dynamics [BOst13], and embracing its classical univariate counterpart (cf. [BGT]) as well as fragmentary multivariate (mostly Euclidean) theories (eg [MeSh], [Res], [Ya]), are groups with a right-invariant metric carrying flows. Following the vector paradigm, they are best seen as *normed groups*. That concept only occasionally appears explicitly in the literature despite its frequent disguised presence, and despite a respectable lineage traceable back to the Pettis closed-graph theorem, to the Birkhoff-Kakutani metrization theorem and further back still to Banach's *Théorie des opérations linéaires*. We collect together known salient features and develop their theory including Steinhaus theory unified by the Category Embedding Theorem [BOst11], the associated themes of subadditivity and convexity, and a topological duality inherent to topological dynamics. We study the latter both for its independent interest and as a foundation for topological regular variation.

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1 Introduction

Group norms, which behave like the usual vector norms, except that scaling is restricted to the basic scalars of group theory (the units ±1 in an abelian context and the exponents ±1 in the non-commutative context), have played a part in the early development of topological group theory. Although ubiquitous they lack a clear and unified exposition. This lack is our motivation here, since they offer the right context for the recent theory of *topological regular variation*. This extends the classical theory (for which see, e.g. [BGT]) from the real line to metrizable topological groups. Normed groups are just groups carrying a right-invariant metric. The basic metrization theorem for groups, the Birkhoff-Kakutani Theorem of 1936 ([Bir], [Kak], see [Kel], Ch.6 Problems N-R, [Klee], [Bour] Part 2, Section 3.1, and [ArMa], compare also [Eng] Exercise 8.1.G and Th. 8.1.21) is usually stated as asserting that a first-countable Hausdorff group has a right-invariant metric. It is properly speaking a 'normability' theorem in the style of Kolmogorov's Theorem ([Kol], or [Ru-FA2], Th. 1.39; in this connection see also [Jam], where strong forms of connectedness are used in an abelian setting to generate norms), as we shall see below. Indeed the metric construction in [Kak] is reminiscent of the more familiar construction of a Minkowski functional (for which see [Ru-FA2] Sect. 1.33), but is implicitly a supremum norm – as defined below; in Rudin's derivation of the metric (for a topological vector space setting, [Ru-FA2] Th. 1.24) this norm is explicit. Early use by A. D. Michal and his collaborators was in providing a canonical setting for differential calculus (see the review [Mich] and as instance [JMW]) and included the noteworthy generalization of the implicit function theorem by Bartle [Bart] (see Section 5). In name the group norm makes an explicit appearance in 1950 in [Pet1] in the course of his classic closed-graph theorem (in connection with Banach's closed-graph theorem and the Banach-Kuratowski category dichotomy for groups). It reappears in the group context in 1963 under the name 'length function', motivated by word length, in the work of R. C. Lyndon [Lyn2] (cf. [LynSch]) on Nielsen's Subgroup Theorem, that a subgroup of a free group is a free group. (Earlier related usage for function spaces is [EH].) The latter name is conventional in geometric group theory despite the parallel usage in algebra (cf. [Far]) and the recent work on norm extension (from a normal subgroup) of Bökamp [Bo].

When a group is topologically complete and also abelian, then it admits a metric which is *bi-invariant*, i.e. is both right- and left-invariant, as [Klee] showed in solving a problem of Banach. This context is of significance for the calculus of regular variation (in the study of products of regularly varying functions with range a normed group) – see [BOst15].

Fresh interest in metric groups dates back to the seminal work of Milnor [Mil] in 1968 on the metric properties of the fundamental group of a manifold and is key to the global study of manifolds initiated by Gromov [Gr1], [Gr2] in the 1980s (and we will see quasi-isometries in the duality theory of normed groups), for which see [BH] and also [Far] for an early account; [PeSp] contains a variety of generalizations and their uses in interpolation theory (but the context is abelian groups).

The very recent [CSC] (see Sect. 2.1.1, Embedding quasi-normed groups into Banach spaces) employs norms in considering Ulam's problem (see [Ul]) on the *global* approximation of nearly additive functions by additive functions. This is a topic related to regular variation, where the weaker concept of *asymptotic* additivity is the key. Recall the classical definition of a regularly varying function, namely a function $h : \mathbb{R} \to \mathbb{R}$ for which the limit

$$\partial_{\mathbb{R}} h(t) := \lim_{x \to \infty} h(tx)h(x)^{-1} \tag{1}$$

exists everywhere; for f Baire, the limit function is a continuous homomorphism (i.e. a multiplicative function). Following the pioneering study of [BajKar] launching a general (i.e., topological) theory of regular variation, [BOst13] has re-interpreted (1), by replacing $|x| \to \infty$ with $||x|| \to \infty$, for functions $h : X \to H$, with tx being the image of x under a T-flow on X (defined in Section 3), and with X, T, H all groups with right-invariant metric (right because of the division on the right) – i.e. normed groups (making ∂h_X a differential at infinity, in Michal's sense [Mi]). In concrete applications the groups may be the familiar Banach groups of functional analyis, the associated flows either the ubiquitous domain translations of Fourier transform theory or convolutions from the related contexts of abstract harmonic analysis (e.g. Wiener's Tauberian theory so relevant to classical regular variation – see e.g. [BGT, Ch. 4]). In all of these one is guaranteed right-invariant metrics. Likewise in the foundations of regular variation the first tool is the group $\mathcal{H}(X)$ of bounded self-homeomorphisms of the group X under a supremum metric (and acting transitively on X); the metric is again right-invariant and hence a group norm. It is thus natural, in view of the applications and the Birkhoff-Kakutani Theorem, to demand right-invariance.

We show in Section 3 and 6 that normed groups offer a natural setting for subadditivity and for (mid-point) convexity.

2 Metric versus normed groups

This section is devoted to group-norms and their associated metrics. We collect here some pertinent information (some of which is scattered in the literature). A central tool for applications is the construction of the subgroup of bounded homeomorphisms of a given group $\mathcal{G}$ of self-homeomorphisms of a topological group X; the subgroup possess a guaranteed right-invariant metric. This is the archetypal example of the symbiosis of norms and metrics, and it bears repetition that, in applications just as here, it is helpful to work simultaneously with a right-invariant metric and its associated group norm.

We say that the group X is *normed* if it has a group-norm as defined below (cf. [DDD]).

Definition. We say that $||\cdot|| : X \to \mathbb{R}_+$ is a *group-norm* if the following properties hold:

(i) Subadditivity (Triangle inequality): $||xy|| \leq ||x|| + ||y||$;

(ii) Positivity: $||x|| > 0$ for $x \neq e$;

(iii) Inversion (Symmetry): $||x^{-1}|| = ||x||$.

If (i) holds we speak of a group semi-norm; if (i) and (iii) and $||e|| = 0$ holds one speaks of a *pseudo-norm* (cf. [Pet1]); if (i) and (ii) holds we speak of a group pre-norm (see [Low] for a full vocabulary).

We say that a group pre-norm, and so also a group-norm, is *abelian*, or more precisely *cyclically permutable,* if

(iv) Abelian norm (cyclic permutation): $||xy|| = ||yx||$ for all x, y.

Other properties we wish to refer to are :

$(\mathrm{i})_K$ for all $x, y : ||xy|| \leq K(||x|| + ||y||)$

$(\mathrm{i})_{\mathrm{ult}}$ for all $x, y : ||xy|| \leq \max\{||x||, ||y||\}$

Remarks 1

1. Mutatis mutandis this is just the usual vector norm, but with scaling restricted to the units ±1. The notation and language thus mimick the vector space counterparts, with subgroups playing the role of subspaces; for example, for a symmetric, subbadditive $p : X \to \mathbb{R}_+$, the set $\{x : p(x) = 0\}$ is a subgroup. Indeed the analysis of Baire subadditive functions (see Section 3) is naturally connected with norms, via regular variation. That is why normed groups occur naturally in regular variation theory.

2. When $(\mathrm{i})_K$, for some constant K, replaces (i), one speaks of quasi-norms (see [CSC], cf. 'distance spaces' [Rach] for a metric analogue). When $(\mathrm{i})_{\mathrm{ult}}$ replaces (i) one speaks of an ultra-norm, or non-Archimedean norm.

3. Note that (i) implies joint continuity of multiplication, while (iii) implies continuity of inversion. (Montgomery [Mon1] shows that joint continuity is implied by separate continuity when the group is locally complete.) See below for the stronger notion of uniform continuity invoked in the Uniformity Theorem of Conjugacy.

4. Abelian groups with ordered norms may also be considered, cf. [JMW].

Remarks 2

Subadditivity implies that $||e|| \geq 0$ and this together with symmetry implies that $||x|| \geq 0$, since $||e|| = ||xx^{-1}|| \leq 2||x||$; thus a group norm cannot take negative values. Subadditivity also implies that $||x^n|| \leq n||x||$, for natural n. The norm is said to be 2-*homogeneous* if $||x^2|| = 2||x||$; see [CSC] Prop. 4.12 (Ch. IV.3 p.38) for a proof that if a normed group is amenable or *weakly commutative* (defined in [CSC] to mean that, for given x, y, there is m of the form 2^n, for some natural number n, with $(xy)^m = x^m y^m$), then it is embeddable as a subgroup of a Banach space. In the case of an abelian group 2-homogeneity corresponds to sublinearity, and here Berz's Theorem characterizes the norm (see [Berz] and [BOst5]). The abelian property implies only that $||xyz|| = ||zxy|| = ||yzx||$, hence its alternative name. Harding [H], in the context of quantum logics, uses this condition to guarantee that the group operations are continuous and calls this a *strong norm.* See [Kel] Ch. 6 Problem O (which notes that a locally compact group with abelian norm has a bi-invariant Haar measure). We note that when X is locally compact continuity of the inverse follows from the continuity of multiplication (see [Ell]). The literature concerning when joint continuity of $(x, y) \to xy$ follows from separate continuity reaches back to Namioka [Nam] (see e.g. [Bou], [HT], [CaMo]).

Remarks 3

If $f : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, subadditive with $f(0) = 0$, then

$$|||x||| := f(||x||)$$

is also a group-norm. See [BOst5] for recent work on Baire (i.e., having the Baire property) subadditive functions.

Proposition 1 (Symmetrization) *If* $|x|$ *is a group pre-norm, then the symmetrization*

$$||x|| := |x| + |x^{-1}|$$

defines a group-norm

Proof. Positivity is clear, likewise symmetry. Also

$$\begin{aligned} ||xy|| &= |xy| + |y^{-1}x^{-1}| \leq |x| + |y| + |y^{-1}| + |x^{-1}| \\ &= ||x|| + ||y||. \qquad \square \end{aligned}$$

Proposition 2. *If* $|| \cdot ||$ *is a group-norm, then* $d(x, y) := ||xy^{-1}||$ *is a right-invariant metric; equivalently,* $\tilde{d}(x, y) := d(x^{-1}, y^{-1}) = ||x^{-1}y||$ *is the conjugate left-invariant metric on the group.*

Conversely, if d *is a right-invariant metric, then* $||x|| := d(e, x) = \tilde{d}(e, x)$ *is a group-norm.*

Thus the metric d *is bi-invariant iff* $||xy^{-1}|| = ||x^{-1}y||$, *i.e. iff the group-norm is abelian.*

Proof. Given a group-norm put $d(x,y) = ||xy^{-1}||$. Then $||xy^{-1}|| = 0$ iff $xy^{-1} = e$, i.e. iff $x = y$. Symmetry follows from inversion as $d(x,y) = ||(xy^{-1})^{-1}|| = ||yx^{-1}|| = d(y,x)$. Finally, d obeys the triangle inequality, since

$$||xy^{-1}|| = ||xz^{-1}zy^{-1}|| \leq ||xz^{-1}|| + ||zy^{-1}||.$$

As for the converse, given a right-invariant metric d, put $||x|| := d(e,x)$. Now $||x|| = d(e,x) = 0$ iff $x = e$. Next $||x^{-1}|| = d(e,x^{-1}) = d(x,e) = ||x||$, and so

$$d(xy,e) = d(x,y^{-1}) \leq d(x,e) + d(e,y^{-1}) = ||x|| + ||y||.$$

Also $d(xa,ya) = ||xaa^{-1}y^{-1}|| = d(x,y)$.

Finally d is bi-invariant iff $d(e,yx^{-1}) = d(x,y) = d(e,x^{-1}y)$ iff $||yx^{-1}|| = ||x^{-1}y||$. Inverting the first term yields the abelian property of the group-norm. □

The two conjugate metrics separately define a left and right uniformity; taken together they define what is known as the ambidextrous uniformity, the only one of the three to be complete – see [Hal-ET, p. 63] and [Br-2]. We will be concerned with special cases of the following definition.

Definition ([Gr1], [Gr2], [BH] Ch. I.8) For constants $\lambda \geq 1, \gamma \geq 0$, the metric spaces X and Y are said to be *(λ-γ)-quasi-isometric* under the mapping $\pi : X \to Y$, if

$$\begin{aligned} \frac{1}{\lambda} d_X(a,b) - \gamma &\leq d_Y(\pi a, \pi b) \leq \lambda d_X(a,b) + \gamma \qquad (a,b \in X), \\ d_Y(y, \pi[X]) &\leq \gamma \qquad (y \in Y). \end{aligned}$$

Corollary. *For π a homomorphism, the normed groups X, Y are (λ-γ)-quasi-isometric under π for the corresponding metrics iff their norms are (λ-γ)-quasi-equivalent, i.e.*

$$\begin{aligned} \frac{1}{\lambda} ||x||_X - \gamma &\leq ||\pi x||_Y \leq \lambda ||x||_X + \gamma \qquad (a,b \in X), \\ d_Y(y, \pi[X]) &\leq \gamma \qquad (y \in Y). \end{aligned}$$

Proof. This follows from $\pi(e_X) = e_Y$ and $\pi(xy^{-1}) = \pi(x)\pi(y)^{-1}$. □

The following result (which we use in [BOst14]) clarifies the relationship between the conjugate metrics and the group structure. We define the ε-*swelling* of a set K in a metric space X, for a given right-invariant metric d_X, to be

$$B_\varepsilon(K) := \{z : d_X(z,k) < \varepsilon \text{ for some } k \in K\}$$

and for the conjugate left-invariant case we can write similarly

$$\tilde{B}_\varepsilon(K) := \{z : \tilde{d}_X(z,k) < \varepsilon \text{ for some } k \in K\}.$$

We write $B_\varepsilon(x_0)$ for $B_\varepsilon(\{x_0\})$, so that

$$B_\varepsilon(x_0) := \{z : ||zx_0^{-1}|| < \varepsilon\}.$$

When $x_0 = e_X$, the ball $B_\varepsilon(e_X)$ is the same under either of the conjugate metrics, as

$$B_\varepsilon(e_X) := \{z : ||z|| < \varepsilon\}.$$

Proposition. *(i) In a locally compact group X, for K compact and for $\varepsilon > 0$ small enough so that the closed ε-ball $B_\varepsilon(e_X)$ is compact, the swelling $B_{\varepsilon/2}(K)$ is pre-compact.*

(ii) $B_\varepsilon(K) = \{wk : k \in K, ||w||_X < \varepsilon\} = B_\varepsilon(e_X)K$, where the notation refers to swellings for d_X a right-invariant metric; similarly, for $\tilde{d}_X$, the conjugate metric, $\tilde{B}_\varepsilon(K) = KB_\varepsilon(e)$.

Proof. (i) If $x_n \in B_{\varepsilon/2}(K)$, then we may choose $k_n \in K$ with $d(k_n, x_n) < \varepsilon/2$. W.l.o.g. k_n converges to k. Thus there exists N such that, for $n > N$, $d(k_n, k) < \varepsilon/2$. For such n, we have $d(x_n, k) < \varepsilon$. Thus the sequence x_n lies in the compact closed ε-ball centred at k and so has a convergent subsequence.

(ii) Let $d_X(x,y)$ be a right-invariant metric, so that $d_X(x,y) = ||xy^{-1}||$. If $||w|| < \varepsilon$, then $d_X(wk,k) = d_X(w,e) = ||w|| < \varepsilon$, so $wk \in B_\varepsilon(K)$. Conversely, if $\varepsilon > d_X(z,k) = d_X(zk^{-1},e)$, then, putting $w = zk^{-1}$, we have $z = wk \in B_\varepsilon(K)$. $\square$

The significance of the following simple corollary is manifold. It explicitly demonstrates that small either-sided translations do not much alter the norm nor either of the pair of conjugate metrics $d, \tilde{d}$. Its main effect is on the analysis of subadditive functions.

Corollary. *With* $||x|| := d_X(x,e)$, *where* d_X *is a right-invariant metric on* X,

$$|(||x|| - ||y||)| \leq ||xy|| \leq ||x|| + ||y||.$$

Proof: By Proposition 2, the triangle inequality and symmetry holds for norms, so $||y|| = ||x^{-1}xy|| \leq ||x^{-1}|| + ||xy|| = ||x|| + ||xy||$. $\square$

We now generalize (1), by letting T, X be subgroups of a normed group G with X invariant under T.

Definition. We say that a function $h : X \to H$ is *slowly varying on* X *over* T if $\partial_X h(t) = e_H$, that is, for each t in T

$$h(tx)h(x)^{-1} \to e_H, \text{ as } ||x|| \to \infty \text{ for } x \in X.$$

We omit mention of X and T when context permits. In practice G will be an internal direct product of two normal subgroups $G = TX$. We may verify the property of h just defined by comparison with a slowly varying function.

Comparison criterion. $h : X \to H$ *is slowly varying iff for some slowly varying function* $g : X \to H$ *and some* $\lambda \in H$,

$$\lim_{||x|| \to \infty} h(x)g(x)^{-1} = \lambda.$$

Proof. Use

$$h(tx)h(x)^{-1} = h(tx)g(tx)^{-1}g(tx)g(x)^{-1}g(x)h(x)^{-1} \to \lambda e_H \lambda^{-1} = e_H. \qquad \square$$

Theorem. *For* d_X *a right-invariant metric on* X, *the norm* $||x|| := d_X(x,e)$, *as a function from* X *to the multiplicative positive reals* $\mathbb{R}_+^*$, *is slowly varying in the mutiplicative sense, i.e., for any* $t \in X$,

$$\lim_{||x|| \to \infty} \frac{||tx||}{||x||} = 1.$$

More generally, for T *a one-parameter subgroup of* X, *any sub-additive Baire function* $p : X \to \mathbb{R}_+^*$ *with*

$$||p||_T := \lim_{x \in T,\, ||x|| \to \infty} \frac{p(x)}{||x||} > 0$$

is multiplicatively slowly varying. (The limit exists by the First Limit Theorem for Baire subadditive functions, see [BOst5]*.)*

Proof: By the corollary, for $x \neq e$,

$$1 - \frac{||t||}{||x||} \leq \frac{||tx||}{||x||} \leq 1 + \frac{||t||}{||x||},$$

which implies slow variation. We regard p as mapping to $\mathbb{R}_+^*$, the strictly positive reals (since $p(x) = 0$ iff $x = e_X$). If $||p||_T > 0$, we may then take $\lambda = ||p||_T$ and the assertion follows from the Comparison criterion above. Explicitly, for $x \neq e$,

$$\frac{p(xy)}{p(x)} = \frac{p(xy)}{||xy||} \cdot \frac{||xy||}{||x||} \cdot \frac{||x||}{p(x)} \to ||p||_T \cdot 1 \cdot \frac{1}{||p||_T} = 1. \qquad \square$$

Corollary. *If* $\pi : X \to Y$ *is a group homomorphism and* $|| \cdot ||_Y$ *is (*1-γ*)-quasi-isometric to* $|| \cdot ||_X$*, then the subadditive function* $p(x) = ||\pi(x)||_Y$ *is slowly varying. For general (*1-γ*)-quasi-isometry the function* p *is extended regularly varying, i.e. satisfies, for some constants* c, d*,*

$$z^d \leq p_*(z) \leq p^*(z) \leq z^c,$$

where

$$p_*(z) = \lim \sup_{||x|| \to \infty} p(zx)p(x)^{-1} \qquad p_*(z) = \lim \inf_{||x|| \to \infty} p(zx)p(x)^{-1}.$$

Proof. Subadditivity of p follows from homomorphism, since $p(xy) = ||\pi(xy)||_Y = ||\pi(x)\pi(y)||_Y \leq ||\pi(x)||_Y + ||\pi(y)||_Y$. Assuming that, for $\lambda = 1$ and $\gamma > 0$, the norm $|| \cdot ||_Y$ is (λ-γ)-quasi-isometric to $|| \cdot ||_X$, we have, for $x \neq e$,

$$1 - \frac{\gamma}{||x||_X} \leq \frac{p(x)}{||x||_X} \leq 1 - \frac{\gamma}{||x||_X}.$$

So

$$\lim_{||x|| \to \infty} \frac{p(x)}{||x||} = 1 \neq 0,$$

and the result follows from the Comparison criterion and the Theorem.

If, for general $\lambda \geq 1$ and $\gamma > 0$, the norm $||\cdot||_Y$ is (λ-γ)-quasi-isometric to $||\cdot||_X$, we have, for $x \neq e$,

$$\lambda^{-1} - \frac{\gamma}{||x||_X} \leq \frac{p(x)}{||x||_X} \leq \lambda - \frac{\gamma}{||x||_X}.$$

So

$$\frac{p(xy)}{p(x)} = \frac{p(xy)}{||xy||} \cdot \frac{||xy||}{||x||} \cdot \frac{||x||}{p(x)} \leq \left(\lambda - \frac{\gamma}{||x||_X}\right) \cdot \frac{||xy||}{||x||} \cdot \left(\lambda^{-1} - \frac{\gamma}{||x||_X}\right)^{-1},$$

giving

$$p^*(y) := \limsup_{x\to\infty} \frac{p(xy)}{p(x)} \leq \lambda^2. \qquad \square$$

Remark. The meaning of the abstract definition of slow variation is open to varying interpretations according to choice of structures. Thus if $X = H = \mathbb{R}$ is construed additively, so that $e_H = e_X = 0$ and $||x|| := |x - 0| = |x|$ in both cases, and with the action tx denoting $t + x$, the function $f(x) := |x|$ is not slowly varying, because $(x+t) - x = t \nrightarrow 0 = e_H$. On the other hand a multiplicative construction on $H = \mathbb{R}_+^*$, for which $e_H = 1$ and $||h||_H := |\log h|$, but with $X = \mathbb{R}$ still additive and tx still meaning $t + x$, yields f as having slow variation (as in the Theorem), in the sense that

$$f(tx)f(x)^{-1} = (x+t)/x \to 1 = e_H \text{ as } x \to \infty.$$

We note that in this context the regularly varying functions h on X behave asymptotically as $h(x) := e^{ax}$, for some constant a.

Note that, for $X = H = \mathbb{R}_+^*$, and with tx meaning $t.x$, since again $||x|| = |\log x|$ is the group norm, we have here

$$f(tx)f(x)^{-1} = ||tx||/||x|| = \frac{|\log tx|}{|\log x|} = \frac{|\log t + \log x|}{|\log x|} \to 1 = e_H, \text{ as } x \to \infty,$$

which again illustrates the content of the Theorem. Here the regularly varying functions take the form $h(x) = x^a$ for some constant a. See [BGT] Ch. 1 for background on additive and multiplicative formulations of regular variation in the classical setting of functions $f : G \to H$ with $G, H = \mathbb{R}$ or $\mathbb{R}_+$.

Definition. We say that $\xi \in X$ is *infinitely divisible* if, for each positive integer n, there is x with $x^n = \xi$. We say that the infinitely divisible element ξ is *embeddable* if, for some one-parameter subgroup T in X, we have $\xi \in T$. When such a T exists it is unique (the elements $\xi^{m/n}$, for m, n integers, are dense in T); we write $T(\xi)$ for it. Clearly any element of a one-parameter subgroup is both infinitely divisible and embeddable. For results on this see [D], Heyer [Hey], McCrudden [McC]. With these definitions, our previous analysis allows the First Limit Theorem to be stated in the context of normed groups.

Proposition. *Let ξ be infinitely divisible and embeddable in $T(\xi)$, a one-parameter subgroup of X. Then for any Baire subadditive $p : X \to \mathbb{R}_+$ and $t \in T(\xi)$,*

$$\partial_{T(\xi)} p(t) := \lim_{s \in T,\ ||s|| \to \infty} \frac{p(ts)}{||s||} = ||p||_T,$$

i.e., treating the subgroup $T(\xi)$ as a direction, the limit function is determined by the direction.

Proof. By subadditivity, $p(s) = p(t^{-1}ts) \leq p(t^{-1}) + p(ts)$, so

$$p(s) - p(t^{-1}) \leq p(ts) \leq p(t) + p(s).$$

For $s \neq e$, divide through by $||s||$ and let $||u|| \to \infty$:

$$||p||_T \leq \partial_T p(t) \leq ||p||_T. \qquad \square$$

Definition (Supremum metric, supremum norm). Let X have a metric d_X. Following [BePe] we denote by $Auth(X)$ the group of auto-homeomorphisms of X under composition. We denote by id_X the identity map $id_X(x) = x$. As before $\mathcal{G}$ is a fixed subgroup of $Auth(X)$, for example $Tr_L(X)$ the group of left-translations τ_x, defined by

$$\tau_x(z) = xz.$$

(We consider this in detail in the next section.) For $g, h \in \mathcal{G}$, define the possibly infinite number

$$d^*_X(g, h) := \sup_{x \in X} d_X(g(x), h(x)).$$

Put

$$\mathcal{H}(X) = \mathcal{H}(X, \mathcal{G}) := \{g \in \mathcal{G} : d^*_X(g, id) < \infty\}.$$

For g, h in $\mathcal{H}(X)$, we call $d^*_X(g, h)$ the *supremum metric* and

$$||h|| = ||h||_{\mathcal{H}(X)} := d^*_X(h, id) = \sup_{x \in X} d_X(h(x), x)$$

the *supremum norm.* Our next result justifies these terms.

Proposition 3 (Group-norm properties in $\mathcal{H}(X)$).
$|| \cdot ||$ is a group-norm: that is, for $h, h' \in \mathcal{H}(X)$,

$$||h|| = 0 \text{ iff } h = e, \; ||h \circ h'|| \leq ||h|| + ||h'|| \text{ and } ||h|| = ||h^{-1}||.$$

Proof. Evidently $d^*(h, id) = \sup_{x \in X} d(h(x), x) = 0$ iff $h(x) = id_X$. We have

$$||h|| = d^*(h, id) = \sup_{x \in X} d(h(x), x) = \sup_{y \in X} d(y, h^{-1}(y)) = ||h^{-1}||.$$

Next note that

$$d^*(id, h \circ h') = \sup_{x \in X} d(hh'(x), x) = \sup_{y \in X} d(h(y), h'^{-1}(y)) = d^*(h, h'^{-1}). \quad (2)$$

But

$$d^*(h, h') = \sup_{x \in X} d(h(x), h'(x)) \leq \sup_{x \in X} [d(h(x), x) + d(x, h'(x))] \leq d^*(h, id) + d^*(h', id) < \infty.$$

$\square$

Theorem 1. *The set $\mathcal{H}(X)$ of bounded auto-homeomorphisms of a metric group X is a group under composition, metrized by the right-invariant supremum metric d^*_X.*

Proof. The identity, id_X, is bounded. For right-invariance (cf. (2)),

$$d^*(g \circ h, g' \circ h) = \sup_{x \in X} d(g(h(x)), g'(h(x)) = \sup_{y \in X} d(g(y), g'(y)) = d^*(g, g'). \quad \square$$

We apply the above construction in the next result to deduce that right-invariance may be arranged if every $x \in X$ has finite right-translation sup-norm:

$$\sup_{z \in X} d_X(xz, z) < \infty.$$

We consider the norm induced by a single left-translation in the next section.

Proposition 4 (Right-sup-norm).
For any metric d_X *on a group* X, *put*

$$\begin{aligned} H &= H_X = \{x \in X : \sup_{z \in X} d_X(xz, z) < \infty\}, \\ ||x||_\infty &: = \sup d_X(xz, z), \text{ for } x \in H. \end{aligned}$$

For $x, y \in H$, *let* $\bar{d}_H(x, y) := \sup_z d_X(xz, yz)$. *Then* $\bar{d}_H$ *is a right-invariant metric on* H, *and* $\bar{d}_H(x, y) = ||xy^{-1}||_\infty$.

If d_X *is right-invariant, then* $H = X$ *and* $\bar{d}_H = d_X$.

Proof. The argument relies implicitly on the natural embedding of X in $Auth(X)$ as $Tr_L(X)$ (made explicit in the next section). For $x \in X$, put

$$||\tau_x||^* := \sup d_X(xz, z).$$

For $x \neq e$, we have $0 < ||\tau_x|| \leq \infty$. By Proposition 3, $\mathcal{H}(X) = \mathcal{H}(X, Tr_L(X)) = \{\tau_x : ||\tau_x||^* < \infty\}$ is a subgroup of $\mathcal{H}(X, Auth(X))$ on which $|| \cdot ||^*$ is thus a norm. Identifiying $\mathcal{H}(X)$ with the subset $H = \{x \in X : ||\tau_x|| < \infty\}$ of X, we see that on H

$$\bar{d}_H(x, y) := \sup_z d_X(xz, yz)$$

defines a right-invariant metric, as $\bar{d}_H(xv, yv) = \sup_z d_X(xvz, yvz) = \sup_z d_X(xz, yz)$. Moreover,

$$||\tau_x||^* = \bar{d}(x, e) = ||\tau_x||_\infty,$$

hence by Proposition 3

$$||\tau_x \tau_y^{-1}||^* = \bar{d}(x, y) = ||xy^{-1}||_\infty,$$

as asserted. If d_X is right-invariant, then of course $||\tau_x||^* = d_X(x, e) = ||x||_X$ and $H = X$. $\square$

Example. Let S, T be normed groups. For $\alpha : S \to T$ we define the possibly infinite number

$$||\alpha|| := \sup\{||\alpha(s)||_T / ||s||_S : s \in S\} = \inf\{M : ||\alpha(s)|| \leq M ||s||\}.$$

α is called bounded if $||\alpha||$ is finite. The bounded elements form a group G under the pointwise multiplication $\alpha\beta(t) = \alpha(t)\beta(t)$. Clearly $||\alpha|| = 0$ implies that $\alpha(t) = e$, for all t. Symmetry is clear. Also

$$||\alpha(t)\beta(t)|| \leq ||\alpha(t)|| + ||\beta(t)|| \leq [||\alpha|| + ||\beta||]||t||,$$

so

$$||\alpha\beta|| \leq ||\alpha|| + ||\beta||.$$

We say that a function $\alpha : S \to T$ is *multiplicative* if α is bounded and

$$\alpha(ss') = \alpha(s)\alpha(s').$$

A function $\gamma : S \to T$ is *asymptotically multiplicative* if $\gamma = \alpha\beta$, where α is multiplicative and bounded and β is bounded. In the commutative situation with S, T normed vector spaces, the norm here reduces to the operator norm. This group norm is studied extensively in [CSC] in relation to Ulam's problem.

Proposition 5 (Magnification metric) *Let* $T = \mathcal{H}(X)$ *with group norm* $||t|| = d_T(t, e_T)$ *and* $\mathcal{A}$ *a subgroup (under composition) of* $Auth(T)$ *(so, for* $t \in T$ *and* $\alpha \in \mathcal{A}$, $\alpha(t) \in \mathcal{H}(X)$ *is a homeomorphism of* X*). For any* $\varepsilon \geq 0$, *put*

$$d^{\varepsilon}_{\mathcal{A}}(\alpha, \beta) := \sup_{||t|| \leq \varepsilon} d^*_T(\alpha(t), \beta(t)).$$

Suppose further that X *distinguishes the maps* $\{\alpha(e_{\mathcal{H}(X)}) : \alpha \in \mathcal{A}\}$, *i.e., for* $\alpha, \beta \in A$, *there is* $z = z_{\alpha,\beta} \in X$ *with* $\alpha(e_{\mathcal{H}(X)})(z) \neq \beta(e_{\mathcal{H}(X)})(z)$.

Then $d^{\varepsilon}_{\mathcal{A}}(\alpha, \beta)$ *is a metric; furthermore,* $d^{\varepsilon}_{\mathcal{A}}$ *is right-invariant for translations by* γ *such that* γ^{-1} *maps the* ε*-ball to the* ε*-ball.*

Proof. To see that this is a metric, note that for $t = e_{\mathcal{H}(X)} = id_T$ we have $||t|| = 0$ and

$$\begin{aligned} d^*_T(\alpha(e_{\mathcal{H}(X)}), \beta(e_{\mathcal{H}(X)})) &= \sup_z d_X(\alpha(e_{\mathcal{H}(X)})(z), \beta(e_{\mathcal{H}(X)})(z)) \\ &\geq d_X(\alpha(e_{\mathcal{H}(X)})(z_{\alpha,\beta}), \beta(e_{\mathcal{H}(X)})(z_{\alpha,\beta})) > 0. \end{aligned}$$

Symmetry is clear. Finally the triangle inequality follows as usual:

$$\begin{aligned} d^{\varepsilon}_{\mathcal{A}}(\alpha, \beta) &= \sup_{||t|| \leq 1} d^*_T(\alpha(t), \beta(t)) \leq \sup_{||t|| \leq 1} [d^*_T(\alpha(t), \gamma(t)) + d^*_T(\gamma(t), \beta(t))] \\ &\leq \sup_{||t|| \leq 1} d^*_T(\alpha(t), \gamma(t)) + \sup_{||t|| \leq 1} d^*_T(\gamma(t), \beta(t)) \\ &= d^{\varepsilon}_{\mathcal{A}}(\alpha, \gamma) + d^{\varepsilon}_{\mathcal{A}}(\gamma, \beta). \end{aligned}$$

One cannot hope for the metric to be right-invariant in general, but if γ^{-1} maps the ε-ball to the ε-ball, one has

$$\begin{aligned} d_{\mathcal{A}}^{\varepsilon}(\alpha\gamma,\beta\gamma) \quad &: \quad = \sup_{||t||\leq\varepsilon} d_T^*(\alpha(\gamma(t)),\beta(\gamma(t)). \\ &= \sup_{||\gamma^{-1}(s)||\leq\varepsilon} d_T^*(\alpha(s),\beta(s)). \qquad \square \end{aligned}$$

In this connection we note the following.

Proposition 6. *In the setting of Prop 5, denote by* $||.||_\varepsilon$ *the norm induced by* $d_{\mathcal{A}}^{\varepsilon}$*; then*

$$\sup_{||t||\leq\varepsilon} ||\gamma(t)||_T - \varepsilon \leq ||\gamma||_\varepsilon \leq \sup_{||t||\leq\varepsilon} ||\gamma(t)||_T + \varepsilon.$$

Proof. By definition, for t with $||t|| \leq \varepsilon$, we have

$$\begin{aligned} ||\gamma||_\varepsilon \quad &: \quad = \sup_{||t||\leq\varepsilon} d_T^*(\gamma(t),t) \leq \sup_{||t||\leq\varepsilon} [d_T^*(\gamma(t),e) + d_T^*(e,t)] \leq \sup_{||t||\leq\varepsilon} ||\gamma(t)||_T + \varepsilon, \\ ||\gamma(t)||_T \quad &= \quad d_T^*(\gamma(t),e) \leq d_T^*(\gamma(t),t) + d_T^*(t,e) \\ &\leq \quad ||t|| + ||\gamma||_\varepsilon \leq \varepsilon + ||\gamma||_\varepsilon. \qquad \square \end{aligned}$$

Theorem 2 (Invariance of Norm Theorem) (for (b) cf. [Klee]).

(a) *The group-norm is abelian (and the metric is bi-invariant) iff*

$$||xy(ab)^{-1}|| \leq ||xa^{-1}|| + ||yb^{-1}||,$$

for all x, y, a, b*, or equivalently,*

$$||uabv|| \leq ||uv|| + ||ab||,$$

for all x, y, a, b*.*

(b) *Hence a metric* d *on the group* X *is bi-invariant iff the Klee property holds:*

$$d(ab,xy) \leq d(a,x) + d(b,y). \qquad \text{(Klee)}$$

In particular, this holds if the group X *is itself abelian.*

(c) *The group norm is abelian iff the norm is preserved under conjugacy (inner automorphisms).*

Proof (a) If the group-norm is abelian, then by the triangle inequality

$$\begin{aligned} ||xyb^{-1}\cdot a^{-1}|| &= ||a^{-1}xyb^{-1}|| \\ &\leq ||a^{-1}x|| + ||yb^{-1}||. \end{aligned}$$

For the converse we demonstrate bi-invariance in the form: $||ba^{-1}|| = ||a^{-1}b||$. In fact it suffices to show that $||yx^{-1}|| \leq ||x^{-1}y||$; for then bi-invariance follows, since taking $x = a, y = b$ we get $||ba^{-1}|| \leq ||a^{-1}b||$, whereas taking $x = b^{-1}, y = a^{-1}$ we get the reverse $||a^{-1}b|| \leq ||ba^{-1}||$. As for the claim, we note that

$$||yx^{-1}|| \leq ||yx^{-1}yy^{-1}|| \leq ||yy^{-1}|| + ||x^{-1}y|| = ||x^{-1}y||.$$

(b) Klee's result is deduced as follows. If d is a bi-invariant metric, then $||\cdot||$ is abelian. Conversely, for d a metric, let $||x|| := d(e,x)$. Then $||.||$ is a group-norm, as

$$d(ee,xy) \leq d(e,x) + d(e,y).$$

Hence d is right-invariant and $d(u,v) = ||uv^{-1}||$. Now we conclude that the group-norm is abelian since

$$||xy(ab)^{-1}|| = d(xy,ab) \leq d(x,a) + d(y,b) = ||xa^{-1}|| + ||yb^{-1}||.$$

Hence d is also left-invariant.

(c) Suppose the norm is abelian. Then for any g, by the cyclic property $||g^{-1}bg|| = ||gg^{-1}b|| = ||b||$. Conversely, if the norm is preserved under automorphism, then we have bi-invariance, since $||ba^{-1}|| = ||a^{-1}(ba^{-1})a|| = ||a^{-1}b||$. □

Remark. Note that, taking $b = v = e$, we have the triangle inequality. Thus the result (a) characterizes maps $||\cdot||$ with the positivity property as group pre-norms which are abelian. (The extended web-site version of this paper offers alternative proofs using a metric formulation.) In regard to conjugacy , see also the Uniformity Theorem for Conjugation in Section 10. We close with the following classical result.

The Normability Theorem for Groups (Kakutani-Birkhoff). *Let X be a first-countable group and let V_n be a balanced local base at e_X with*

$$V_{n+1}^4 \subseteq V_n.$$

Let $r = \sum_{n=1}^{\infty} c_n(r)2^{-n}$ *be a terminating representation of the dyadic number* r*, and put*

$$A(r) := \sum_{n=1}^{\infty} c_n(r)V_n.$$

Then

$$p(x) := \inf\{r : x \in A(r)\}$$

is a group-norm. If further X *is locally compact, non-compact, then* p *may be arranged such that* p *is unbounded on* X*, but bounded on compact sets.*

For a proof see that offered in [Ru-FA2] for Th. 1.24 (p 18-19), which derives a metrization of a topological vector space in the form $d(x, y) = p(x-y)$ and makes no use of the scalar field, That proof may be rewritten verbatim with xy^{-1} substituting for the additive notation $x - y$ (cf. Proposition 1).

Remarks.

1. If the group-norm is abelian, then we have the *commutator inequality*

$$||[x, y]|| \leq 2||x^{-1}y||,$$

because

$$||[x, y]|| = ||x^{-1}y^{-1}xy|| \leq ||x^{-1}y|| + ||y^{-1}x|| = 2||x^{-1}y||.$$

The triangle inequality gives only

$$||[x, y]|| = ||x^{-1}y^{-1}xy|| \leq ||x^{-1}y^{-1}|| + ||xy|| = ||xy|| + ||yx||.$$

2. Taking $u = f(tx), v = f(x)^{-1}$ etc., assuming the Klee Property, we have

$$\begin{aligned} ||f(tx)g(tx)[f(x)g(x)]^{-1}|| &= ||f(tx)g(tx)g(x)^{-1}f(x)^{-1}|| \\ &\leq ||f(tx)f(x)^{-1}|| + ||g(tx)g(x)^{-1}||, \end{aligned}$$

showing that the product of two slowly varying functions is slowly varying, since

$$f(tx)f(t)^{-1} \to e \text{ iff } ||f(tx)f(t)^{-1}|| \to 0.$$

3 Subadditivity

Recall from [Kucz] p. 140 the definitions of *upper and lower hulls* of a function p :

$$\begin{aligned} M_p(x) &= \lim_{r\to 0+} \sup\{p(z) : z \in B_r(x)\}, \\ m_p(x) &= \lim_{r\to 0+} \inf\{p(z) : z \in B_r(x)\}. \end{aligned}$$

(Usually these are of interest for convex functions p.) These definitions remain valid for a normed group. (Note that e.g. $\inf\{p(z) : z \in B_r(x)\}$ is a decreasing function of r.) We understand the balls here to be defined by a *right-invariant* metric, i.e.

$$B_r(x) := \{y : d(x, y) < r\} \text{ with } d \text{ right-invariant.}$$

These are subadditive functions if the group G is $\mathbb{R}^d$. We reprove some results from Kuczma [Kucz], thus verifying the extent to which they may be generalized to normed groups. Only our first result appears to need the Klee property (bi-invariance of the metric); fortunately this result is not needed in the sequel. The Main Theorem below concerns the behaviour of $p(x)/||x||$.

Lemma 1 (cf. [Kucz] L. 1 p. 403). *For a normed group G with the Klee group, m_p and M_p are subadditive.*

Proof. For $a > m_p(x)$ and $b > m_p(y)$ and $r > 0$, let $d(u, x) < r$ and $d(v, y) < r$ satisfy

$$\inf[p(z) : z \in B_r(x)] \le p(u) < a, \text{ and } \inf[p(z) : z \in B_r(y)] \le p(v) < b.$$

Then, by the Klee property,

$$d(xy, uv) \le d(x, u) + d(y, v) < 2r.$$

Now

$$\inf[p(z) : z \in B_{2r}(xy)] \le p(uv) \le p(u) + p(v) < a + b,$$

hence

$$\inf[p(z) : z \in B_{2r}(xy)] \le \inf[p(z) : z \in B_r(x)] + \inf[p(z) : z \in B_r(x)],$$

and the result follows on taking limits as $r \to 0+$. □

Lemma 2 (cf. [Kucz] L. 2 p. 403). *For a normed group G, if $p : G \to \mathbb{R}$ is subadditive, then*

$$m_p(x) \leq M_p(x) \text{ and } M_p(x) - m_p(x) \leq M_p(e).$$

Proof. Only the second assertion needs proof. For $a > m_p(x)$ and $b < M_p(x)$, there exist $u, v \in B_r(x)$ with

$$a > p(u) \geq m_p(x), \text{ and } b < p(v) \leq M_p(x).$$

So

$$b - a < p(v) - p(u) \leq p(vu^{-1}u) - p(u) \leq p(vu^{-1}) + p(u) - p(u) = p(vu^{-1}).$$

Now

$$||vu^{-1}|| \leq ||v|| + ||u|| < 2r,$$

so $vu^{-1} \in B_{2r}(e)$ and hence

$$p(vu^{-1}) \leq \sup[p(z) : z \in B_{2r}(e)].$$

Hence, with r fixed, taking a, b to their respective limits,

$$M_p(x) - m_p(x) \leq \sup[p(z) : z \in B_{2r}(e)].$$

Taking limits as $r \to 0+$, we obtain the second inequality. □

Lemma 3. *For any subadditive function $f : G \to \mathbb{R}$, if f is locally bounded above at a point, then it is locally bounded at every point.*

Proof. We repeat the proof in [Kucz] p. 404 Th. 2, thus verifying that it continues to hold in a normed group.

Supppose that p is locally bounded above at t_0 by K. We first show that f is locally bounded above at e. Suppose otherwise that for some $t_n \to e$ we have $p(t_n) \to \infty$. Now $t_n t_0 \to e t_0 = t_0$ and so

$$p(t_n) = p(t_n t_0 t_0^{-1}) \leq p(t_n t_0) + p(t_0^{-1}) \leq K + p(t_0^{-1}),$$

a contradiction. Hence p is locally bounded above at e, i.e. $M_p(e) < \infty$. But $0 \leq M_p(x) - m_p(x) \leq M_p(e)$, hence both $M_p(x)$ and $m_p(x)$ are finite for every x. That is, p is locally bounded above and below at each x. □

Proposition (cf. [Kucz] p 404 Th 3). *For a Baire group G and a Baire function $f : G \to \mathbb{R}$, if f is subadditive, then f is locally bounded.*

Proof. By the Baire assumptions for some k, $H^k = \{x : |f(x)| < k\}$ is non-meagre Suppose that f is not locally bounded; then it is not locally bounded above at some point u, i.e. there exists $u_n \to u$ with

$$f(u_n) \to +\infty.$$

By the Category Embedding Theorem ([BOst11], and Section 4), for some $k \in \omega$, $t \in H^k$ and an infinite $\mathbb{M}$, we have

$$\{u_n t : n \in \mathbb{M}\} \subseteq H^k.$$

For n in $\mathbb{M}$, we have

$$f(u_n) = f(u_n t t^{-1}) \leq f(u_n t) + f(t^{-1}) \leq k + f(t^{-1}),$$

which contradicts $f(u_n) \to +\infty$. □

Definition. We say that a normed group G has a *vanishingly small word-net* (which may be also *compactly generated,* as appropriate) if, for any $\varepsilon > 0$, there is $\eta > 0$ such that, for all δ with $0 < \delta < \eta$ there is a set (a compact set) of generators Z_δ in $B_\delta(e)$ and a constant M_δ such that, for all x with $||x|| > M_\delta$, there is some word $w(x) = z_1 ... z_{n(x)}$ using generators in Z_δ with $||z_i|| = \delta(1 + \varepsilon_i)$, with $|\varepsilon_i| < \varepsilon$, where

$$d(x, w(x)) < \delta$$

and

$$1 - \varepsilon \leq \frac{n(x)\delta}{||x||} \leq 1 + \varepsilon.$$

Thus $\mathbb{R}^d$ has a vanishingly small compactly generated word-net and hence so does the sequence space l_2.

Main Theorem. *Let G be a normed group with a vanishingly small word-net. Let $p : G \to \mathbb{R}_+$ be Baire, subadditive with*

$$\beta := \lim \sup_{||x|| \to 0+} \frac{p(x)}{||x||} < \infty.$$

Then

$$\lim \sup_{||x|| \to \infty} \frac{p(x)}{||x||} \leq \beta < \infty.$$

Proof. Let $\varepsilon > 0$. Let $b = \beta + \varepsilon$. Hence on $B_\delta(e)$ for δ small enough to gurantee the existence of Z_δ and M_δ we have also

$$\frac{p(x)}{||x||} \leq b.$$

By the Proposition, we may assume that p is bounded by some constant K in $B_\delta(e)$. Let $||x|| > M_\delta$.

Choose a word $w(x) = z_0 z_1 ... z_n$ with $||z_i|| = \delta(1 + \varepsilon_i)$ with $|\varepsilon_i| < \varepsilon$, with

$$p(x_i) < b||x_i|| = b\delta(1 + \varepsilon_i)$$

and

$$d(x, w(x)) < \delta,$$

i.e.

$$x = w(x)s$$

for some s with $||s|| < \delta$ and

$$1 - \varepsilon \leq \frac{n(x)\delta}{||x||} \leq 1 + \varepsilon.$$

Now

$$\begin{array}{rcl} p(x) & = & p(ws) \leq p(w) + p(r) = \sum p(z_i) + p(s) \\ & \leq & \sum b\delta(1 + \varepsilon_i) + p(s) \\ & = & nb\delta(1 + \varepsilon) + K. \end{array}$$

So

$$\frac{p(x)}{||x||} \leq \frac{n\delta}{||x||} b(1 + \varepsilon) + \frac{M}{||x||}.$$

Hence we obtain

$$\frac{p(x)}{||x||} \leq b(1 + \varepsilon)^2 + \frac{M}{||x||}.$$

So in the limit

$$\lim \sup_{||x|| \to \infty} \frac{p(x)}{||x||} < \beta,$$

as asserted. □

We note a related result, which requires the following definition. For p subadditive, put

$$p_*(x) = \liminf_{y \to x} p(y), \qquad p^*(x) := \limsup_{y \to x} p(y).$$

These are subadditive and lower (resp. upper) semicontinuous with $p_*(x) \leq p(x) \leq p^*(x)$.

Mueller's Theorem ([Mue] Th. 3). *Let p be subadditive on a locally compact group G and suppose*

$$\liminf_{x \to e} p^*(x) \leq 0.$$

Then p is continuous almost everywhere.

4 Theorems of Steinhaus type and Dichotomy

If ψ_n converges to the identity, then, for large n, each ψ_n is almost an isometry. Indeed, as we shall see in Section 10, by the Proposition on Permutation metrics, we have

$$d(x,y) - 2||\psi_n|| \leq d(\psi_n(x), \psi_n(y)) \leq d(x,y) + 2||\psi_n||.$$

This motivates our next result; we need to recall a definition and the Category Embedding Theorem from [BOst11], whose proof we reproduce here for completeness. In what follows, the words *quasi everywhere (q.e.)*, or *for quasi all points* mean *for all points off a meagre set* (see [Kah]).

Definition (**weak category convergence**). A sequence of homeomorphisms ψ_n satisfies the *weak category convergence* condition (wcc) if:

For any non-empty open set U, there is an non-empty open set $V \subseteq U$ such that, for each $k \in \omega$,

$$\bigcap_{n \geq k} V \backslash \psi_n^{-1}(V) \text{ is meagre.} \tag{wcc}$$

Equivalently, for each $k \in \omega$, there is a meagre set M such that, for $t \notin M$,

$$t \in V \Longrightarrow (\exists n \geq k)\ \psi_n(t) \in V.$$

Category Embedding Theorem. *Let X be a Baire space. Suppose given homeomorphisms $\psi_n : X \to X$ for which the weak category convergence condition* (wcc) *is met. Then, for any non-meagre Baire set T, for locally quasi all $t \in T$, there is an infinite set $\mathbb{M}_t$ such that*

$$\{\psi_m(t) : m \in \mathbb{M}_t\} \subseteq T.$$

Proof. Suppose T is Baire and non-meagre. We may assume that $T = U \backslash M$ with U non-empty and M meagre. Let $V \subseteq U$ satisfy (wcc).

Since the functions h_n are homeomorphisms, the set

$$M' := M \cup \bigcup_n h_n^{-1}(M)$$

is meagre. Put

$$W = \mathbf{h}(V) := \bigcap_{k \in \omega} \bigcup_{n \geq k} V \cap h_n^{-1}(V) \subseteq V \subseteq U.$$

Then $V \cap W$ is co-meagre in V. Indeed

$$V \backslash W = \bigcup_{k \in \omega} \bigcap_{k \geq n} V \backslash h_n^{-1}(V),$$

which by assumption is meagre.

Let $t \in V \cap W \backslash M'$ so that $t \in T$. Now there exists an infinite set $\mathbb{M}_t$ such that, for $m \in \mathbb{M}_t$, there are points $v_m \in V$ with $t = h_m^{-1}(v_m)$. Since $h_m^{-1}(v_m) = t \notin h_m^{-1}(M)$, we have $v_m \notin M$, and hence $v_m \in T$. Thus $\{h_m(t) : m \in \mathbb{M}_t\} \subseteq T$ for t in a co-meagre set, as asserted. $\square$

Examples. In $\mathbb{R}$ we may consider $\psi_n(t) = t + z_n$ with $z_n \to z_0 := 0$. It is shown in [BOst11] that for this sequence the condition (wcc) is satisfied in both the usual topology and the density topology on $\mathbb{R}$. This remains true in $\mathbb{R}^d$, where the specific instance of the theorem is referred to as the Kestelman-Borwein-Ditor Theorem; see the next section ([Kes], [BoDi]; compare also the Oxtoby-Hoffmann-Jørgensen zero-one law for Baire groups, [HJ] p. 356, [Oxt], cf. [RR-1]). In fact in any metrizable group$X$ with right-invariant metric d_X, for a null sequence tending to the identity $z_n \to z_0 := e_X$, the mapping defined by $\psi_n(x) = z_n x$ converges to the identity (see [BOst13], Corollary to Ford's Theorem); here too (wcc) holds. This follows from the next result, which extends the proof of [BOst11].

First Proposition on weak category convergence. *If ψ_n converges to the identity, then ψ_n satisfies the weak category convergence condition* (wcc).

Proof. It is more convenient to prove the equivalent statement that ψ_n^{-1} satisfies the category convergence condition.

Put $z_n = \psi_n(z_0)$, so that $z_n \to z_0$. Let k be given.

Suppose that $y \in B_\varepsilon(z_0)$, i.e. $r = d(y, z_0) < \varepsilon$. For some $N > k$, we have $\varepsilon_n = d(\psi_n, id) < \frac{1}{3}(\varepsilon - r)$, for all $n \geq N$. Now

$$\begin{aligned} d(y, z_n) &\leq d(y, z_0) + d(z_0, z_n) \\ &= d(y, z_0) + d(z_0, \psi_n(z_0)) \leq r + \varepsilon_n. \end{aligned}$$

For $y = \psi_n(x)$ and $n \geq N$,

$$\begin{aligned} d(z_0, x) &\leq d(z_0, z_n) + d(z_n, y) + d(y, x) \\ &= d(z_0, z_n) + d(z_n, y) + d(x, \psi_n(x)) \\ &\leq \varepsilon_n + (r + \varepsilon_n) + \varepsilon_n < \varepsilon. \end{aligned}$$

So $x \in B_\varepsilon(z_0)$, giving $y \in \psi_n(B_\varepsilon(z_0))$. Thus

$$y \notin \bigcap_{n \geq N} B_\varepsilon(z_0) \backslash \psi_n(B_\varepsilon(z_0)) \supseteq \bigcap_{n \geq k} B_\varepsilon(z_0) \backslash \psi_n(B_\varepsilon(z_0)).$$

It now follows that

$$\bigcap_{n \geq k} B_\varepsilon(z_0) \backslash \psi_n(B_\varepsilon(z_0)) = \varnothing.$$

□

As a corollary we have the following important result known for topological groups (see [RR-TG], Rogers [Jay-Rog] p. 48, and [Kom1] for the topological vector space setting) and here proved in the metric setting.

Piccard-Pettis Theorem (Piccard [Pic1], [Pic2], Pettis [Pet1], [RR-TG] cf. [BOst14]). *In a normed group, for A non-meagre, the sets AA^{-1} and $A^{-1}A$ have non-empty interior.*

Proof. Suppose otherwise. We consider the set AA^{-1} and refer to the right-invariant metric $d(x,y) = ||xy^{-1}||$. Suppose the theorem is false. Then, for each integer $n = 1, 2, ..,$ there is $z_n \in B_{1/n}(e)\backslash AA^{-1}$; hence $z_n \to z_0 = e$. By Proposition 1 of Section 2, $\psi_n(x) := z_n x$ converges to the identity (as the metric is right-invariant) and so satisfies the (wcc); hence, there is $a \in A$ such that for infinitely many n

$$\psi_n(a) \in A, \text{ i.e. } z_n a \in A, \text{ or } z_n \in AA^{-1},$$

a contradiction. Reference to the conjugate metric secures the same result for $A^{-1}A$. □

One says that a set A is *thick* if e is an interior point of AA^{-1} (see e.g. [HJ] Section 2.4). The next result (proved essentially by the same means) applied to the additive group $\mathbb{R}$ implies the Kesteman-Borwein-Ditor ([BOst11]) theorem on the line. The name used here refers to a similar (weaker) property studied in Probability Theory (in the context of probabilities regarded as a semigroup under convolution, for which see [PRV], or [Par] 3.2 and 3.5, [BlHe], [Hey]). We need a definition.

Definition. Say that a set A in G is *right-shift compact* if, for any sequence of points a_n in A, there is a point t and a subsequence $\{a_n : n \in \mathbb{M}_t\}$ such that $a_n t$ converges through $\mathbb{M}_t$ to a point $a_0 t$ in A; similarly for *left-shift compact.* Evidently, finite Cartesian products of shift-compact sets are shift-compact. Thus a right-shift compact set A is pre-compact. (If the subsequence $a_m t$ converges to $a_0 t$, for m in $\mathbb{M}_t$, then likewise a_m converges to a_0, for m in $\mathbb{M}_t$.)

Proposition. *In a normed group, if a subgroup S is locally right-shift compact, then S is closed and locally compact. Conversely, a closed, locally compact subgroup is locally right-shift compact.*

Proof. Suppose that $a_n \to a_0$ with $a_n \in S$. If $a_m t \to a_0 t \in S$ down a subset M then $a_0 t(a_m t)^{-1} = a_0 a_m^{-1} \in S$ for $m \in M$. Hence also $a_0 = a_0 a_m^{-1} a_m \in S$ for $m \in M$. Thus S is closed. □

Remark. Suppose that $a_n = (a_n^i) \in A = \prod A_i$. Pick t_i and inductively infinite $\mathbb{M}_i \subseteq \mathbb{M}_{i-1}$ so that $a_n^i t_i \to a_0^i t_i$ along $n \in \mathbb{M}_i$ with $a_n^i t_i \in A_i$ for $n \in \omega$.

Diagonalize $\mathbb{M}_i$ by setting $\mathbb{M} := \{m_i\}$, where $m_{n+1} = \min\{m \in \mathbb{M}_{n+1} : m > m_n\}$. Then the subsequence $\{a_m : m \in \mathbb{M}\}$ satisfies, for each J finite,

$$\mathrm{pr}_J ta_m \subseteq \prod_{j \in J} A_j \text{ for eventually all } m \in \mathbb{M}.$$

Shift-Compactness Theorem. *In a normed group G, for A precompact, Baire and non-meagre, the set A is right-shift compact, i.e., for any sequence $a_n \in A$, there are $t \in G$ and $a \in A$ such that $a_n t \in A$ and $a_n t \to a$ down a subsequence. Likewise the set A is left-shift compact.*

Proof. Suppose $a_n \in A \subseteq \bar{A}$ with $\bar{A}$ compact. W.l.o.g. $a_n \to a_0 \in \bar{A}$. Hence $z_n := a_n a_0^{-1} \to e_G$. By the First Proposition on weak category convergence above, $\psi_n(x) := z_n x$ converges to the identity. Hence, for some $a \in A$ and infinite $\mathbb{M}$, we have $\{z_m a : m \in \mathbb{M}\} \subseteq A$. Taking $t = a_0^{-1}a$, we thus have $a_n t \in A$ and $a_n t \to a \in A$ along $\mathbb{M}$. Replace A by A^{-1} to obtain the other-handed result. $\square$

The following theorem asserts that a 'covering property modulo shift' is satisfied by bounded shift-compact sets. It will be convenient to make the following

Definitions. 1. Say that $\mathcal{D}:= \{D_1, ..., D_h\}$ *shift-covers* X, or is a *shifted-cover* of X if, for some $d_1, ..., d_h$ in G,

$$(D_1 - d_1) \cup ... \cup (D_h - d_h) = X.$$

Say that X is *compactly shift-covered* if every open cover $\mathcal{U}$ of X contains a finite subfamily $\mathcal{D}$ which shift-covers X.

2. We say that $\mathcal{D}:= \{D_1, ..., D_h\}$ *strongly shift-covers* A, or is a *strong shifted-cover* of A if, there are *arbitrarily small* $d_1, ..., d_h$ in $\mathbb{R}$ such that

$$(D_1 - d_1) \cup ... \cup (D_h - d_h) \supseteq A.$$

Say that A is *compactly strongly shift-covered* if every open cover $\mathcal{U}$ of A contains a finite subfamily $\mathcal{D}$ which strongly shift-covers A.

Example. Note that $A \subseteq \mathbb{R}$ is a dense-open (open in the density topology) iff each point of A is a density point of A. Suppose a_0 is a limit point of such a set A in the usual topology; then, for any $\varepsilon > 0$, we may find a point $\alpha \in A$ to within $\varepsilon/2$ of a_0 and hence some $t \in A$ within $\varepsilon/2$ of the point α such that some subsequence $t + a_m$ is included in A, with limit $t + a_0$ and with $|t| < \varepsilon$. That is, a dense-open set is strongly shift-compact.

Compactness Theorem (Compactness modulo shift, [BOst8]**).** *Let A be a right-shift compact subset of a separable normed group G. Then A is compactly shift-covered, i.e. for any norm-open cover $\mathcal{U}$ of A, there is a finite subset $\mathcal{V}$ of $\mathcal{U}$, one for each member of $\mathcal{V}$, such that the corresponding translates of $\mathcal{V}$ cover A.*

Proof. Let $\mathcal{U}$ be an open cover of A. Since G is second-countable we may assume that $\mathcal{U}$ is a countable family. Write $\mathcal{U} = \{U_i : i \in \omega\}$. Let $Q = \{q_j : j \in \omega\}$ enumerate a dense subset of G. Suppose, contrary to the assertion, that there is no finite subset $\mathcal{V}$ of $\mathcal{U}$ such that translates of $\mathcal{V}$, each translated by one element of Q, cover A. For each n, choose $a_n \in A$ not covered by $\{U_i - q_j : i, j < n\}$. As A is precompact, we may assume, by passing to a subsequence (if necessary), that a_n converges to some point a_0, and also that, for some t, the sequence $a_n t$ lies entirely in A. Let U_i in $\mathcal{U}$ cover $a_0 t$. Without loss of generality we may assume that $a_n t \in U_i$ for all n. Thus $a_n \in U_i t^{-1}$ for all n. Thus we may select $V := U_i q_j$ to be a translation of U_i such that $a_n \in V = U_i q_j$ for all n. But this is a contradiction, since a_n is not covered by $\{U_{i'} q_{j'} : i', j' < n\}$ for $n > \max\{i, j\}$. $\square$

The above proof of the compactness theorem for shift-covering may be improved to strong shift-covering, with only a minor modification (replacing Q with a set $Q^\varepsilon = \{q_j^\varepsilon : j \in \omega\}$ which enumerates, for given $\varepsilon > 0$, a dense subset of the ε ball about e), yielding the following.

Strong Compactness Theorem (Strong Compactness modulo shift, cf. [BOst8]**).** *Let A be a strongly right-shift compact subset of a separable normed group G. Then A is compactly strongly shift-covered, i.e. for any norm-open cover $\mathcal{U}$ of A, there is a finite subset $\mathcal{V}$ of $\mathcal{U}$ and arbitrarily small translations, one for each member of $\mathcal{V}$, such that the corresponding translates of $\mathcal{V}$ cover A.*

Next we turn to the Steinhaus theorem, which we will derive in Section 7 more directly as a corollary of the Category Embedding Theorem. For completeness we recall in the proof below its connection with the Weil topology introduced in [We].

Definition ([Hal-M] Section 62 p. 257 and 273).

1. A *measurable group* $(X, \mathcal{S}, m)$ is a σ-finite measure space with X a group and m a non-trivial measure such that both $\mathcal{S}$ and m is left-invariant and the mapping $x \to (x, xy)$ is measurability preserving.

2. A measurable group X is *separated* if for each $x \neq e_X$ in X, there is a measurable $E \subset X$ of finite, positive measure and $\varepsilon > 0$ such that $\mu(E \triangle xE) < \varepsilon$.

Steinhaus Theorem (cf. Comfort [Com] Th. 4.6 p. 1175). *Let X be a normed locally compact group which is separated under its Haar measure. For measurable A having positive finite Haar measure, the sets AA^{-1} and $A^{-1}A$ have non-empty interior.*

Proof. For X separated, we recall (see [Hal-M] Sect. 62 and [We]) that the Weil topology on X, under which X is a topological group, is generated by the neighbourhood base at e_X comprising sets of the form $N_{E,\varepsilon} := \{x \in X : \mu(E \triangle xE) < \varepsilon\}$, with $\epsilon > 0$ and E measurable and of finite positive measure. Recall from [Hal-M] Sect. 62 the following results: (Th. F) a measurable set with non-empty interior has positive measure; (Th. A) a set of positive measure contains a set of the form GG^{-1}, with G measurable and of finite, positive measure; and (Th. B) for such G, $N_{G\varepsilon} \subseteq GG^{-1}$ for all small enough $\varepsilon > 0$. Thus a measurable set has positive measure iff it is non-meagre in the Weil topology. Thus if A is measurable and has positive measure it is non-meagre in the Weil topology. Moreover, by [Hal-M] Sect 61, Sect. 62 Ths. A and B, the metric open sets of X are generated by sets of the form $N_{E,\varepsilon}$ for some Borelian-$(\mathcal{K})$ set E of positive, finite mesure. By the Piccard-Pettis Theorem (from the Category Embedding Theorem) AA^{-1} contains a non-empty Weil neighbourhood $N_{E,\varepsilon}$. $\square$

Remark. See Section 6 below (and also 7 for extensions to products AB) for an alternative proof via the density topology drawing on Mueller's Haar measure density theorem [Mue] and a category-measure theorem of Martin [Mar].

The Subgroup Dichotomy Theorem (Banach-Kuratowski Theorem) ([Ban-G] Satz 1, [Kur-1] Ch. VI. 13. XII; cf. [Kel] Ch. 6 Pblm P; cf. [BGT] Cor. 1.1.4 and also [BCS] and [Be] for the measure variant).

Let X be a normed group which is non-meagre and let A any Baire subgroup. Then A is either meagre or clopen in X.

Proof. Suppose that A is non-meagre. We show that e is an interior point of A, from which it follows that A is open. Suppose otherwise. Then there is a sequence $z_n \to e$ with $z_n \in B_{1/n}(e)\backslash A$. Now for some $a \in A$ and infinite M we have $z_n a \in A$ for all $n \in M$. But A is a subgroup, hence $z_n = z_n a a^{-1} \in A$ for $n \in M$, a contradiction.

Now suppose that A is not closed. Let a_n be a sequence in A with limit x. Then $a_n x^{-1} \to e$. Now for some $a \in A$ and infinite M we have $z_n x^{-1} a \in A$ for all $n \in M$. But A is a subgroup, so z_n^{-1} and a^{-1} are in A and hence, for all $n \in M$, we have $x^{-1} = z_n^{-1} z_n x^{-1} a a^{-1} \in A$. Hence $x \in A$, as A is a subgroup. $\square$

Remark. Banach's proof is purely topological so applies to topological groups (though originally stated for metric groups) and relies on the mapping $x \to ax$ being a homeomorphism, likewise Kuratowski's proof which proceeds via another dichotomy as detailed below.

Kuratowski Dichotomy ([Kur-B], [Kur-1], [McSh] Cor. 1)**.** *Suppose $H \subseteq Auth(X)$ acts transitively on X, and $Z \subseteq X$ is Baire and has the property that for each $h \in H$*

$$Z = h(Z) \text{ or } Z \cap h(Z) = \varnothing,$$

i.e. under each $h \in H$, either Z is invariant or Z and its image are disjoint. Then, either H is meagre or is clopen.

The result below generalizes the category version of the Steinhaus Theorem [St] of 1920, first stated explicitly by Piccard [Pic1] in 1939, and restated in [Pet1] in 1950; in the current form it may be regarded as a 'localized-refinement' of [RR-TG].

Generalized Piccard-Pettis Theorem ([Pic1], [Pic2], [Pet1], [Pet2], [BGT] Th. 1.1.1, [BOst3], [RR-TG], cf. [Kel] Ch. 6 Prb. P). *Let be X be a homogenous space. Suppose that ψ_u converges to the identity, as $u \to u_0$, and that A is Baire and non-meagre. Then, for some $\delta > 0$, we have*

$$A \cap \psi_u(A) \neq \varnothing, \text{ for all } u \text{ with } d(u, u_0) < \delta,$$

or, equivalently, for some $\delta > 0$

$$A \cap \psi_u^{-1}(A) \neq \varnothing, \text{ for all } u \text{ with } d(u, u_0) < \delta.$$

Proof. We may suppose that $A = V\backslash M$ with M meagre and V open. Hence, for any $v \in V\backslash M$, there is some $\varepsilon > 0$ with

$$B_\varepsilon(v) \subseteq U.$$

By definition of convergence, there is $\delta > 0$ such that, for u with $d(u, u_0) < \delta$, we have

$$d^*(\psi_u, id) < \varepsilon/2.$$

Hence, for any such u and any y in $B_{\varepsilon/2}(v)$, we have

$$d(\psi_u(y), y) < \varepsilon/2.$$

From this it follows that

$$W := \psi_u(B_{\varepsilon/2}(z_0)) \cap B_{\varepsilon/2}(z_0) \neq \varnothing,$$

and

$$W' := \psi_u^{-1}(B_{\varepsilon/2}(z_0)) \cap B_{\varepsilon/2}(z_0) \neq \varnothing.$$

For fixed u with $d(u, u_0) < \delta$, the set

$$M' := M \cup \psi_u(M) \cup \psi_u^{-1}(M)$$

is meagre. Let $w \in W\backslash M'$ (or $w \in W'\backslash M'$, as the case may be). Since $w \in B_\varepsilon(z_0)\backslash M \subseteq V\backslash M$, we have

$$w \in V\backslash M \subseteq A.$$

Similarly, $w \in \psi_u(B_\varepsilon(z_0))\backslash\psi_u(M) \subseteq \psi_u(V)\backslash\psi_u(M)$. Hence

$$\psi_u^{-1}(w) \in V\backslash M \subseteq A.$$

In this case, as asserted,

$$A \cap \psi_u^{-1}(A) \neq \varnothing.$$

In the other case ($w \in W'\backslash M'$), one obtains similarly

$$\psi_u(w) \in V\backslash M \subseteq A.$$

Here too

$$A \cap \psi_u^{-1}(A) \neq \varnothing.$$

□

Remarks.

1. In the theorem above it is possible to work with a weaker condition, namely local convergence at z_0, where one demands that for some neighbourhood $B_\eta(z_0)$ and some K

$$d(\psi_u(z),z) \leq Kd(u,u_0), \text{ for } z \in B_\eta(z_0).$$

This implies that, for any $\varepsilon > 0$, there is $\delta > 0$ such that, for $z \in B_\delta(z_0)$,

$$d(\psi_u(z),z) < \varepsilon, \text{ for } z \in B_\delta(z_0).$$

2. The Piccard-Pettis Theorem for topological groups (named by Kelley, [Kel] Ch. 6 Pblm P-(b), the Banach-Kuratowski-Pettis Theorem, say BKPT for short) asserts the category version of the Steinhaus Theorem [St] that, for A Baire and non-meagre, the set $A^{-1}A$ is a neighbourhood of the identity; our version of the Piccard theorem as stated implies this albeit only in the context of metric groups. Let d_X be a right-invariant metric on X and take $\psi_u(x) = ux$ and $u_0 = e$. Then ψ_u converges to the identity (see [BOst13] Section 3), and so the theorem implies that $B_\delta(e) \subseteq A^{-1} \cap A$ for some $\delta > 0$; indeed $a' \in A \cap \psi_u(A)$ for $u \in B_\delta(e)$ means that $a' \in A$ and, for some $a \in A$, also $ua = a'$ so that $u = a^{-1}a' \in A^{-1}A$. It is more correct to name the following important and immediate corollary the BKPT, since it appears in this formulation in [Ban-G], [Kur-1], derived by different means, and was used by Pettis in [Pet1] to deduce his Steinhaus-type theorem.

McShane's Interior Points Theorem ([McSh] Cor. 3). *Let $T : X^2 \to X$ such that $T_a(x) := T(x,a)$ is a self-homeomorphism for each $a \in X$ and such that for each pair (x_0, y_0) there is a homeomorphism $\varphi : X \to X$ with $y_0 = \varphi(x_0)$ satisfying*

$$T(x,\varphi(x)) = T(x_0,y_0), \text{ for all } x \in X.$$

Let A and B be second category with B Baire. Then the image $T(A,B)$ has interior points and there are $A_0 \subseteq A, B_0 \subseteq B$, with $A\backslash A_0$ and $B\backslash B_0$ meagre and $T(A_0,B_0)$ open.

5 The Kestelman-Borwein-Ditor Theorem: a bitopological approach

Definition (Genericity). Suppose Γ is $\mathcal{L}$ or $\mathcal{B}a$, the class of measurable sets or Baire sets (i.e. sets with the Baire property). We will say that $P \in \Gamma$ holds *for generically all* t if $\{t : t \notin P\}$ is null/meagre according as Γ is $\mathcal{L}$ or $\mathcal{B}a$.

In this section we develop a bi-topological approach to a generalization of the following result. An alternative approach is given in the next section.

Theorem (Kestelman-Borwein-Ditor Theorem). *Let* $\{z_n\} \to 0$ *be a null sequence of reals. If* T *is measurable and non-null/Baire and non-meagre, then for generically all* $t \in T$ *there is an infinite set* $\mathbb{M}_t$ *such that*

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

A stronger form still is derived in [BOst9] (the Generic Reflection Theorem); see also [BOst3] Section 2.1 Note 3, [BOst4] Section 2.1 Note 1. For proofs see the original papers [Kes] and [BoDi]; for a unified treatment in the real-variable case see [BOst9].

Let $(X, \mathcal{S}, m)$ be a probability space which is totally-finite. Let m^* denote the outer measure

$$m^*(E) := \inf\{m(F) : E \subset F \in \mathcal{S}\}.$$

Let the family $\{K_n(x) : x \in X\} \subset \mathcal{S}$ satisfy
(i) $x \in K_n(x)$,
(ii) $m(K_n(x)) \to 0$.

Relative to a fixed family $\{K_n(x) : x \in X\}$ define the upper and lower (outer) density at x of any set E by

$$\begin{aligned} \overline{D}^*(E, x) &= \sup \limsup_n m^*(E \cap K_n(x))/m(K_n(x)), \\ \underline{D}^*(E, x) &= \inf \liminf_n m^*(E \cap K_n(x))/m(K_n(x)). \end{aligned}$$

By definition $\overline{D}^*(E, x) \geq \underline{D}^*(E, x)$. When equality holds, one says that the density of E exists at x, and the common value is denoted by $D^*(E, x)$. If E

is measurable the star associated with the outer measure m^* is omitted. If the density is 1 at x, then x is a density point; if the density is 0 at x then x is a dispersion point of E.

We say that a *(weak) density theorem* holds for $\{K_n(x) : x \in X\}$ if for every set (every measurable set) A almost every point of A is an (outer) density point of A.

Martin [Mar] shows that the family

$$\mathcal{U} = \{U : \overline{D}^*(X\backslash U, x) = 0, \text{ for all } x \in U\}$$

forms a topology, the density topology on X, with the following property.

Density Topology Theorem. *If a density theorem holds for* $\{K_n(x) : x \in X\}$ *and* U *is d-open, then every point of* U *is a density point of* U *and so* U *is measurable. Furthermore, a measurable set such that each point is a density point is d-open.*

We note that the idea of a density topology was introduced slightly earlier by Goffman ([GoWa],[GNN]); see also Tall [T]. It can be traced to the work of Denjoy [Den] in 1915. Recall that a function is approximately continuous in the sense of Denjoy iff it is continuous under the density topology: [LMZ], p.1.

Category-Measure Theorem ([Mar] Th. 4.11). *Suppose* X *is a probability space and a density theorem holds for* $\{K_n(x) : x \in X\}$. *A necessary and sufficient condition that a set be nowhere dense in the d-topology is that it have measure zero. Hence a necessary and sufficient condition that a set be meagre is that it have measure zero. In particular the topological space* $(X, \mathcal{U})$ *is a Baire space.*

We now see that the preceeding theorem is applicable to a Haar measure on a locally compact group X by reference to the following result. Here bounded means pre-compact (covered by a compact set).

Haar measure density theorem ([Mue]; cf. [Hal-M] p. 268). *Let* A *be a* σ*-bounded subset and* μ *a left-invariant Haar measure of a locally compact group* X. *Then there exists a sequence* U_n *of bounded measurable neigbourhoods of* e_X *such that* $m^*(A \cap U_n x)/m^*(U_n x) \to 1$ *for almost all* x *out of a measurable cover of* A.

We now offer a generalization of a result from [BOst11].

Second Proposition on weak category convergence. *Let X be a normed locally compact group with left-invariant Haar measure m. Let V be m-measurable and non-null. For any null sequence $\{z_n\} \to e$ and each $k \in \omega$,*

$$H_k = \bigcap_{n \geq k} V \backslash (V \cdot z_n) \textit{ is of } m\textit{-measure zero, so meagre in the } d\textit{-topology.}$$

That is, the sequence $h_n(x) := xz_n^{-1}$ satisfies the weak category convergence condition (wcc)

Proof. Suppose otherwise. We write Vz_n for $V \cdot z_n$, etc. Now, for some k, $m(H_k) > 0$. Write H for H_k. Since $H \subseteq V$, we have, for $n \geq k$, that $\emptyset = H \cap h_n^{-1}(V) = H \cap (Vz_n)$ and so a fortiori $\emptyset = H \cap (Hz_n)$. Let u be a metric density point of H. Thus, for some bounded (Borel) neighbourhood $U_\nu u$ we have

$$\mu[H \cap U_\nu u] > \frac{3}{4} m[U_\nu u].$$

Fix ν and put

$$\delta = m[U_\nu u].$$

Let $E = H \cap U_\nu u$. For any z_n, we have $m[(Ez_n) \cap U_\nu u z_n] = m[E] > \frac{3}{4}\delta$. By Theorem A of [Hal-M] p. 266, for all large enough n, we have

$$m(U_\nu u \triangle U_\nu u z_n) < \delta/4.$$

Hence, for all n large enough we have $|(Ez_n) \backslash U_\nu u| \leq \delta/4$. Put $F = (Ez_n) \cap U_\nu u$; then $m[F] > \delta/2$.

But $\delta \geq m[E \cup F] = m[E] + m[F] - m[E \cap F] \geq \frac{3}{4}\delta + \frac{1}{2}\delta - m[E \cap F]$. So

$$m[H \cap (Hz_n)] \geq m[E \cap F] \geq \frac{1}{4}\delta,$$

contradicting $\emptyset = H \cap (Hz_n)$. This establishes the claim. □

As a corollary of the Category Embedding Theorem the Proposition now yields the following result.

Theorem (Generalized Kestelman-Borwein-Ditor Theorem 1). *Let X be a normed locally compact group. Let $\{z_n\} \to e_X$ be a null sequence* *in X. If T is Haar measurable and non-null, resp. Baire and non-meagre, then for generically all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{tz_m : m \in \mathbb{M}_t\} \subseteq T.$$

This theorem in turn yields an important conclusion.

Kodaira's Theorem ([Kod] corollary to Satz 18. p. 98, cf. [Com] Th. 4.17 p. 1182). *Let X be a normed locally compact group and $f : X \to Y$ a homorphism into a separable normed group Y. Then f is Haar-measurable iff f is Baire under the density topology iff f is continuous under the norm topology.*

Proof. Suppose that f is measurable. Then under the d-topology f is a Baire function. Hence by the classical Baire Continuity Theorem (see, e.g. Section 7 below), since Y is second-countable, f is continuous on some co-meagre set T. Now suppose that f is not continuous at e_X. Hence, for some $\varepsilon > 0$ and some $z_n \to z_0 = e_X$ (in the sense of the norm on X), we have $||f(z_n)|| > \varepsilon$, for all n. By the Kestelman-Borwein-Ditor Theorem, there is $t \in T$ and an infinite $\mathbb{M}_t$ such that $tz_n \to t = tz_0 \in T$. Hence, for n in $\mathbb{M}_t$, we have

$$f(t)f(z_n) = f(tz_n) \to f(tz_0) = f(t),$$

i.e. $f(z_n) \to e_Y$, a contradiction. □

Remarks.

1. Comfort [Com] Th. 4.17 proves this result for both X and Y locally compact, with the hypothesis that Y is σ-compact and f measurable with respect to two Haar measures on X and Y. That proof employs Steinhaus' Theorem and the Weil topology. (Under the density topology, Y will not be second-countable.) When Y is metrizable this implies that Y is separable; of course if f is a continuous surjection, Y will be locally-compact.

2. The theorem reduces measurability to the Baire property and in so doing resolves a long-standing issue in the foundations of regular variation; hitherto the theory was established on two alternative foundations employing either measurable functions, or Baire functions, for its scope, with historical preference for measurable functions in connection with integration. We refer to [BGT] for an exposition of the theory which characterizes regularly varying

functions of either type by a reduction to an underlying homomorphism of the corresponding type relying on its continuity and then represents either type by very well behaved functions. Kodaira's theorem shows that the broader topological class may be given priority. See in particular [BGT] p. 5,11 and [BOst11].

3. The Kestelman-Borwein-Ditor Theorem inspires the following definitions which we will find useful in the next section

Definitions. Call a set T *subuniversal* if for any null sequence $z_n \to e_G$ there is $t \in G$ and infinite $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subset T.$$

Call a set T *generically subuniversal* if for any null sequence $z_n \to e_G$ there is $t \in G$ and infinite $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subset T \text{ and } t \in T.$$

Thus the Kestelman-Borwein-Ditor Theorem asserts that a set T which is Baire non-meagre, or measurable non-null is (generically) subuniversal. The term subuniversal is coined from Kestelman's definition of set being 'universal for null sequences' ([Kes] Th. 2) which required $\mathbb{M}_t$ above to be co-finite rather than infinite. By the Shift-compactness Theorem, a generically subuniversal subset of a normed group is shift-compact (Section 4).

6 The Subgroup Theorem

In this section G is a normed locally compact group with left-invariant Haar measure. We shall be concerned with two topologies on G : the norm topology and the density topology. Under the latter the *binary* group operation need not be jointly continuous (see Heath and Poerio [HePo]); nevertheless a right-shift $x \to xa$, for a constant, is continuous, and so we may say that the density topology is *right-invariant*. We note that if S is measurable and non-null then S^{-1} is measurable and non-null under the corresponding right-invariant Haar and hence also under the original left-invariant measure. We may thus say that the density topology is *inversion-invariant*. This motivates

Theorem (Topological, or Category, Interior Point Theorem). *Let $\{z_n\} \to e$ be a null sequence (in the norm topology). Let G be given a* *right-invariant and inversion-invariant topology* τ*, under which it is a Baire space and the homeomorphisms* $h_n(x) = xz_n$ *satisfy* (wcc)*. For* S *Baire and non-meagre in* τ*, the difference set* $S^{-1}S$*, and likewise* SS^{-1}*, is an open neighbourhood of* e*.*

Proof. Suppose otherwise. Then for each positive integer n we may select

$$z_n \in B_{1/n}(e) \backslash (S^{-1}S).$$

Since $\{z_n\} \to e$ (in the norm topology), the Category Embedding Theorem applies, and gives an $s \in S$ and an infinite $\mathbb{M}_s$ such that

$$\{h_m(s) : m \in \mathbb{M}_s\} \subseteq S.$$

Then for any $m \in \mathbb{M}_s$,

$$sz_m \in S \text{ , i.e. } z_m \in S^{-1}S,$$

a contradiction. Replacing S by S^{-1} we obtain the corresponding result for SS^{-1}. □

Corollary (Piccard Theorem, Piccard [Pic1], [Pic2]). *For* S *Baire and non-meagre in the norm topology, the difference sets* SS^{-1} *and* $S^{-1}S$ *have* e *as interior point.*

First Proof. Apply the preceeding Theorem , since by the First Proposition on weak category convergence (Section 4), the (wcc) condition holds. □

Second Proof. Suppose otherwise. Then, as before, for each positive integer n we may select $z_n \in B_{1/n}(e) \backslash (S^{-1}S)$. Since $z_n \to e$, by the Kestelman-Borwein-Ditor Theorem, for quasi all $s \in S$ there is an infinite $\mathbb{M}_s$ such that $\{sz_m : m \in \mathbb{M}_s\} \subseteq S$. Then for any $m \in \mathbb{M}_s$, $sz_m \in S$, i.e. $z_m \in SS^{-1}$, a contradiction. □

Corollary (Steinhaus' Theorem, [St], [We]; cf. Comfort [Com] Th. 4.6 p. 1175, Beck et al. [BCS]). *For* S *of positive measure, the difference sets* $S^{-1}S$ *and* SS^{-1} *have* e *as interior point.*

Proof. Arguing as in the first proof above, by Second Proposition on weak category convergence (Section 5), the wcc holds and S, in the density topology, is Baire and non-meagre (by the Category-Measure Theorem of Section 5). The measure-theoretic form of the second proof above also applies. □

The following corollary to Steinhaus' Theorem (and its Baire category version) have important consequences in the Euclidean case. We will say that the group G is (weakly) *Archimedean* if for each $r > 0$ and each $g \in G$ there is $n = n(g)$ such that $g \in B^n$ where $B := \{x : ||x|| < r\}$ is the r-ball.

Theorem (Category [Measure] Subgroup Theorem). *For a Baire [measurable] subgroup S of a weakly Archimedean locally compact group G, the following are equivalent:*
(i) $S = G$,
(ii) *S is non-meagre [non-null].*

Proof. By the Topological/Category Interior Point Theorem, for some r-ball B,

$$B \subseteq SS^{-1} \subseteq S,$$

and hence $G = \bigcup_n B^n = S$. □

We will see in the next section a generalization of the Pettis extension of Piccard's result asserting that, for S, T Baire non-meagre, the product ST contains interior points. As our approach will continue to be bitopological, we will deduce also the Steinhaus result that, for S, T non-null and measurable, ST contains interior points.

7 The Semigroup Theorem

In this section G is a normed group which is locally compact. The aim here is to prove a generalization to the normed group setting of the following classical result due to Hille and Phillips [H-P] Th. 7.3.2 (cf. Beck et al. [BCS] Th. 2, [Be]) in the measurable case, and to Bingham and Goldie [BG] in the Baire case; see [BGT] Cor. 1.1.5.

Theorem (Category [Measure] Semigroup Theorem). *For an additive Baire [measurable] semigroup S of $\mathbb{R}$, the following are equivalent:*
(i) *S contains an interval,*
(ii) $S \supseteq (s, \infty)$, *for some s,*
(iii) *S is non-meagre [non-null].*

We will need a strengthening of the Kestelman-Borwein-Ditor Theorem of Section 5. First we capture a key similarity (their topological 'common basis', adapting a term from logic) between the Baire and measure cases. Recall ([Rog2] p. 460) the usage in logic, whereby a set B is a basis for a class $\mathcal{C}$ of sets whenever any member of $\mathcal{C}$ contains a point in B.

Theorem (Common Basis Theorem). *For V, W Baire non-meagre in G equipped with either the norm or the density topology, there is $a \in G$ such that $V \cap (aW)$ contains a non-empty open set modulo meagre sets common to both, up to translation. In fact, in both cases, up to translation, the two sets share a norm $\mathcal{G}_\delta$ subset which is non-meagre in the norm case and non-null in the density case.*

Proof. In the norm topology case if V, W are Baire non-meagre, we may suppose that $V = I \backslash M_0 \cup N_0$ and $W = J \backslash M_1 \cup N_1$, where I, J are open sets. Take $V_0 = I \backslash M_0$ and $W_0 = J \backslash M_1$. If v and w are points of V_0 and W_0, put $a := vw^{-1}$. Thus $v \in I \cap (aJ)$. So $I \cap (aJ)$ differs from $V \cap (aW)$ by a meagre set. Since $M_0 \cup N_0$ may be expanded to a meagre $\mathcal{F}_\sigma$ set M, we deduce that $I \backslash M$ and $J \backslash M$ are non-meagre $\mathcal{G}_\delta$-sets.

In the density topology case, if V, W are measurable non-null let V_0 and W_0 be the sets of density points of V and W. If v and w are points of V_0 and W_0, put $a := vw^{-1}$. Then $v \in T := V_0 \cap (aW_0)$ and so T is non-null and v is a density point of T. Hence if T_0 comprises the density points of T, then $T \backslash T_0$ is null, and so T_0 differs from $V \cap (aW)$ by a null set. Evidently T_0 contains a non-null closed, hence $\mathcal{G}_\delta$-subset (as T_0 is measurable non-null, by regularity of Lebesgue measure). $\square$

Theorem (Conjunction Theorem). *For V, W Baire non-meagre/measurable non-null, there is $a \in G$ such that $V \cap (aW)$ is Baire non-meagre/ measurable non-null and for any null sequence $z_n \to e_G$ and quasi all (almost all) $t \in V \cap (aW)$ there exists an infinite $\mathbb{M}_t$ such that*

$$\{tz_m : m \in \mathbb{M}_t\} \subset V \cap (aW).$$

Proof. In either case applying Theorem 9, for some a the set $T := V \cap (aW)$ is Baire non-meagre/measurable non-null. We may now apply the Kestelman-Borwein-Ditor Theorem to the set T. Thus for almost all $t \in T$ there is an infinite $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subset T \subset V \cap (aW). \qquad \square$$

This result motivates a further strengthening of generic subuniversality (compare Section 5).

Definitions. Let S be generically subuniversal.

1. Call T *similar* to S if for every null sequence $z_n \to e_G$ there is $t \in S \cap T$ and $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subset S \cap T.$$

Thus S is similar to T and both are generically subuniversal.

Call T *weakly similar* to S if if for every null sequence $z_n \to 0$ there is $s \in S$ and $\mathbb{M}_s$ such that

$$\{sz_m : m \in \mathbb{M}_s\} \subset T.$$

Thus again T is subuniversal.

2. Call S *subuniversally self-similar,* or just *self-similar* (up to inversion-translation), if for some $a \in G$ and some $T \subset S$, S is similar to aT^{-1}.

Call S *weakly self-similar* (up to inversion-translation) if for some $a \in G$ and some $T \subset S$, S is weakly similar to aT^{-1}.

Theorem (Self-similarity Theorem). *For S Baire non-meagre/measurable non-null, S is self-similar.*

Proof. Fix a null sequence $z_n \to 0$. If S is Baire non-meagre/measurable non-null then so is S^{-1}; thus we have for some a that $T := S \cap (aS^{-1})$ is likewise Baire non-meagre/measurable non-null and so for quasi all (almost all) $t \in T$ there is an infinite $\mathbb{M}_t$ such that

$$\{tz_m : m \in \mathbb{M}_t\} \subset T \subset S \cap (aS^{-1}),$$

as required. □

Theorem (Semigroup Theorem). *If S, T are generically subuniversal with T (weakly) similar to S, then ST^{-1} contains a ball about the identity e_G. Hence if S is generically subuniversal and (weakly) self-similar, then SS has interior points. Hence for $G = \mathbb{R}^d$, if additionally S is a semigroup, then S contains an open sector.*

Proof. For S, T (weakly) similar, we claim that ST^{-1} contains $B_\delta(e)$ for some $\delta > 0$. Suppose not: then for each positive n there is z_n with

$$z_n \in B_{1/n}(e) \backslash (ST^{-1}).$$

Now z_n^{-1} is null so there is s in S and infinite $\mathbb{M}_s$ such that

$$\{z_m^{-1}s : m \in \mathbb{M}_t\} \subset T.$$

For any m in $\mathbb{M}_t$ pick $t_m \in T$ so that $z_m^{-1}s = t_m$; then we have

$$z_m^{-1} = t_m s^{-1} \quad \text{so} \quad z_m = st_m^{-1},$$

a contradiction. Thus for some $\delta > 0$ we have $B_\delta(e) \subset ST^{-1}$.

For S self-similar, say S is similar to $T := aS^{-1}$, for some a, then $B_\delta(e)a \subset ST^{-1}a = S(aS^{-1})^{-1}a = SSa^{-1}a$, i.e. SS has non-empty interior. □

For applications see [BOst-RVWL]. By the Common Basis Theorem, replacing T by T^{-1}, we obtain as an immediate corollary of Theorem 12 a new proof of two classical results, extending the Steinhaus and Piccard Theorem and Kominek's Vector Sum Theorem.

Theorem (**Product Set Theorem**, Steinhaus [St] measure case, Pettis [Pet2] Baire case, cf. [Kom1] in the setting of topological vector spaces and [Be] and [BCS] in the group setting).

If S, T are Baire non-meagre/measurable non-null, then ST contains interior points.

8 Convexity

This section begins by developing natural conditions under which the Portmanteau theorem of convex functions (cf. [BOst6]) remains true when reformulated for a normed group setting, and then deduces generalizations of classical automatic continuity theorems for convex functions on a group.

Definitions.

1. A group G will be called 2-*divisible* (or quadratically closed) if the equation $x^2 = g$ for $g \in G$ always has a unique solution in the group to be denoted $g^{1/2}$. See [Lev] for a proof that any group may be embedded as a subgroup in an overgroup where the equations over G are soluble (compare also [Lyn1]).

2. In an arbitrary group, say that a subset C is $\frac{1}{2}$-*convex* if, for all x, y

$$x, y \in C \Longrightarrow \sqrt{xy} \in C,$$

where$\sqrt{xy}$ signifies some element z with $z^2 = xy$. We recall the following results.

Theorem (**Eberlein-McShane Theorem**, [Eb], [McSh]). *Let X be a 2-divisible topological group of second category. Then any $\frac{1}{2}$-convex Baire set has a non-empty interior. If X is abelian and each sequence defined by $x_{n+1}^2 = x_n$ converges to e_X then the interior of a $\frac{1}{2}$-convex set C is dense in C.*

Theorem (**Convex Minorant Theorem**, [McSh]). *Let X be 2-divisible abelian topological group. Let f and g be real-valued functions defined on a non-meagre subset C with f convex and g Baire such that*

$$f(x) \leq g(x), \text{ for } x \in C.$$

Then f is continuous on the interior of C.

Definition. We say that the function $h : G \to R$ is $\frac{1}{2}$-*convex* on the $\frac{1}{2}$–convex set C if, for $x, y \in C$,

$$h(\sqrt{xy}) \leq \frac{1}{2}\left(h(x) + h(y)\right),$$

with$\sqrt{xy}$ as above.

Example. For $G = R_+^*$ the function $h(x) = x$ is convex on G, since

$$2xy \leq x^2 + y^2.$$

Averaging Lemma. *A non-meagre set T is 'averaging', that is, for any given point $u \in T$ and for any sequence $\{u_n\} \to u$, there are $v \in G$ (a right-averaging translator) and $\{v_n\} \subseteq T$ such that, for infinitely many $n \in \omega$, we have*

$$u_n^2 = v_n v.$$

There is likewise a left-averaging translator such that for some $\{w_n\} \subseteq T$ such that, for infinitely many $n \in \omega$, we have

$$u_n^2 = w w_n.$$

Proof. Define null sequences by

$$z_n = u_n u^{-1}, \text{ and } \tilde{z}_n = u^{-1} u_n.$$

We are to solve

$$\begin{aligned} u_n^2 v^{-1} &= v_n \in T, \\ u\tilde{z}_n z_n u v^{-1} &= v_n \in T, \\ \tilde{z}_n z_n u v^{-1} &= u^{-1} v_n \in T' = u^{-1} T. \end{aligned}$$

Now put $\psi_n(x) := \tilde{z}_n z_n x$; then

$$d(x, \tilde{z}_n z_n x) = d(e, \tilde{z}_n z_n) = ||\tilde{z}_n z_n|| \leq ||\tilde{z}_n|| + ||z_n|| \to 0.$$

By the Category Embedding Theorem, for some $\tau \in T' = u^{-1}T$, we have with $\tau = u^{-1}t$ and for infinitely many n

$$\begin{aligned} u^{-1} v_n &: = \tilde{z}_n z_n \tau \in T' = u^{-1} T, \\ u\tilde{z}_n z_n \tau &= v_n \in T, \\ u\tilde{z}_n z_n u u^{-1} \tau &= v_n \in T, \\ u_n^2 u^{-1} \tau &= v_n \in T, \\ u_n^2 &= v_n \tau^{-1} u = v_n v \end{aligned}$$

(with $v = \tau^{-1} u = t^{-1} u^2 \in T^{-1} u^2$).

As for the remaining assertion, note that $u_n^{-1} \to u^{-1}, v_n^{-1} \in T^{-1}$ and

$$u_n^{-2} = v^{-1} v_n^{-1}.$$

Thus noting that T^{-1} is non-meagre (since inversion is a homeomorphism) and replacing T^{-1} by T we obtain the required assertion by a right-averaging translator. □

Note the connection between the norms of the null sequences is only by way of the conjugate metrics:

$$||z_n|| = d(e, u_n u^{-1}) = d(u, u_n), \text{ and } ||\tilde{z}_n|| = d(e, u^{-1} u_n) = d(u_n^{-1}, u^{-1}) = \tilde{d}(u_n, u).$$

Whilst we may make no comparisons between them, both norms nevertheless converge to zero.

Definition. We say that $f : G \to H$ is *locally Lipschitz* at g if, for some neighbourhood N_g of g and for some constants K_g and all x, y in N_g,

$$\left|\left| f(x) f(y)^{-1} \right|\right|_H \leq K_g ||xy^{-1}||_G.$$

We say that $f : G \to H$ is *locally bi-Lipschitz* at g if, for some neighbourhood N_g of g and for some positive constants K_g, κ_g, and all x, y in N_g,

$$\kappa_g ||xy^{-1}||_G \leq \left|\left|f(x)f(y)^{-1}\right|\right|_H \leq K_g ||xy^{-1}||_G.$$

If $f : G \to H$ is invertible, this asserts that both f and its inverse f^{-1} are locally Lipschitz at g and $f(g)$ respectively.

We say that the *norm* on G is *n-Lipschitz* if the function $f_n(x) := x^n$ from G to G is *locally Lipschitz* at all $g \neq e$, i.e. for each there is a neighbourhood N_g of g and positive constants κ_g, K_g so that

$$\kappa_g ||xy^{-1}||_G \leq \left|\left|x^n y^{-n}\right|\right|_G \leq K_g ||xy^{-1}||_G.$$

In an abelian context the power function is a homomorphism; we note that [HJ] p. 381 refers to a semigroup being *modular* when each f_n (defined as above) is an injective homomorphism. The condition on the right with $K = n$ is automatic, and so one need require only that for some positive constant κ

$$\kappa ||g|| \leq ||g^n||.$$

Note that if $x^n = y^n$ then $(xy^{-1})^n = e$ and so $||xy^{-1}|| = 0$, i.e. the power function is injective. If the group is divisible then clearly the power function is an isomorphism.

We note that in the additive group of reals x^2 fails to be locally bi-Lipschitz at the origin (since its derivative there is zero): see [Bart]. However, the following are bi-Lipschitz.

Examples.

1. In $\mathbb{R}^d$ with additive notation, we have $||x^2|| := ||2x|| = 2||x||$, so the norm is 2-Lipschitz.

2. In $\mathbb{R}^*_+$ we have $||x^2|| := |\log x^2| = 2|\log x| = 2||x||$ and again the norm is 2-Lipschitz.

3. In a Klee group the mapping $f(x) := x^n$ is uniformly (locally) Lipschitz, since

$$\left|\left|x^n y^{-n}\right|\right|_G \leq n||xy^{-1}||_G.$$

Reflecting Lemma. *Suppose the norm is everywhere locally* 2*-Lipschitz. Then, for T non-meagre, T is reflecting i.e. there are $w \in G$ (a right-reflecting translator) and $\{v_n\} \subseteq T$ such that, for infinitely many $n \in \omega$, we have*

$$v_n^2 = u_n w.$$

There is likewise a left-reflecting translator.

Proof. Let $T^2 := \{g : g = t^2 \text{ for some } t \in T\}$. By assumption, T^2 is non-meagre. With $u_n = uz_n$, put $S = T^2$ and notice that $u_n w \in S$ iff $uz_n w \in S$ iff $z_n w \in u^{-1}S$. Now $u^{-1}S$ is non-meagre and $\psi_n(x) := z_n x$ as usual converges to the identity, so the existence of $w \in u^{-1}S$ is assured such that $z_n w = u^{-1}v_n^2$. □

Remarks. 1. Note that the assertion here is

$$u_n^{-1}v_n = wv_n^{-1},$$

so that

$$d(v_n, w) = d(v_n^{-1}, u_n^{-1}) = \tilde{d}(v_n, u_n) \approx \tilde{d}(v_n, u),$$

or

$$d(v_n, w) \approx \tilde{d}(v_n, u),$$

suggesting the terminology of reflection.

2. Boundedness theorems for reflecting and averaging sets follow as in [BOst6] since the following are true in any group, as we see below.

Theorem. *For f a $\frac{1}{2}$-convex function, if f is locally bounded above at x_0 then it is locally bounded below at x_0 (and hence locally bounded at x_0).*

Proof. Say f is bounded above in $B := B_\delta(x_0)$ by M. Consider $u \in \tilde{B}_\delta(x_0)$. Thus $\tilde{d}(x_0, u) = ||u^{-1}x_0|| < \delta$. Put $t = u^{-1}x_0^2$; then $tx_0^{-1} = u^{-1}x_0$, and so

$$d(t, x_0) = ||tx_0^{-1}|| = ||u^{-1}x_0|| = \tilde{d}(u, x_0) < \delta.$$

Then $t \in B$, and since $x_0^2 = ut$ we have

$$2f(x_0) \leq f(u) + f(t) \leq f(u) + M,$$

or

$$f(u) \geq 2f(x_0) - M.$$

Thus $2f(x_0) - M$ is a lower bound for f on the open set $\tilde{B}_\delta(x_0)$. □

As a corollary a suitably rephrased Bernstein-Doetsch Theorem ([Kucz], [BOst6]) is thus true.

Bernstein-Doetsch Theorem. *For f a $\frac{1}{2}$-convex function, if f is locally bounded above at x_0, then f is continuous at at x_0.*

Proof. We repeat the 'Second proof' of [Kucz] p. 145. Choose $y_n \to x_0$ with $f(y_n) \to m_f(x_0)$ and $z_n \to x_0$ with $f(z_n) \to M_f(x_0)$. Let $u_n := y_n^2 x_n^{-1}$. Thus $y_n^2 = u_n x_n$ and so

$$2f(y_n) \leq f(u_n) + f(z_n),$$

i.e. $f(u_n) \geq 2f(y_n) - f(z_n)$. Hence in the limit we obtain

$$M_f(x_0) \geq \liminf f(u_n) \geq 2M_f(x_0) - m_f(x_0).$$

One thus has that $M_f(x_0) \leq m_f(x_0)$. But $m_f(x_0) \leq f(x_0) \leq M_f(x_0)$, and both hull values are finite (by the preceeding theorem). Thus $m_f(x_0) = f(x_0) = M_f(x_0)$, from which continuity follows. □

We now consider the transferability of upper and lower local boundedness. Our proofs work directly with definitions (so are not modelled after those in Kuczma [Kucz]). We do not however consider domains other than the whole metric group. For clarity of proof structure we give separate proofs for the two cases, first when G is abelian and later for general G.

Theorem (Local upper boundedness). *For f a $\frac{1}{2}$-convex function defined on an abelian group G, if f is locally bounded above at some point x_0, then f is locally bounded above at all points.*

Proof. *Case* (i) *The Abelian case.* Say f is bounded above in $B := B_\delta(x_0)$ by M. Given a fixed point t, put $z = z_t := x_0^{-1}t^2$, so that $t^2 = x_0 z$. Consider any $u \in B_{\delta/2}(t)$. Write $u = st$ with $||s|| < \delta/2$. Now put $y = s^2$; then $||y|| = ||s^2|| \leq 2||s|| < \delta$. Hence $yx_0 \in B_\delta(x_0)$. Now

$$u^2 = (st)^2 = s^2 t^2 = yx_0 z,$$

as the group is abelian. So

$$f(u) \leq \frac{1}{2} f(yx_0) + \frac{1}{2} f(z) \leq \frac{1}{2} M + \frac{1}{2} f(z_t).$$

That is, $\frac{1}{2}(M + f(z_t))$ is an upper bound for f in $B_{\delta/2}(x_0)$.

Case (ii) *The general case.* Now we consider the general case. As before, suppose f is bounded above in $B := B_\delta(x_0)$ by M, and let t be a given a fixed point; put $z = z_t := x_0^{-1}t^2$ so that $t^2 = x_0 z$.

For this fixed t the mapping $y \to \alpha(y) := ytyt^{-1}y^{-2}$ is continuous with $\alpha(e) = e$, so $\alpha(y)$ is $o(y)$ as $||y|| \to 0$. Now

$$sts = [stst^{-1}s^{-2}]s^2t = \alpha(s)s^2t,$$

and we may suppose that, for some $\eta < \delta/2$, we have $||\alpha(s)|| < \delta/2$, for $||s|| < \eta$. Note that

$$stst = \alpha(s)s^2t^2.$$

Consider any $u \in B_r(t)$ with $r = \min\{\eta, \delta/2\}$. Write $u = st$ with $||s|| < r \leq \delta/2$. Now put $y = s^2$. Then $||y|| = ||s^2|| \leq 2||s|| < \delta$ and $||o(s)y|| \leq \eta + \delta/2 < \delta$. Hence $o(s)yx_0 \in B_\delta(x_0)$. Now

$$u^2 = stst = \alpha(s)s^2t^2 = \alpha(s)yx_0z.$$

Hence, by convexity,

$$f(u) \leq \frac{1}{2}f(o(s)yx_0) + \frac{1}{2}f(z) \leq \frac{1}{2}M + \frac{1}{2}f(z_t). \qquad \square$$

As an immediate corollary of the last theorem and the Bernstein-Doetsch Theorem we have the following result.

Dichotomy Theorem for convex functions ([Kucz] p. 147). *For* $\frac{1}{2}$-*convex* f *(so in particular for additive* f*) either* f *is continuous everywhere, or it is discontinuous everywhere.*

The definition below requires continuity of ‘square-rooting’ – taken in the form of an algebraic closure property of degree 2 in a group G, expressed as the solvability of certain ‘quadratic equations’ over the group. Its status is clarified later by reference to Bartle’s Inverse Function Theorem. We recall that a group is n-divisible if $x^n g = e$ is soluble for each $g \in G$. (In the absence of algebraic closure of any degree an extension of G may be constructed in which these equations are solvable – see for instance Levin [Lev].)

Definition. We say that the normed group G is *locally convex* at $\tau = t^2$ if, for any $\varepsilon > 0$, there is $\delta > 0$ such that for all g with $||g|| < \varepsilon$, the equation

$$xtxt = gt^2,$$

equivalently $xtxt^{-1} = g$, has its solutions satisfying $||x|| < \delta$.

Thus G is locally convex at e if, for any $\varepsilon > 0$, there is $\delta > 0$ such that for all g with $||g|| < \varepsilon$, the equation

$$x^2 = g$$

has its solutions with $||x|| < \delta$.

Remark. Putting $u = xt$ the local convexity equation reduces to $u^2 = gt^2$, asserting the local existence of square roots (local 2-divisibility). If G is abelian the condition at t reduces to the condition at e.

Theorem (Local lower boundedness: abelian case). *Let G be a locally convex abelian group with a 2-Lipschitz norm, i.e. $g \to g^2$ is a bi-Lipschitz isomorphism such that, for some $\kappa > 0$,*

$$\kappa||g|| \leq ||g^2|| \leq 2||g||.$$

For f a $\frac{1}{2}$-convex function, if f is locally bounded below at some point, then f is locally bounded below at all points.

Proof. *Case* (i) *The Abelian case.* We change the roles of t and x_0 in the preceeding abelian theorem, treating t as a reference point, albeit now for lower boundedness, and x_0 as some arbitrary other fixed point. Suppose that f is bounded below by L on $B_\delta(t)$. Let $yx_0 \in B_{\kappa\delta}(x_0)$, so that $0 < ||y|| < \kappa\delta$. Choose s such that $s^2 = y$. Then,

$$\kappa||s|| \leq ||y|| < \kappa\delta,$$

so $||s|| < \delta$. Thus $u = st \in B_\delta(t)$. Now the identity $u^2 = s^2t^2 = yx_0z$ implies that

$$\begin{aligned} L &\leq f(u) \leq \frac{1}{2}f(yx_0) + \frac{1}{2}f(z_t), \\ 2L - f(z_t) &\leq f(yx_0), \end{aligned}$$

i.e. that $2L - f(z_t)$ is a lower bound for f on $B_{\kappa\delta}(x_0)$.

Case (ii) *The general case.* Now we consider the general case. Suppose as before that f is bounded below by L on $B_\delta(t)$. Since the map $\alpha(\sigma) := \sigma t\sigma t^{-1}\sigma^{-2}$ is continuous and $\alpha(e) = e$, we may choose η such that $||\alpha(\sigma)|| < \kappa\delta/2$, for $||\sigma|| < \eta$. Now choose $\varepsilon > 0$ such that, for each y with $||y|| < \varepsilon$, the solution $u = \sigma t$ to

$$u^2 = yt^2$$

has $||\sigma|| < \eta$. Let $r = \min\{\kappa\delta/2, \varepsilon\}$.

Let $yx_0 \in B_r(x_0)$; then $0 < ||y|| < \kappa\delta/2$ and $||y|| < \varepsilon$. As before put $z = z_t := x_0^{-1}t^2$ so that $t^2 = x_0 z$. Consider $u = \sigma t$ such that $u^2 = yx_0 z$; thus we have

$$u^2 = \sigma t\sigma t = yx_0 z = yx_0 x_0^{-1}t^2 = yt^2.$$

Hence $||\sigma|| < \eta$ (as $||y|| < \varepsilon$). Now we write

$$u^2 = \sigma t\sigma t = [\sigma t\sigma t^{-1}\sigma^{-2}]\sigma^2 t^2 = \alpha(\sigma)\sigma^2 t^2 = yt^2.$$

We compute that

$$y = \alpha(\sigma)\sigma^2$$

and

$$\kappa\delta/2 \geq ||y|| = ||\alpha(\sigma)\sigma^2|| \geq ||\sigma^2|| - ||\alpha(\sigma)|| \geq \kappa||\sigma|| - ||\alpha(\sigma)||,$$

so

$$||\sigma|| \leq \delta/2 + ||\alpha(\sigma)||/\kappa < \delta/2 + \delta/2 < \delta.$$

Thus $u \in B_\delta(t)$. Now the identity $u^2 = yx_0 z$ together with convexity implies as usual that

$$\begin{aligned} L &\leq f(u) \leq \frac{1}{2}f(yx_0) + \frac{1}{2}f(z_t), \\ 2L - f(z_t) &\leq f(yx_0), \end{aligned}$$

i.e. $2L - f(z_t)$ is a lower bound for f on $B_{\kappa\delta}(x_0)$. □

The local 2-divisibility assumption at t^2 asserts that $f_t(\sigma) := \sigma t\sigma t^{-1}$ is invertible locally at e. Bartle's theorem below guarantees that f_t has uniform local inverse under a smoothness assumption, i.e. that for $||\sigma|| = ||f_t^{-1}(y)|| < \delta$, for all small enough y, say for $||y|| < \kappa\delta$. To state the theorem we need some definitions.

Definitions.

1. f is said to have a *derivative* at x_0 if there is a continuous homomorphism $f'(x_0)$ such that

$$\lim_{||u||\to 0+} \frac{1}{||u||} ||f(ux_0)f(x_0)^{-1}[f'(x_0)(u)]^{-1}|| = 0.$$

2. f is *of class C'* on the open set U if it has a derivative at each point u in U and, for each x_0 and each $\varepsilon > 0$, there is $\delta > 0$ such that, for all x_1, x_2 in $B_\delta(x_0)$ both

$$||f'(x_1)(u)[f'(x_2)(u)]^{-1}|| < \varepsilon ||u||$$

and

$$||f(x_1)f(x_2)^{-1}f'(x_0)(x_1x_2^{-1})^{-1}|| < \varepsilon ||x_1x_2^{-1}||.$$

The two conditions may be rephrased relative to the right-invariant metric d on the group as

$$d(f'(x_1)(u), f'(x_2)(u)) < \varepsilon ||u||,$$

and

$$d(f(x_1)f(x_2)^{-1}, f'(x_0)(x_1x_2^{-1}) < \varepsilon d(x_1, x_2).$$

3. Suppose that $y_0 = f(x_0)$. Then f is *smooth* at x_0 if there are positive numbers α, β such that if $0 < d(y, y_0) < \beta$ then there is x such that $y = f(x)$ and $d(x, x_0) \leq \alpha \cdot d(y, y_0)$. If f is invertible, then this asserts that

$$d(f^{-1}(y), f^{-1}(y_0)) \leq \alpha \cdot d(y, y_0).$$

Example. Let $f(x) = tx$ with t fixed. Here f is smooth at x_0 if there are positive numbers α, β such that

$$||xx_0^{-1}|| \leq \alpha ||tx(tx_0)^{-1}|| = \alpha ||txx_0^{-1}t^{-1}||.$$

Note that in a Klee group $||txx_0^{-1}t^{-1}|| = ||t^{-1}txx_0^{-1}|| = ||xx_0^{-1}||$.

Theorem (**Bartle's Inverse Function Theorem**, [Bart] Th. 2.4). *Suppose that*

(i) *f is of class C' in the ball $B_r(x_0) = \{x \in G : ||xx_0^{-1}|| < r\}$, for some $r > 0$, and*

(ii) *$f'(x_0)$ is smooth (at e and so anywhere).*

Then f is smooth at x_0 and hence open.

If also the derivative $f'(x_0)$ is an isomorphism, then f has a uniformly continuous local inverse at x_0.

Corollary. *If $f_t(\sigma) := \sigma t \sigma t^{-1}$ is of class C' on $B_r(e)$ and $f_t'(e)$ is smooth, then G is locally convex at t.*

Proof. Immediate since $f_t(e) = e$. □

We are now in a position to state generalizations of two results derived in the real line case in [BOst6].

Proposition. *Let G be any locally convex group with a 2-Lipschitz norm. If f is $\frac{1}{2}$-convex and bounded below on a reflecting subset S of G, then f is locally bounded below on G.*

Proof. Suppose not. Let T be a reflecting subset of S. Let K be a lower bound on T. If f is not locally bounded from below, then at any point u in $\bar{T}$ there is a sequence $\{u_n\} \to u$ with $\{f(u_n)\} \to -\infty$. For some $w \in G$, we have $v_n^2 = wu_n \in T$, for infinitely many n. Then

$$K \leq f(v_n) \leq \frac{1}{2}f(w) + \frac{1}{2}f(u_n), \text{ or } 2K - f(w) \leq f(u_n),$$

i.e. $f(u_n)$ is bounded from below, a contradiction. □

Theorem (Generalized Mehdi Theorem cf. [Meh] Th. 3**).** *A $\frac{1}{2}$-convex function $f : G \to \mathbb{R}$ on a normed group, bounded above on an averaging subset S, is continuous on G.*

Proof. Let T be an averaging core of S. Suppose that f is not continuous, but is bounded above on T by K. Then f is not locally bounded above at some point of $u \in \bar{T}$. Then there is a null sequence $z_n \to e$ with $f(u_n) \to \infty$, where $u_n = uz_n$. Select $\{v_n\}$ and w in G so that, for infinitely many n, we have

$$u_n^2 = wv_n.$$

But for such n,we have

$$f(u_n) \leq \frac{1}{2}f(w) + \frac{1}{2}f(v_n) \leq \frac{1}{2}f(w) + \frac{1}{2}K,$$

contradicting the unboundedness of $f(u_n)$. □

The Generalized Mehdi Theorem, together with the Averaging Lemma, implies the classical result below and its generalizations.

Theorem (**Császár-Ostrowski Theorem** [Csa], [Kucz] p. 210). *A convex function $f : \mathbb{R} \to \mathbb{R}$ bounded above on a set of positive measure/non-meagre set is continuous.*

Topological Császár-Ostrowski Theorem. *A $\frac{1}{2}$-convex function $f : G \to \mathbb{R}$ on a normed group, bounded above on a non-meagre subset, is continuous.*

Reference to the Generalized Borwein-Ditor Theorem yields the following.

Haar-measure Császár-Ostrowski Theorem. *A $\frac{1}{2}$-convex function $f : G \to \mathbb{R}$ on a normed group carrying a Radon measure, bounded above on a set of positive measure, is continuous.*

9 Automatic continuity: the Jones-Kominek Theorem

This section is dedicated to generalizations to normed groups and to a more general class of topological groups of the following result for the real line.

Theorems of Jones and Kominek. *Let f be additive on $\mathbb{R}$ and either have a continuous restriction, or a bounded restriction, $f|T$, where T is some analytic set spanning $\mathbb{R}$. Then f is continuous.*

The result follows from the Expansion Lemma and Darboux's Theorem (see below) that an additive function bounded on an interval is continuous. In fact the bounded case above (*Kominek's Theorem,* [Kom2]) implies the continuous case (*Jones's Theorem,* [Jones1], [Jones2]), as was shown in [BOst7]. [OC] develops limit theorems for sequences of functionals whose properties are given on various kinds of spanning sets including spanning in the sense of linear rational combinations.

Before stating the current generalizations we begin with some preliminaries on analytic subsets of a topological group.

We recall ([Jay-Rog], p. 11, or [Kech] Ch. III for the Polish space setting) that in a Hausdorff space X a *K-analytic set* is a set A that is the image under a compact-valued, upper semi-continuous map from $\mathbb{N}^{\mathbb{N}}$; if this mapping takes values that are singletons or empty, the set A is said to be *analytic*. In either case A is *Lindelöf.* (The topological notion of K-analyticity was introduced by Choquet, Frolik, Sion and Rogers under variant definitions, eventually found to be equivalent, as a consequence of a theorem of Jayne, see [Jay-Rog] Sect. 2.8 p. 37 for a discussion.) If the space X is a topological group, then the subgroup $\langle A\rangle$ spanned (generated) by an analytic subset A is also analytic and so Lindelöf (for which, see below); note the result due to Loy [Loy] and Christensen [Ch] that an analytic Baire group is Polish (cf. [HJ] Th. 2.3.6 p. 355). Note that a Lindelöf group need not be metric; see for example the construction due to Oleg Pavlov [Pav]. If additionally the group X is metric, then $\langle A\rangle$ is separable, and so in fact this K-analytic set is analytic (a continuous image of $\mathbb{N}^{\mathbb{N}}$ – see [Jay-Rog] Th. 5.5.1 (b), p. 110).

Definition. We say that a set S is *Souslin-$\mathcal{H}$* if it is of the form

$$S=\bigcup_{\alpha\in\omega^{\omega}}\cap_{n=1}^{\infty}H(\alpha|n),$$

with each $H(\alpha|n)\in\mathcal{H}$. We will often take $\mathcal{H}$ to be $\mathcal{F}(X)$, the family of closed subsets of the space X.

We recall that a set is *meagre* if it is a countable union of nowhere dense sets, a set is *Baire* if it is open modulo a meagre set, or equivalently if it is closed modulo a meagre set (cf. Engelking [Eng] especially p.198 Section 3.9 and Exercises 3.9.J, although we prefer ‘meagre’ to ‘of first category’).

Definition. Let G be any group. For any positive integer n and for any subset S let $S^{(n)}$ denote the set of S-words of length n. Say that a subset H of G *spans* G *(*in the sense of group theory), or *finitely generates the goup* G, if for any $g\in G$, there are $h_1,...,h_n$ in H such that

$$g=h_1^{\varepsilon_1}\cdot...\cdot h_n^{\varepsilon_n},\text{ with }\varepsilon_i=\pm1.$$

(If H is *symmetric*, so that $h^{-1}\in H$ iff $h\in H$, there is no need for inverses.)

We begin with results concerning K-analytic groups.

Proposition. *The span of a K-analytic set is K-analytic; likewise for analytic sets.*

Proof. Since $f(v,w) = vw$ is continuous, $S^{(2)} = f(S \times S)$ is K-analytic by [Jay-Rog] Th 2.5.1 p. 23. Similarly all the sets $S^{(n)}$ are K-analytic. Hence the span, namely $\bigcup_{n\in\mathbb{N}} S^{(n)}$ is such ([Jay-Rog], Th. 2.5.4 p. 23). $\square$

Intersection Theorem ([Jay-Rog] Th 2.5.3, p. 23). *The intersection of a* K*-analytic set with a Souslin-*$\mathcal{F}(X)$ *in a Hausdorff space* X *is* K*-analytic.*

Projection Theorem ([RW] and [Jay-Rog] Th 2.6.6, p. 30). *Let* X *and* Y *be topological spaces with* Y *a* K*-analytic set. Then the projection on* X *of a Souslin-*$\mathcal{F}(X \times Y)$ *is Souslin-*$\mathcal{F}(X)$*.*

Nikodym's Theorem ([Nik]; [Jay-Rog] p. 42) *The Baire sets of a space* X *are closed under the Souslin operation. Hence Souslin-*$\mathcal{F}(X)$ *sets are Baire.*

Definitions.

1. Say that a function $f : X \to Y$ between two topological spaces is $\mathcal{H}$*-Baire*, for $\mathcal{H}$ a class of sets in Y, if $f^{-1}(H)$ has the Baire property for each set H in $\mathcal{H}$. Thus f is $\mathcal{F}(Y)$-Baire if $f^{-1}(F)$ is Baire for all closed F in Y. Since

$$f^{-1}(Y \backslash H) = X \backslash f^{-1}(H),$$

f is $\mathcal{F}(Y)$-Baire iff it is $\mathcal{G}(Y)$-Baire, when we will simply say that f is *Baire* ('f has the Baire property' is the alternative usage).

2. One must distinguish between functions that are $\mathcal{F}(Y)$-Baire and those that lie in the smallest family of functions closed under pointwise limits of sequences and containing the continuous functions (for a modern treatment see [Jay-Rog] Sect. 6). We follow tradition in calling these last *Baire-measurable.*

3. We will say that a function is *Baire-continuous* if it is continuous when restricted to some co-meagre set. In the real line case and with the density topology, this is Denjoy's approximate continuity ([LMZ], p.1); recall ([Kech], 17.47) that a set is (Lebesgue) measurable iff it has the Baire property under the density topology.

The connections between these concepts are given in the theorems below. See the cited papers for proofs.

Banach-Neeb Theorem ([Ban-T] Th. 4 pg. 35, and Vol I p. 206; [Ne]).

(i) *A Baire-measurable* $f : X \to Y$ *with* X *a Baire space and* Y *metric is Baire-continuous; and*

(ii) *a Borel-measurable* $f : X \to Y$ *with* X, Y *metric and* Y *separable is Baire-measurable.*

Remarks. In fact Banach shows that a Baire-measurable function is Baire-continuous on each perfect set ([Ban-T] Vol. II p. 206). Neeb assumes in addition that Y is arcwise connected, but as Pestov [Pes] remarks the arcwise connectedness may be dropped by referring to a result of Hartman and Mycielski [HM] that a separable metrizable group embeds as a subgroup of an arcwise connected separable metrizable group.

Baire Continuity Theorem. *A Baire function* $f : X \to Y$ *is Baire continuous in the following cases:*

(i) Baire condition (see e.g. [HJ] Th. 2.2.10 p. 346): Y *is a second-countable space*

(ii) Emeryk-Frankiewicz-Kulpa ([EFK]): X *is Čech-complete and* Y *has a base of cardinality not exceeding the continuum;*

(iii) Pol condition ([Pol]): f *is Borel,* X *is Borelian-K and* Y *is metrizable and of nonmeasurable cardinality;*

(iv) Hansell condition ([Han]): f *is* σ*-discrete and* Y *is metric.*

We will say that the pair (X, Y) *enables Baire continuity* if the spaces X, Y satisfy either of the two conditions (i) or (ii). In the applications below Y is usually the additive group of reals $\mathbb{R}$, so satisfies (i). Building on [EFK], Fremlin ([Frem] Section 9), characterizes a space X such that every Baire function $f : X \to Y$ is Baire-continuous for all metric Y in the language of 'measurable spaces with negligibles'; reference there is made to disjoint families of negligible sets all of whose subfamilies have a measurable union. For a discussion of discontinuous homomorphisms, especially counterexamples on $C(X)$ with X compact (e.g. employing Stone-Čech compactifications, $X = \beta\mathbb{N}\backslash\mathbb{N}$), see [Dal] Section 9.

Remarks. Hansell's condition, requiring the function f to be σ-discrete, is implied by f being analytic when X is absolutely analytic (i.e. Souslin-$\mathcal{F}(X)$ in any complete metric space X into which it embeds). Frankiewicz [Fr] considers implications of the Axiom of Constructibility.

The following result provides a criterion for verifying that f is Baire.

Theorem (Souslin criterion). *Let X and Y be Hausdorff topological groups with Y a K-analytic set. If $f : X \to Y$ has Souslin-$\mathcal{F}(X \times Y)$ graph, then f is Baire.*

Proof. Let $G \subseteq X \times Y$ be the graph of f which is Souslin-$\mathcal{F}(X \times Y)$. For F closed in Y, we have

$$f^{-1}(F) = \mathrm{pr}_X[G \cap (X \times F)],$$

which, by the Intersection Theorem, is the projection of a *Souslin-*$\mathcal{F}(X \times Y)$ set. By the Projection Theorem, $f^{-1}(F)$ is Souslin-$\mathcal{F}(X)$. Closed sets have the Baire property by definition, so by Nikodym's Theorem $f^{-1}(F)$ has the Baire property. □

Before stating our next theorem we recall a classical result in the sharper form resulting from the enabling condition (ii) above.

Banach-Mehdi Theorem ([Ban-T] 1.3.4, p. 40, [Meh], [HJ] Th. 2.2.12 p. 348, or [BOst14])**.** *An additive Baire continuous function $f : X \to Y$ between complete metric groups is continuous, when Y is separable, or has base of cardinality less than the continuum.*

The Souslin criterion and the next theorem together have as immediate corollary the classical Souslin-graph Theorem; in this connection recall (see the corollary of [HJ] Th. 2.3.6 p. 355) that a normed group which is Baire and analytic is Polish.

Theorem (Baire Homomorphism Theorem). *Let X and Y be topological groups with Y a K-analytic group and X non-meagre. If $f : X \to Y$ is a Baire homomorphism, then f is continuous.*

Corollary 1 (Souslin-graph Theorem, Schwartz [Schw], cf. [Jay-Rog] p.50)*. Let X and Y be topological groups with Y a K-analytic group and X non-meagre. If $f : X \to Y$ is a homomorphism with Souslin-$\mathcal{F}(X \times Y)$ graph, then f is continuous.*

Proofs. Here we refer to the proof in [Jay-Rog] of the Souslin-graph theorem; that proof may be construed as having two steps: one establishing the Souslin criterion, the other the Baire homomorphism theorem. □

Corollary 2 (Generalized Jones Theorem: Thinned Souslin-graph Theorem). *Let X and Y be topological groups with X non-meagre and Y a K-analytic set. Let S be an K-analytic set spanning X and $f : X \to Y$ a homomorphism with restriction to S continous on S. Then f is continuous.*

Proof. Since f is continuous on S, the graph $\{(x, y) \in S \times Y : y = f(x)\}$ is closed in $S \times Y$ and so is K-analytic by [Jay-Rog] Th. 2.5.3. Now $y = f(x)$ iff, for some $n \in \mathbb{N}$, there is $(y_1, ..., y_n) \in Y^n$ and $(s_1, ..., s_n) \in S^n$ such that $x = s_1 \cdot ... \cdot s_n$, $y = y_1 \cdot ... \cdot y_n$, and, for $i = 1, .., n$, $y_i = f(s_i)$. Thus $G := \{(x, y) : y = f(x)\}$ is K-analytic. Formally,

$$G = \mathrm{pr}_{X\times Y}\left[\bigcup_{n\in\mathbb{N}}\left[M_n \cap (X \times Y \times S^n \times Y^n) \cap \bigcap_{i\le n} G_{i,n}\right]\right],$$

where

$$M_n := \{(x, y, s_1, ..., s_n, y_1, ..., y_n) : y = y_1 \cdot ... \cdot y_n \text{ and } x = s_1 \cdot ... \cdot s_n\},$$

and

$$G_{i,n} := \{(x, y, s_1, ..., s_n, y_1, ..., y_n) \in X\times Y\times X^n\times Y^n : y_i = f(s_i)\}, \text{for } i = 1, ..., n.$$

Here each set M_n is closed and each $G_{i,n}$ is K-analytic. Hence, by the Intersection and Projection Theorems, the graph G is K-analytic. By the Souslin-graph theorem f is thus continuous. □

This is a new proof of the Jones Theorem. We now consider results for the more special normed group context. Here again one should note the corollary of [HJ] Th. 2.3.6 p. 355 that a normed group which is Baire and analytic is Polish.

Our first result has a proof which is a minor adaptation of the proof in [BoDi]. We recall that a Hausdorff topological space is paracompact ([Eng] Ch. 5, or [Kel] Ch. 6, especially Problem Y) if every open cover has a locally finite open refinement and that (i) Lindelöf spaces and (ii) metrizable spaces are paracompact. Paracompact spaces are normal, hence topological groups need not be paracompact, as exemplified again by the example due to Oleg Pavlov [Pav] quoted earlier or by the example of van Douwen [vD] (see also [Com] Section 8.4 p. 1222); however, L. G. Brown [Br-2] shows that a locally complete group is paracompact (and this includes the locally compact case, cf. [Com] Th. 2.9 p. 1161). The assumption of paracompactness is thus natural.

Theorem (Generalized Borwein-Ditor Theorem 2). *Let G be a paracompact topological group equipped with a locally-finite, inner regular Borel measure m (Radon measure) which is left-invariant, resp. right-invariant, (for example, G locally compact, equipped with a Haar measure).*

If A is a (Borel) measurable set with $0 < m(A) < \infty$ and $z_n \to e$, then, for m-almost $a \in A$, there is an infinite set $\mathbb{M}_a$ such that the corresponding right-translates, resp. left-translates, of z_n are in A, i.e., in the first case

$$\{z_n a : n \in \mathbb{M}_a\} \subseteq A.$$

Proof. Without loss of generality we consider right-translation of the sequence $\{z_n\}$. Since G is paracompact, it suffices to prove the result for A open and of finite measure. By inner-regularity A may be replaced by a σ-compact subset of equal measure. It thus suffices to prove the theorem for K compact with $m(K) > 0$ and $K \subseteq A$. Define a decreasing sequence of compact sets $T_k := \bigcup_{n \geq k} z_n^{-1} K$, and let $T = \bigcap_k T_k$. Thus $x \in T$ iff, for some infinite $\mathbb{M}_x$,

$$z_n x \in K \text{ for } m \in \mathbb{M}_x,$$

so that T is the set of 'translators' x for the sequence $\{z_n\}$. Since K is closed, for $x \in T$, we have $x = \lim_{n \in \mathbb{M}_x} z_n x \in K$; thus $T \subseteq K$. Hence, for each k,

$$m(T_k) \geq m(z_k^{-1} K) = m(K),$$

by left-invariance of the measure. But, for some n, $T_n \subseteq A$. (If $z_n^{-1} k_n \notin A$ on an infinite set $\mathbb{M}$ of n, then since $k_n \to k \in K$ we have $z_n^{-1} k_n \to k \in A$, but $k = \lim z_n^{-1} k_n \notin A$, a contradiction since A is open.) So, for some n, $m(T_n) < \infty$, and thus $m(T_k) \to m(T)$. Hence $m(K) \geq m(T) \geq m(K)$. So $m(K) = m(T)$ and thus almost all points of K are translators. $\square$

Remark. It is quite consistent to have the measure left-invariant and the metric right-invariant.

Analytic Dichotomy Lemma (Spanning). *Let G be a connected, normed group. Suppose that an analytic set $T \subseteq G$ spans a set of positive measure or a non-meagre set. Then T spans G.*

Proof. In the category case, the result follows from the Banach-Kuratowski Dichotomy Theorem of Section 4 ([Ban-G, Satz 1], [Kur-1, Ch. VI. 13. XII], [Kel, Ch. 6 Prob. P p. 211]) by considering S, the subgroup generated by T; since T is analytic, S is analytic and hence Baire, and, being non-meagre, is clopen and hence all of G, as the latter is a connected group.

In the measure case, by the Steinhaus Theorem of Section 4 ([St], [BGT, Th. 1.1.1], [BOst3]), T^2 has non-empty interior, hence is non-meagre. The result now follows from the category case. $\square$

Our next lemma follows directly from Choquet's Capacitability Theorem [Choq] (see especially [Del2, p. 186], and [Kech, Ch. III 30.C]). For completeness, we include the brief proof. Incidentally, the argument we employ goes back to Choquet's theorem, and indeed further, to [ROD] (see e.g. [Del1, p. 43]).

Compact Contraction Lemma. *In a normed group carrying a Radon measure, for T analytic, if $T \cdot T$ has positive Radon measure, then for some compact subset S of T, $S \cdot S$ has positive measure.*

Proof. We present a direct proof (see below for our original inspiration in Choquet's Theorem). As T^2 is analytic, we may write ([Jay-Rog]) $T^2 = h(H)$, for some continuous h and some $\mathcal{K}_{\sigma\delta}$ subset of the reals, e.g. the set H of the irrationals, so that $H = \bigcap_i \bigcup_j d(i,j)$, where $d(i,j)$ are compact and, without loss of generality, the unions are each increasing: $d(i,j) \subseteq d(i,j+1)$. The map $g(x,y) := xy$ is continuous and hence so is the composition $f = g \circ h$. Thus $T \cdot T = f(H)$ is analytic. Suppose that $T \cdot T$ is of positive measure. Hence, by the capacitability argument for analytic sets ([Choq], or [Si, Th.4.2 p. 774], or [Rog1, p. 90], there referred to as an 'Increasing sets lemma'), for some compact set A, the set $f(A)$ has positive measure. Indeed if $|f(H)| > \eta > 0$, then the set A may be taken in the form $\bigcap_i d(i,j_i)$, where the indices j_i are chosen inductively, by reference to the increasing union, so that $|f[H \cap \bigcap_{i<k} d(i,j_i)]| > \eta$, for each k. (Thus $A \subseteq H$ and $f(A) = \bigcap_i f[H \cap \bigcap_{i<k} d(i,j_i)]$ has positive measure, cf. [EKR].)

The conclusion follows as $S = h(A)$ is compact and $S \cdot S = g(S) = f(A)$. $\square$

Note. The result may be deduced indirectly from the Choquet Capacitability Theorem by considering the capacity $I : G^2 \to \mathbb{R}$, defined by $I(X) = m(g(X))$, where, as before, $g(x,y) := xy$ is continuous and m

denotes a Radon measure on G (on this point see [Del2, Section 1.1.1, p. 186]). Indeed, the set T^2 is analytic ([Rog2, Section 2.8, p. 37-41]), so $I(T^2) = \sup I(K^2)$, where the supremum ranges over compact subsets K of T. Actually, the Capacitability Theorem says only that $I(T^2) = \sup I(K_2)$, where the supremum ranges over compact subsets K_2 of T^2, but such a set may be embedded in K^2 where $K = \pi_1(K) \cup \pi_2(K)$, with π_i the projections onto the axes of the product space.

Corollary. *For T analytic and $\varepsilon_i \in \{\pm 1\}$, if $T^{\varepsilon_1} \cdot ... \cdot T^{\varepsilon_d}$ has positive measure (measure greater than η) or is non-meagre, then for some compact subset S of T, the compact set $K = S^{\varepsilon_1} \cdot ... \cdot S^{\varepsilon_d}$ has $K \cdot K$ of positive measure (measure greater than η).*

Proof. In the measure case the same approach may be used based now on the continuous function $g(x_1, ..., x_d) := x_1^{\varepsilon_1} \cdot ... \cdot x_d^{\varepsilon_d}$ ensuring that K is of positive measure (measure greater than η). In the category case, if $T' = T^{\varepsilon_1} \cdot ... \cdot T^{\varepsilon_d}$ is non-meagre then, by the Steinhaus Theorem ([St], or [BGT, Cor. 1.1.3]), $T' \cdot T'$ has non-empty interior. The measure case may now be applied to T' in lieu of T. (Alternatively one may apply the Pettis-Piccard Theorem, as in the Analytic Dichotomy Lemma.) □

Theorem (Compact Spanning Approximation). *For T analytic in X, if the span of T is non-null or is non-meagre, then there exists a compact subset of T which spans X.*

Proof. If T is non-null or non-meagre, then T spans all the reals (by the Analytic Dichotomy Lemma); then for some $\varepsilon_i \in \{\pm 1\}$, $T^{\varepsilon_1} \cdot ... \cdot T^{\varepsilon_d}$ has positive measure/ is non-meagre. Hence for some K compact $K^{\varepsilon_1} \cdot ... \cdot K^{\varepsilon_d}$ has positive measure/ is non-meagre. Hence K spans some and hence all reals. □

Analytic Covering Lemma ([Kucz, p. 227], cf. [Jones2, Th. 11]). *Given normed groups G and H, and T analytic in G, let $f : G \to H$ have continuous restriction $f|T$. Then T is covered by a countable family of bounded analytic sets on each of which f is bounded.*

Proof. For $k \in \omega$ define $T_k := \{x \in T : ||f(x)|| < k\} \cap B_k(e_G)$. Now $\{x \in T : ||f(x)|| < k\}$ is relatively open and so takes the form $T \cap U_k$ for some open subset U_k of G. The Intersection Theorem shows this to be analytic since U_k is an $\mathcal{F}_\sigma$ set and hence Souslin-$\mathcal{F}$. $\square$

Expansion Lemma ([Jones2, Th. 4], [Kom2, Th. 2], and [Kucz, p. 215]). *Suppose that S is Souslin-$\mathcal{H}$, i.e. of the form*

$$S = \bigcup_{\alpha \in \omega^\omega} \cap_{n=1}^{\infty} H(\alpha|n),$$

with each $H(\alpha|n) \in \mathcal{H}$, for some family of analytic sets $\mathcal{H}$ on which f is bounded. If S spans the normed group G, then, for each n, there are sets $H_1, ..., H_k$ each of the form $H(\alpha|n)$, such that for some integers $r_1, ..., r_k$

$$T = H_1 \cdot ... \cdot H_k$$

has positive measure/ is non-meagre, and so $T \cdot T$ has non-empty interior.

Proof. For any $n \in \omega$ we have

$$S \subseteq \bigcup_{\alpha \in \omega^\omega} H(\alpha|n).$$

Enumerate the countable family $\{H(\alpha|n) : \alpha \in \omega^n\}$ as $\{T_h : h \in \omega\}$. Since S spans G, we have

$$G = \bigcup_{h \in \omega} \bigcup_{\mathbf{k} \in \mathbb{N}^h} \left(T_{k_1} \cdot ... \cdot T_{k_h}\right).$$

As each T_k is analytic, so too is the continuous image

$$T_{k_1} \cdot ... \cdot T_{k_h},$$

which is thus measurable. Hence, for some $h \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{N}^h$ the set

$$T_{k_1} \cdot ... \cdot T_{k_h}$$

has positive measure/ is non-meagre. $\square$

Definition. We say that S is a *pre-compact set* if its closure is compact. We will say that f is a *pre-compact function* if $f(S)$ is pre-compact for each pre-compact set S.

Analytic Automaticity Theorem for Metric Groups (Jones-Kominek Theorem). *Let be G be either a non-meagre normed group, or a group supporting a Radon measure, and let H be K-analytic (hence Lindelöf, and so second countable in our metric setting). Let $h : G \to H$ be a homomorphism between metric groups and let T be an analytic set in G which finitely generates G.*

(i) (Jones condition) *If h is continuous on T, then h is continuous.*

(ii) (Kominek condition) *If h is pre-compact on T, then h is precompact.*

Proof. As in the Analytic Covering Lemma, write

$$T = \bigcup_{k \in \mathbb{N}} T_k.$$

(i) If h is not continuous, suppose that $h(x_n)$ does not converge to $h(x_0)$. Since

$$G = \bigcup_{m \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} T_k^{(m)},$$

G is a union of analytic sets and hence analytic ([Jay-Rog] Th. 2.5.4 p. 23). Now, for some m, k the span $T_k^{(m)}$ is non-meagre, as is the span of a compact subset $S_k^{(m)}$ for some $S_k \subseteq T_k$. So for some shifted subsequence $tx_n \to tx_0$, where t and x_0 lies in $S_k^{(m)}$. Thus there is an infinite set $\mathbb{M}$ such that, for $n \in \mathbb{M}$,

$$tx_n = t_n^1 ... t_n^m \text{ with } t_n^i \in S_k.$$

W.l.o.g., as S_k is compact,

$$t_n^{(i)} \to t_0^{(i)} \in S_k \subset T,$$

and so

$$tx_n = t_n^1 ... t_n^m \to t_0^1 ... t_0^m = tx_0 \text{ with } t_0^i \in S_k \subset T.$$

Hence, as $t_n^i \to t_0^i \subset T$, we have, for $n \in \mathbb{M}$,

$$\begin{aligned} h(t)h(x_n) &= h(tx_n) = h(t_n^1 ... t_n^m) = h(t_n^1) ... h(t_n^m) \\ &\to h(t_0^1) ... h(t_0^m) = h(t_0^1 ... t_0^m) \\ &= h(tx_0) = h(t)h(x_0). \end{aligned}$$

Thus

$$h(x_n) \to h(x_0),$$

a contradiction.

(ii) If $\{h(x_n)\}$ is not precompact with $\{x_n\}$ precompact, by the same argument, for some $S_k^{(n)}$ and some infinite set $\mathbb{M}$, we have $tx_n = t_n^1 ... t_n^m$ and $t_n^i \to t_0^i \subset T$, for $n \in \mathbb{M}$. Hence $h(tx_n) = h(t)h(x_n)$ is precompact and so $h(x_n)$ is precompact, a contradiction. $\square$

The following result connects the preceeding theorem to Darboux's Theorem.

Definition. Say that a homomorphism between normed groups is $\mathbb{N}$-*homogeneous* if $||f(x^n)|| = n||f(x)||$, for any x and $n \in \mathbb{N}$. Thus any homomorphism into the additive reals is $\mathbb{N}$-homogeneous. More generally, say that the norm is $\mathbb{N}$-*subhomogeneous* if there are constants κ_n with $\kappa_n \to \infty$ such that for all elements z of the group

$$\kappa_n||z|| \leq ||z^n||,$$

or equivalently

$$||z^{1/n}|| \leq \frac{1}{\kappa_n}||z||.$$

Thus $z^{1/n} \to e$; a related condition was considered by McShane in [McSh] (cf. the Eberlein-McShane Theorem of the preceeding Section). In keeping with the convention of functional analysis (appropriately to our usage of norm) the next result refers to a locally bounded homomorphism as bounded.

Generalized Darboux Theorem ([Dar]) *A bounded homomorphism from a normed group to an* $\mathbb{N}$*-subhomogeneous normed group is continuous; in particular, a bounded, additive function is continuous.*

Proof. Suppose that $f : G \to H$ is a homomorphism to a normed $\mathbb{N}$-subhomogeneous group H; thus $||f(x^n)|| \geq \kappa_n||f(x)||$, for any $x \in G$ and $n \in \mathbb{N}$. Suppose that f is bounded by M and, for $||x|| < \eta$, we have

$$||f(x)|| < M.$$

Let $\varepsilon > 0$ be given. Choose N such that $\kappa_N > M/\varepsilon$, i.e. $M/\kappa_N < \varepsilon$. Now $x \to x^N$ is continuous, hence there is $\delta = \delta_N(\eta) > 0$ such that, for $||x|| < \delta$,

$$||x^N|| < \eta.$$

Consider x with $||x|| < \delta_N(\eta)$. Then $\kappa_N||f(x)|| \leq ||f(x)^N|| = ||f(x^N)|| < M$. So for x with $||x|| < \delta_N(\eta)$ we have

$$||f(x)|| < M/\kappa_N < \varepsilon,$$

proving continuity at e. □

Compare [HJ] Th 2.4.9 p. 382.

The Main Theorem of [BOst7] may be given a combinatorial restatement in the group setting. We need some further definitions.

Definition. For G a metric group, let $\mathcal{C}(G) = \mathcal{C}(\mathbb{N}, G) := \{\mathbf{x} \in G^{\mathbb{N}} : \mathbf{x}$ is convergent$\}$. For $x \in \mathcal{C}(G)$ we write

$$L(\mathbf{x}) = \lim_n x_n.$$

We make $\mathcal{C}(G)$ into a group by setting

$$\mathbf{x} \cdot \mathbf{y} := \langle x_n y_n : n \in \mathbb{N} \rangle.$$

Thus $\mathbf{e} = \langle e_G \rangle$ and $\mathbf{x}^{-1} = \langle x_n^{-1} \rangle$. We identify G with the subgroup of constant sequences, that is

$$T = \{\langle g : n \in \mathbb{N} \rangle : g \in G\}.$$

The natural action of G or T on $\mathcal{C}(G)$ is then $t\mathbf{x} := \langle tx_n : n \in \mathbb{N} \rangle$. Thus $\langle g \rangle = g\mathbf{e}$, and then $t\mathbf{x} = t\mathbf{e} \cdot \mathbf{x}$.

Definition. For G a group, a set $\mathcal{G}$ of convergent sequences $\mathbf{u} = \langle u_n : n \in \mathbb{N} \rangle$ in $c(G)$ is a *G-ideal in the sequence space* $\mathcal{C}(G)$ if it is a subgroup closed under the mutiplicative action of G, and will be termed *complete* if it is closed under subsequence formation. That is, a complete G-ideal in $\mathcal{C}(G)$ satisfies

(i) $\mathbf{u} \in \mathcal{G}$ implies $t\mathbf{u} = \langle tu_n \rangle \in \mathcal{G}$, for each t in G,

(ii) $\mathbf{u}, \mathbf{v} \in \mathcal{G}$ implies that $\mathbf{u}\mathbf{v}^{-1} \in \mathcal{G}$,

(iii) $\mathbf{u} \in \mathcal{G}$ implies that $\mathbf{u}_{\mathbb{M}} = \{u_m : m \in \mathbb{M}\} \in \mathcal{G}$ for every infinite $\mathbb{M}$.

If $\mathcal{G}$ isatisfies (i) and $\mathbf{u}, \mathbf{v} \in \mathcal{G}$ implies only that $\mathbf{u}\mathbf{v} \in \mathcal{G}$, we say that $\mathcal{G}$ is a *G-subideal* in $\mathcal{C}(G)$.

Remarks.

0. If $\mathcal{G}$ is merely an ideal then $\mathcal{G}^* = \{\mathbf{u}_{\mathbb{M}} : \text{for } \mathbf{u} \in t \text{ and } \mathbb{M} \subset \mathbb{N}\}$ is a complete G-ideal; indeed $t\mathbf{u}_{\mathbb{M}} = (t\mathbf{u})_{\mathbb{M}}$ and $\mathbf{u}_{\mathbb{M}}\mathbf{v}_{\mathbb{M}}^{-1} = (\mathbf{u}\mathbf{v}^{-1})_{\mathbb{M}}$ and $\mathbf{u}_{\mathbb{M}\mathbb{M}'} = \mathbf{u}_{\mathbb{M}'}$ for $\mathbb{M}' \subset \mathbb{M}$.

1. We speak of Euclidean sequential structures when G is the vector space $\mathbb{R}^d$ regarded as an additive group.

2. The conditions (i) and (ii) assert that $\mathcal{G}$ is similar in structure to a left-ideal, being closed under multiplication by G and a subgroup of $\mathcal{C}(G)$.

3. We refer only to the combinatorial properties of $\mathcal{C}(G)$; but one may give $\mathcal{C}(G)$ a pseudo-norm by setting

$$||x||_c := d_G(Lx, e) = ||Lx||.$$

The corresponding pseudo-metric is

$$d(x, y) := \lim d_G(x_n, y_n) = d_G(Lx, Ly).$$

We may take equivalence of sequences with identical limit; then $\mathcal{C}(G)^{\sim}$ becomes a normed group. However, in our theorem below we do not wish to refer to such an equivalence.

Definitions. For a family $\mathcal{F}$ of functions from G to H, we denote by $\mathcal{F}(T)$ the family $\{f|T : f \in \mathcal{F}\}$ of functions in $\mathcal{F}$ restricted to $T \subseteq G$. Let us denote a convergent sequence with limit x_0, by $\{x_n\} \to x_0$. We say the property $\mathcal{Q}$ of functions (property being regarded set-theoretically, i.e. as a family of functions from G to H) is *sequential on* T if

$$f \in \mathcal{Q} \text{ iff } (\forall \{x_n : n > 0\} \subseteq T)[(\{x_n\} \to x_0) \implies f|\{x_n : n > 0\} \in \mathcal{Q}(\{x_n : n > 0\})].$$

If we further require the limit point to be enumerated in the sequence, we call $\mathcal{Q}$ *completely sequential on* T if

$$f \in \mathcal{Q} \text{ iff } (\forall \{x_n\} \subseteq T)[(\{x_n\} \to x_0) \implies f|\{x_n\} \in \mathcal{Q}(\{x_n\})].$$

Our interest rests on properties that are completely sequential; our theorem below contains a condition referring to completely sequential properties, that is, the condition is required to hold on convergent sequences with limit included (so on a compact set), rather than on arbitrary sequences.

Note that if $\mathcal{Q}$ is (completely) sequential then $f|\{x_n\} \in \mathcal{Q}(\{x_n\})$ iff $f|\{x_n : n \in \mathbb{M}\} \in \mathcal{Q}(\{x_n : n \in \mathbb{M}\})$, for every infinite $\mathbb{M}$.

Definition. Let $h : G \to H$, with G, H metric groups. Say that a sequence $\mathbf{u} = \{u_n\}$ is $\mathcal{Q}$-*good* for h if

$$h|\{u_n\} \in \mathcal{Q}|\{u_n\},$$

and put

$$\mathcal{G}_{h\mathcal{Q}} = \{\mathbf{u} : h|\{u_n\} \in \mathcal{Q}|\{u_n\}\}.$$

If $\mathcal{Q}$ is completely sequential, then $\mathbf{u}$ is $\mathcal{Q}$-good for h iff every subsequence of $\mathbf{u}$ is $\mathcal{Q}$-good for h, so that $\mathcal{G}_{h\mathcal{Q}}$ is a G-ideal iff it is a complete G-ideal. One then has:

Lemma. *If $\mathcal{Q}$ is completely sequential and $\mathcal{F}$ preserves $\mathcal{Q}$ under shift and multiplication and division on compacts, then $\mathcal{G}_{h\mathcal{Q}}$ for $h \in \mathcal{F}$ is a G-ideal.*

Theorem (Analytic Automaticity Theorem - combinatorial form). *Suppose that functions of $\mathcal{F}$ having $\mathcal{Q}$ on G have $\mathcal{P}$ on G, where $\mathcal{Q}$ is a property of functions from G to H that is completely sequential on G.*

Suppose that, for all $h \in \mathcal{F}$, $\mathcal{G}_{h\mathcal{Q}}$, the family of $\mathcal{Q}$-good sequences is a G-ideal. Then, for any analytic set T spanning G, functions of $\mathcal{F}$ having $\mathcal{Q}$ on T have $\mathcal{P}$ on G.

This theorem is applied with $G = \mathbb{R}^d$ and $H = \mathbb{R}$ in [BOst6] to subadditive functions, convex functions, and to regularly varying functions, defined on $\mathbb{R}^d$ to derive automatic properties such as automatic continuity, automatic local boundedness and automatic uniform boundedness.

10 Duality in normed groups

In this section we use the generic notation of S for a group with metric d_S; recall from Section 2 that $Auth(S)$ denotes the autohomeomorphisms of S; $\mathcal{H}(S)$ denotes the bounded elements of $Auth(S)$. We write $\mathcal{A} \subseteq \mathcal{H}(S)$ for a subgroup of automorphisms of S; thus $\mathcal{A}$ is a topological group metrized by the supremum metric

$$d_T(t_1, t_2) = \sup_{s \in S} d_X(t_1(s), t_2(s)).$$

Note that $e_{\mathcal{A}} = id_S$. The purpose of this notation is to embrace the two cases:
(i) $S = X$ and $\mathcal{A} = \mathcal{H}(X)$, and

(ii) $S = \mathcal{H}(X)$ and $\mathcal{A} = \mathcal{H}(\mathcal{H}(X))$.

In what follows, we regard the group $\mathcal{H}(X)$ as the topological dual of X and verify that (X, d_X) is embedded in the second dual $\mathcal{H}(\mathcal{H}(X))$. As an application one may use this duality to clarify, in the context of a non-autonomous differential equation with initial conditions, the link between its solutions trajectories and flows of its varying 'coefficient matrix'. See [Se1] and [Se2], which derive the close relationship for a general non-autonomous differential equation $u' = f(u, t)$ with $u(0) = x \in X$, between its trajectories in X and local flows in the function space Φ of translates f_t of f (where $f_t(x, s) = f(x, t + s)$.

One may alternatively capture the topological duality as algebraic complementarity – see [Ost-knit] for details. A summary will suffice here. One first considers the commutative diagram below where initially the maps are only *homeomorphisms* (herein $T \subseteq \mathcal{H}(X)$ and $\Phi^T(t, x) = (t, tx)$ and $\Phi^X(x, t) = (t, xt)$ are embeddings). Then one extends the diagram to a diagram of *isomorphisms,* a change facilitated by forming the direct product group $G := T \times X$. Thus $G = T_G X_G$ where T_G and X_G are normal subgroups, commuting elementwise, and isomorphic respectively to T and X; moreover, the subgroup T_G, acting multiplicatively on X_G, represents the T-flow on X and simultaneously the multiplicative action of X_G on G represents the X-flow on $T_X = \{t_x : t \in T, x \in X\}$, the group of right-translates of T, where $t_x(u) = \theta_x(t)(u) = t(ux)$. If G has an invariant metric d_G, and T_G and X_G are now regarded as groups of translations on G, then they may be metrized by the supremum metric d_G^*, whereupon each is isometric to itself as subgroup of G. Our approach here suffers a loss of elegance, by dispensing with G, but gains analytically by working directly with d_X and d_X^*.

$$\begin{tikzcd}
(t,x) \arrow[r, leftrightarrow, "\Phi^T"] \arrow[d, leftrightarrow] & (t,tx) \arrow[d, leftrightarrow] \\
(x,t) \arrow[r, leftrightarrow, "\Phi^X"] & (t,xt)
\end{tikzcd}$$

Here the two vertical maps may, and will, be used as identifications, since $(t, tx) \leftrightarrows (t, x) \leftrightarrows (t, xt)$ are bijections (more in fact is true, see [Ost-knit]).

Definitions. Let X be a topological group with right-invariant metric d_X.

1. The *left-translation group* $Tr_L(X)$ comprises the maps $\tau_x : X \to X$, defined by $\tau_x(z) = xz$ with composition as multiplication. The map τ_x is continuous with continuous inverse $\tau_x^{-1}(y) = \tau_{x^{-1}}(y) = x^{-1}y$. The identity is $id_X = \tau_e$, where $e = e_X$. Thus $Tr_L(X) \subseteq Auth(X)$. We have, as d_X is right-invariant, that

$$||\tau_x||_T := d_T(\tau_x, \tau_e) := \sup_{z \in X} d_X(xz, z) = d_X(x, e_X) = ||x||_X.$$

The appearance of a *supremum norm* derived from a supremum metric on $T = Tr_L(X)$ here is natural, in that for any metric d_X (now not necessarily right-invariant), the metric

$$\bar{d}(x, y) = \sup_{z \in X} d_X(xz, yz)$$

is right-invariant on the subgroup $H = \{x \in X : \bar{d}(x, e) < \infty\}$ (by the Proposition and Corollary on group-norm properties of the previous section). Returning to our theme, we have $Tr_L(X) \subseteq \mathcal{H}(X)$; $Tr_L(X)$ is a subgroup and $\tau : X \to Tr_L(X)$ is an isomorphism, because

$$\tau_x \circ \tau_y(z) = \tau_x(\tau_y(z)) = x(\tau_y(z) = xyz = \tau_{xy}(z).$$

Moreover, τ is an isometry, as d_X is right-invariant; indeed, we have

$$d_T(\tau_x, \tau_y) = \sup_z d_X(xz, yz) = d_X(x, y).$$

2. We now lift the isomorphism τ to $\mathcal{H}(X)$. That is, for $x \in X$, and $s \in \mathcal{H}(X)$ we define the translation map xs, more properly written $\xi_x(s)$, where $\xi_x(s)(z) = s(\tau_x^{-1}(z)) = s(x^{-1}z)$. If $T \subseteq \mathcal{H}(X)$ is τ-invariant, we may of course regard τ as operating on T. For example, if $T = Tr_L(X)$, we have $\xi_x(\tau_y)(z) = \tau_y\tau_x^{-1}(z)$, so $\xi_x(\tau_y) = \tau_{yx^{-1}}$.

Denote the set of translation maps by $\Xi = \{\xi_x : x \in X\} \subseteq Auth(X)$. Now Ξ is a group (under composition) with identity $e_\Xi = \xi_e$, where $e = e_X$. Note that

$$\xi_x(e_S)(e_X) = x^{-1},$$

so the mapping $x \to \xi_x$ from X to Ξ is bijective. Also

$$\begin{aligned} (\xi_x \circ \xi_y(s))(z) &= \xi_x(\xi_y(s))(z) = (\xi_y(s))(x^{-1}z) \\ &= s(y^{-1}x^{-1}z) = s((xy)^{-1}z) = \xi_{xy}(s)(z), \end{aligned}$$

so ξ is an isomorphism from X to Ξ and so $\xi_x^{-1} = \xi_{x^{-1}}$.

A comparison with the normed vector space context and the metrization of the translations $x \to t(z + x)$ for a linear map t suggests that in order to metrize Ξ by reference to $\xi_x(t)$ we need to take account of $||t||$. A natural metric here for any $\varepsilon \geq 0$ is thus the magnification metric

$$d_T^\varepsilon(\xi_x, \xi_y) := \sup_{||t|| \leq \varepsilon} d_T(\xi_x(t), \xi_y(t)). \tag{3}$$

By Proposition 5 this is a metric; indeed with $t = e_{\mathcal{H}(X)} = id_X$ we have $||t|| = 0$ and, since d_X is assumed right-invariant, for $x \neq y$, we have with $z_{xy} = e$ that $d_X(x^{-1}z, y^{-1}z) = d_X(x^{-1}, y^{-1}) > 0$. The presence of the case $\varepsilon = 0$ is not fortuitous; see [Ost-knit] for an explanation via an isomorphism theorem. We trace the dependence on $||t||$ in the following Proposition. We refer to Gromov's notion [Gr1], [Gr2] of quasi-isometry under π, in which π is a mapping between spaces. In a first application we take π to be a self-homeomorphism, in particular a left-translation; in the second $\pi(x) = \xi_x(t)$ with t fixed is an evaluation map appropriate to a dual embedding. We begin with a theorem promised in Section 2.

Uniformity Theorem for Conjugation. *Let $\Gamma : G^2 \to G$ be the conjugation $\Gamma(g, x) := g^{-1}xg$.*

Under a bi-invariant Klee metric, for all a, b, g, h ,

$$d_G(a, b) - 2d_G(g, h) \leq d_G(gag^{-1}, hbh^{-1}) \leq 2d_G(g, h) + d_G(a, b),$$

and hence conjugation is uniformly continuous.

Proof. Referring to the Klee property, via the cyclic property we have

$$\begin{aligned} d_G(gag^{-1}, hbh^{-1}) &= ||gag^{-1}hb^{-1}h^{-1}|| = ||h^{-1}gag^{-1}h^{-1}b^{-1}|| \\ &\leq ||h^{-1}g|| + ||ag^{-1}h^{-1}b^{-1}|| \\ &\leq ||h^{-1}g|| + ||ab^{-1}|| + ||h^{-1}g||. \end{aligned}$$

Then substitute $g^{-1}ag$ for a etc., g^{-1} for g etc., to obtain

$$d_G(a, b) \leq 2d_G^*(g^{-1}, h^{-1}) + d_G(gag^{-1}, hbh^{-1}).$$

But d_G is bi-invariant, so

$$d_G(g^{-1}, h^{-1}) = \tilde{d}_G(g, h) = d_G(g, h). \qquad \square$$

Proposition (Permutation metric). *For* $\pi \in \mathcal{H}(X)$*, let* $d_\pi(x,y) := d_X(\pi(x),\pi(y))$*. Then* d_π *is a metric, and*

$$d_X(x,y) - 2||\pi|| \le d_\pi(x,y) \le d_X(x,y) + 2||\pi||.$$

In particular, if d_X *is right-invariant and* $\pi(x)$ *is the left-translation* $\tau_z(x) = zx$*, then*

$$d_X(x,y) - 2||z|| \le d_z(x,y) = d_X(zx,zy) \le d_X(x,y) + 2||z||.$$

Proof. By the triangle inequality,

$$d_X(\pi(x),\pi(y)) \le d_X(\pi(x),x) + d_X(x,y) + d_X(y,\pi(y)) \le 2||\pi|| + d_X(x,y).$$

Likewise,

$$\begin{aligned} d_X(x,y) &\le d_X(x,\pi(x)) + d_X(\pi(x),\pi(y)) + d_X(\pi(y),y) \\ &\le 2||\pi|| + d_X(\pi(x),\pi(y)). \end{aligned}$$

If $\pi(x) := zx$, then $||\pi|| = \sup d(zx,x) = ||z||$ and the result follows. $\square$

Recall from Proposition 2 that for d a metric on a group X, we write $\tilde{d}(x,y) = d(x^{-1},y^{-1})$ for the *conjugate metric*. The conjugate metric $\tilde{d}$ is left-invariant iff the metric d is right-invariant. Under such circumstances both metrics induce the same norm (since $d(e,x) = d(x^{-1},e)$, as we have seen above). In what follows note that $\xi_x^{-1} = \xi_{x^{-1}}$.

Theorem 3 (**Quasi-isometric duality**). *If the metric* d_X *on* X *is right-invariant and* $t \in \mathcal{H}(X)$*, then*

$$\tilde{d}_X(x,y) - 2||t||_{\mathcal{H}(X)} \le d_T(\xi_x(t),\xi_y(t)) \le \tilde{d}_X(x,y) + 2||t||_{\mathcal{H}(X)},$$

and hence, for each $\varepsilon \ge 0$*, the magnification metric* (3) *satisfies*

$$\tilde{d}_X(x,y) - 2\varepsilon \le d_T^\varepsilon(\xi_x,\xi_y) \le \tilde{d}_X(x,y) + 2\varepsilon.$$

Equivalently, in terms of conjugate metrics,

$$d_X(x,y) - 2\varepsilon \le \tilde{d}_T^\varepsilon(\xi_x,\xi_y) \le d_X(x,y) + 2\varepsilon.$$

Hence,

$$||x|| - 2\varepsilon \le ||\xi_x||_\varepsilon \le ||x|| + 2\varepsilon,$$

and so $||x_n|| \to \infty$ *iff* $d_T(\xi_{x(n)}(t), \xi_e(t)) \to \infty$.

Proof. We follow a similar argument to that for the permutation metric. By right-invariance,

$$\begin{aligned} d_X(t(x^{-1}z), t(y^{-1}z)) &\leq d_X(t(x^{-1}z), x^{-1}z) + d_X(x^{-1}z, y^{-1}z) + d_X(y^{-1}z, t(y^{-1}z)) \\ &\leq 2||t|| + d_X(x^{-1}, y^{-1}), \end{aligned}$$

so

$$d_T(\xi_x(t), \xi_y(t)) = \sup_z d_X(t(x^{-1}z), t(y^{-1}z)) \leq 2||t|| + d_X(x, e_X).$$

Now, again by right-invariance,

$$d_X(x^{-1}, y^{-1}) \leq d(x^{-1}, t(x^{-1})) + d(t(x^{-1}), t(y^{-1})) + d(t(y^{-1}), y^{-1}).$$

But

$$d(t(x^{-1}), t(y^{-1})) \leq \sup_z d_X(t(x^{-1}z), t(y^{-1}z)),$$

so

$$d_X(x^{-1}, y^{-1}) \leq 2||t|| + \sup_z d_X(t(x^{-1}z), t(y^{-1}z)) = 2||t|| + d_T(\xi_x(t), \xi_y(t)),$$

as required. □

We thus obtain the following result.

Topological Duality Theorem.

For X *a normed group, the second dual* Ξ *is a normed group isometric to* X *which, for any* $\varepsilon \geq 0$*, is* ε*-quasi-isometric in relation to* $\tilde{d}_T^{\varepsilon}(\xi_x, \xi_y)$ *and the* $||\cdot||^{\varepsilon}$ *norm.*

Proof. We metrize Ξ by setting $d_{\Xi}(\xi_x, \xi_y) = d_X(x, y)$. This makes Ξ an isometric copy of X and ε-quasi-isometric copy in relation to the conjugate metric $\tilde{d}_T^{\varepsilon}(\xi_x, \xi_y)$ given for any $\varepsilon \geq 0$ by

$$\tilde{d}_T^{\varepsilon}(\xi_x, \xi_y) := \sup_{||t|| \leq \varepsilon} d_T(\xi_x^{-1}(t), \xi_y^{-1}(t)).$$

In particular for $\varepsilon = 0$ we have

$$d_T(\xi_x^{-1}(e), \xi_y^{-1}(e)) = \sup_z d_X(xz, yz) = d(x, y).$$

Assuming d_X is right-invariant, d_Ξ is right-invariant, since

$$d_\Xi(\xi_x\xi_z, \xi_y\xi_z) = d_\Xi(\xi_{xz}, \xi_{yz}) = d_X(xz, yz) = d_X(x, y). \qquad \square$$

Remark. For $S = Tr_L(X)$, in view of $\xi_x(\tau_y)(z) = \tau_y\tau_x^{-1}(z)) = \tau_{yx^{-1}}(z)$, we observe that, for d_X right-invariant,

$$\begin{aligned} \sup_w d_H(\xi_x(\tau_w), \xi_y(\tau_w)) &= \sup_w \sup_z d_X(\xi_x(\tau_w)(z), \xi_y(\tau_w)(z)) \\ &= \sup_w \sup_z d_X(wx^{-1}z, wy^{-1}z) = \sup_w d_X(vxx^{-1}, vxy^{-1}) \\ &= \sup_v d_X(vy, vx), \end{aligned}$$

possibly with infinite value. (Here we have written $w = vx$.) Now $\bar{d}_X(y, x) = \sup_v d_X(vy, vx)$ is also a metric, which is left-invariant on the bounded elements under the related norm (cf. Proposition 4 of the previous section). Of course, if d_X were bi-invariant (both right- and left-invariant), we would have $\sup_w d_H(\xi_x(\tau_w), \xi_y(\tau_w)) = d_X(x, y)$.

11 Divergence in the bounded subgroup

For S a space and $\mathcal{A}$ a subgroup of $Auth(S)$, let $\varphi : \mathcal{A} \times S \to S$ be a continuous flow. We will write $\alpha(s) := \varphi^\alpha(s) = \varphi(\alpha, s)$. This is consistent with $\mathcal{A}$ being a subgroup of $Auth(S)$. As explained at the outset of Section 2, we have in mind two pairs $(\mathcal{A}, S)$, as follows.

Example 1 Take $S = X$ to be a topological group and $\mathcal{A} = T \subseteq \mathcal{H}(X)$ to be a subgroup of automorphisms of X. Then T is a topological group with supremum metric

$$d_T(t_1, t_2) = \sup_x d_X(t_1(x), t_2(x)).$$

Note that here $e_T = id_X$.

Example 2. $(\mathcal{A}, S) = (\Xi, T) = (X, T)$. Here X is identified with its second dual Ξ (of the preceding section).

Given a flow $\varphi(t, x)$ on $T \times X$, with T closed under translation, the action defined by

$$\varphi(\xi_x, t) := \xi_{x^{-1}}(t)$$

is continuous, hence a flow on $\Xi \times T$, which is identified with $X \times T$. Note that $\xi_{x^{-1}}(t)(e_X) = t(x)$, i.e. projection onto the e_X coordinate retrieves the T-flow φ. Here, for $\xi = \xi_{x^{-1}}$, writing $x(t)$ for the translate of t, we have

$$\xi(t) := \varphi^{\xi}(t) = \varphi(\xi, t) = x(t),$$

so that φ may be regarded as a X-flow on T.

We now formalize the notion of a sequence converging to the identity and divergent sequence. These are critical to the definition of regular variation [BOst13].

Definition. Let $\psi_n : X \to X$ be auto-homeomorphisms.
We say that a sequence ψ_n in $\mathcal{H}(X)$ *converges to the identity* if

$$||\psi_n|| = d^*(\psi_n, id) := \sup_{t \in X} d(\psi_n(t), t) \to 0.$$

Thus, for all t, we have $z_n(t) := d(\psi_n(t), t) \le ||\psi_n||$ and $z_n(t) \to 0$. Thus the sequence $||\psi_n||$ is bounded.

Illustrative examples. In $\mathbb{R}$ we may consider $\psi_n(t) = t + z_n$ with $z_n \to 0$. In a more general context, we note that a natural example of a convergent sequence of homeomorphisms is provided by a flow parametrized by *discrete time* (thus also termed a 'chain') towards a sink. If $\psi : \mathbb{N} \times X \to X$ is a flow and $\psi_n(x) = \psi(n, x)$, then, for each t, the orbit $\{\psi_n(t) : n = 1, 2, ...\}$ is the image of the real null sequence $\{z_n(t) : n = 1, 2, ...\}$.

Proposition. (i) *For a sequence ψ_n in $\mathcal{H}(X)$, ψ_n converges to the identity iff ψ_n^{-1} converges to the identity.*

(ii) Suppose X has abelian norm. For $h \in \mathcal{H}(X)$, if ψ_n converges to the identity then so does $h^{-1}\psi_n h$.

Proof. Only (ii) requires proof, and that follows from $||h^{-1}\psi_n h|| = ||hh^{-1}\psi_n|| = ||\psi_n||$, by the cyclic property, as in Proposition 7(c) of the previous section. □

Definitions.

1. Again let $\varphi_n : X \to X$ be auto-homeomorphisms. We say that the sequence φ_n in $\mathcal{G}$ *diverges uniformly* if for for any $M > 0$ we have, for ultimately all n, that

$$d(\varphi_n(t), t) \ge M, \text{ for all } t.$$

Equivalently, putting

$$d_*(h,h') = \inf_{x\in X} d(h(x), h'(x)),$$

$$d_*(\varphi_n, id) \to \infty.$$

2. More generally, let $\mathcal{A} \subseteq \mathcal{H}(S)$ with $\mathcal{A}$ a metrizable topological group. We say that α_n is a *pointwise divergent sequence in* $\mathcal{A}$ if, for each $s \in S$,

$$d_S(\alpha_n(s), s) \to \infty,$$

equivalently, $\alpha_n(s)$ does not contain a bounded subsequence.

3. We say that α_n is a *uniformly divergent sequence in* $\mathcal{A}$ if

$$||\alpha_n||_{\mathcal{A}} := d_{\mathcal{A}}(e_{\mathcal{A}}, \alpha_n) \to \infty,$$

equivalently, α_n does not contain a bounded subsequence.

Examples. In $\mathbb{R}$ we may consider $\varphi_n(t) = t + x_n$ where $x_n \to \infty$. In a more general context, a natural example of a uniformly divergent sequence of homeomorphisms is again provided by a flow parametrized by discrete time from a source to infinity. If $\varphi : \mathbb{N} \times X \to X$ is a flow and $\varphi_n(x) = \varphi(n,x)$, then, for each x, the orbit $\{\varphi_n(x) : n = 1,2,...\}$ is the image of the divergent real sequence $\{y_n(x) : n = 1,2,...\}$, where $y_n(x) := d(\varphi_n(x), x) \geq d_*(\varphi_n, id)$.

Remark. Our aim is to offer analogues of the topological vector space characterization of boundedness: for a bounded sequence of vectors $\{x_n\}$ and scalars $\alpha_n \to 0$ ([Ru-FA2] cf. Th. 1.30) $\alpha_n x_n \to 0$. But here $\alpha_n x_n$ is interpreted in the spirit of duality as $\alpha_n(x_n)$ with the homeomorphisms α_n converging to the identity.

Theoretical examples motivated by duality

1. Evidently, if $S = X$, the pointwise definition reduces to functional divergence in $\mathcal{H}(X)$ defined pointwise:

$$d_X(\alpha_n(x), x) \to \infty.$$

The uniform version corresponds to divergence in the supremum metric in $\mathcal{H}(X)$.

2. If $S = T$ and $\mathcal{A} = X = \Xi$, we have, by the Quasi-isometric Duality Theorem, that

$$d_T(\xi_{x(n)}(t), \xi_e(t)) \to \infty$$

iff

$$d_X(x_n, e_X) \to \infty,$$

and the assertion is ordinary divergence in X. Since

$$d_\Xi(\xi_{x(n)}, \xi_e) = d_X(x_n, e_X),$$

the uniform version also asserts that

$$d_X(x_n, e_X) \to \infty.$$

Recall that $\xi_x(s)(z) = s(\tau_x^{-1}(z)) = s(x^{-1}z)$, so the interpretation of Ξ as having the action of X on T was determined by

$$\varphi(\xi_x, t) = \xi_{x^{-1}}(t)(e) = t(x).$$

One may write

$$\xi_{x(n)}(t) = t(x_n).$$

When interpreting $\xi_{x(n)}$ as x_n in X acting on t, note that

$$d_X(x_n, e_X) \leq d_X(x_n, t(x_n)) + d_X(t(x_n), e_X) \leq ||t|| + d_X(t(x_n), e_X),$$

so, as expected, the divergence of x_n implies the divergence of $t(x_n)$.

We return to a study of $\mathcal{H}(X)$. We let $\mathcal{H}_u(X)$ denote the subgroup of uniformly continuous homeomorphisms, that is homeomorphisms α satisfying the condition that, for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$d(\alpha(x), \alpha(x')) < \varepsilon, \text{ for } d(x, x') < \delta. \tag{4}$$

Lemma 1 (Compare [dGMc] Cor. 2.13).
(i) *For fixed $\xi \in \mathcal{H}(X)$, the mapping $\tau \to \tau\xi$ is continuous.*
(ii) *For fixed $\alpha \in \mathcal{H}_u(X)$, the mapping $\beta \to \alpha\beta$ is uniformly continuous.*
(iii) *The mapping $(\alpha, \beta) \to \alpha\beta$ is continuous on $\mathcal{H}_u(X) \times \mathcal{H}_u(X)$.*

Proof. (i) We have

$$d^*(\tau\xi, \sigma\xi) = \sup d(\tau(\xi(t)), \sigma(\xi(t))) = \sup d(\tau(s), \sigma(s)) = d^*(\tau, \sigma).$$

(ii) For $\alpha \in \mathcal{H}_u(X)$ and given $\varepsilon > 0$, choose $\delta > 0$, so that (4) holds. Then, for β, γ with $d^*(\beta, \gamma) < \delta$, we have $d(\beta(t), \gamma(t)) < \delta$ for each t, and hence

$$d^*(\alpha\beta, \alpha\gamma) = \sup d(\alpha(\beta(t)), \alpha(\gamma(t))) \leq \varepsilon.$$

(iii) Again, for $\alpha \in \mathcal{H}_u(X)$ and given $\varepsilon > 0$, choose $\delta > 0$, so that (4) holds. Thus, for β, η with $d^*(\beta, \eta) < \delta$, we have $d(\beta(t), \eta(t)) < \delta$ for each t. Hence for ξ with $d^*(\alpha, \xi) < \varepsilon$ we obtain

$$\begin{aligned} d(\alpha(\beta(t)), \xi(\eta(t))) &\leq d(\alpha(\beta(t)), \alpha(\eta(t))) + d(\alpha(\eta(t)), \xi(\eta(t))) \\ &\leq \varepsilon + d^*(\alpha, \xi) \leq \varepsilon + \varepsilon. \end{aligned}$$

Consequently, we have

$$d^*(\alpha\beta, \xi\eta) = \sup d(\alpha(\beta(t)), \xi(\eta(t))) \leq 2\varepsilon.$$

□

Comment. See also [AdC] for a discussion of the connection between choice of metric and uniform continuity. The following result is of interest and extends to a countable family of autohomeomorphisms.

Proposition (**deGroot-McDowell Lemma,** [dGMc], Lemma 2.2). *Given an autohomeomorphism α of X, the metric on X may be replaced by a topologically equivalent one such that α is uniformly continuous.*

The next definition extends our earlier one from sequential to continuous limits.

Definition. Let $\{\psi_u : u \in I\}$ for I an open interval be a family of homeomorphisms (cf. [Mon2]). Let $u_0 \in I$. Say that ψ_u *converges to the identity as* $u \to u_0$ if

$$\lim_{u \to u_0} ||\psi_u|| = 0.$$

This property is preserved under topological conjugacy; more precisely we have the following result, whose proof is routine and hence omitted.

Lemma 2. *Let σ be a homeomorphism which is uniformly continuous, and write $u_0 = \sigma z_0$.*

If $\{\psi_z : z \in B_\varepsilon(z_0)\}$ converges to the identity as $z \to z_0$, then as $u \to u_0$ so does the conjugate $\{\psi_u = \sigma\psi_z\sigma^{-1} : u \in B_\varepsilon(u_0), u = \sigma z\}$.

Lemma 3 *Suppose that the homeomorphisms* $\{\varphi_n\}$ *are uniformly divergent,* $\{\psi_n\}$ *are convergent and* σ *is bounded, i.e. is in* $\mathcal{H}(X)$*. Then* $\{\varphi_n\sigma\}$ *is uniformly divergent and likewise* $\{\sigma\varphi_n\}$*. In particular* $\{\varphi_n\psi_n\}$ *is uniformly divergent, and likewise* $\{\varphi_n\sigma\psi_n\}$*, for any bounded homeomorphism* $\sigma \in \mathcal{H}(X)$*.*

Proof. Consider $s := ||\sigma|| = \sup d(\sigma(x), x) > 0$. For any M, from some n onwards we have

$$d_*(\varphi_n, id) = \inf_{x\in X} d(\varphi_n(x), x) > M,$$

i.e.

$$d(\varphi_n(x), x) > M,$$

for all x. For such n, we have $d_*(\varphi_n\sigma, id) > M - s$, i.e. for all t we have

$$d(\varphi_n(\sigma(t)), t)) > M - s.$$

Indeed, otherwise at some t this last inequality is reversed, and then

$$\begin{aligned} d(\varphi_n(\sigma(t)), \sigma(t)) &\le d(\varphi_n(\sigma(t)), t) + d(\sigma(t), t) \\ &\le M - s + s = M. \end{aligned}$$

But this contradicts our assumption on n with $x = \sigma(t)$. Hence $d_*(\varphi_n\sigma, id) > M - s$ for all large enough n.

The other cases follow by the same argument, with the interpretation that now $s > 0$ is arbitrary; then we have for all large enough n that $d(\psi_n(x), x) < s$, for all x. □

Remark. Lemma 3 says that the filter of sets (countably) generated from the sets

$$\{\varphi|\varphi : X \to X \text{ is a homeomorphism and } ||\varphi|| \ge n\}$$

is closed under composition with elements of $\mathcal{H}(X)$.

We now return to the notion of divergence.

Definition. We say that pointwise (resp. uniform) divergence is *unconditional* in $\mathcal{A}$ if, for any (pointwise/uniform) divergent sequence α_n,
(i) for any bounded σ, the sequence $\sigma\alpha_n$ is (pointwise/uniform) divergent;

and,
(ii) for any ψ_n convergent to the identity, $\psi_n \alpha_n$ is (pointwise/uniform) divergent.

Remarks. In clause (ii) each of the functions ψ_n has a bound depending on n. The two clauses could be combined into one by requiring that if the bounded functions ψ_n converge to ψ_0 in the supremum norm, then $\psi_n \alpha_n$ is (pointwise/uniform) divergent.

By Lemma 3 uniform divergence in $\mathcal{H}(X)$ is unconditional. We move to other forms of this result.

Proposition. *If the metric on $\mathcal{A}$ is left- or right-invariant, then uniform divergence is unconditional in $\mathcal{A}$.*

Proof. If the metric $d = d_{\mathcal{A}}$ is left-invariant, then observe that if β_n is a bounded sequence, then so is $\sigma\beta_n$, since

$$d(e, \sigma\beta_n) = d(\sigma^{-1}, \beta_n) \le d(\sigma^{-1}, e) + d(e, \beta_n).$$

Since $||\beta_n^{-1}|| = ||\beta_n||$, the same is true for right-invariance. Further, if ψ_n is convergent to the identity, then also $\psi_n \beta_n$ is a bounded sequence, since

$$d(e, \psi_n \beta_n) = d(\psi_n^{-1}, \beta_n) \le d(\psi_n^{-1}, e) + d(e, \beta_n).$$

Here we note that, if ψ_n is convergent to the identity, then, so is ψ_n^{-1} by continuity of inversion (or by metric invariance). The same is again true for right-invariance. □

The case where the subgroup $\mathcal{A}$ of autohomeomorphisms is the translations Ξ, though immediate, is worth noting.

Theorem 1 (The case $\mathcal{A} = \Xi$). *If the metric on the group X is left- or right-invariant, then uniform divergence is unconditional in $\mathcal{A} = \Xi$.*

Proof. We have already noted that Ξ is isometrically isomorphic to X. □

Remarks.

1. If the metric is bounded, there may not be any divergent sequences.

2. We already know from Lemma 3 that uniform divergence in $\mathcal{A} = \mathcal{H}(X)$ is unconditional.

3. The unconditionality condition (i) corresponds directly to the technical condition placed in [BajKar] on their filter $\mathcal{F}$. In our metric setting, we thus employ a stronger notion of limit to infinity than they do. The filter implied by the pointwise setting is generated by sets of the form

$$\bigcap_{i\in I}\{\alpha : d_X(\alpha_n(x_i), x_i) > M \text{ ultimately}\} \text{ with } I \text{ finite.}$$

However, whilst this is not a countably generated filter, its projection on the x-coordinate:

$$\{\alpha : d_X(\alpha_n(x), x) > M \text{ ultimately}\},$$

is.

4. When the group is locally compact, 'bounded' may be defined as 'pre-compact', and so 'divergent' becomes 'unbounded'. Here divergence is unconditional (because continuity preserves compactness).

The supremum metric need not be left-invariant; nevertheless we still do have unconditional divergence.

Theorem 2. *For $\mathcal{A} \subseteq \mathcal{H}(S)$, pointwise divergence in $\mathcal{A}$ is unconditional.*

Proof. For fixed $s \in S$, $\sigma \in \mathcal{H}(S)$ and $d_X(\alpha_n(s), s))$ unbounded, suppose that $d_X(\sigma\alpha_n(s), s))$ is bounded by K. Then

$$\begin{aligned} d_S(\alpha_n(s), s)) &\leq d_S(\alpha_n(s), \sigma(\alpha_n(s))) + d_S(\sigma(\alpha_n(s)), s) \\ &\leq ||\sigma||_{\mathcal{H}(S)} + K, \end{aligned}$$

contradicting that $d_S(\alpha_n(s), s))$ is unbounded. Similarly, for ψ_n converging to the identity, if $d_S(\psi_n(\alpha_n(x)), x)$ is bounded by L, then

$$\begin{aligned} d_S(\alpha_n(s), s)) &\leq d_S(\alpha_n(s), \psi_n(\alpha_n(s))) + d_S(\psi_n(\alpha_n(s)), s) \\ &\leq ||\psi_n||_{\mathcal{H}(S)} + L, \end{aligned}$$

contradicting that $d_S(\alpha_n(s), s))$ is unbounded. □

Corollary 1. *Pointwise divergence in $\mathcal{A} \subseteq \mathcal{H}(X)$ is unconditional.*

Corollary 2. *Pointwise divergence in $\mathcal{A} = \Xi$ is unconditional.*

Proof. In Theorem 2, take $\alpha_n = \xi_{x(n)}$. Then unboundedness of $d_T(\xi_{x(n)}(t), t)$ implies unboundedness of $d_T(\sigma\xi_{x(n)}(t), t)$ and of $d_T(\psi_n\xi_{x(n)}(t)), t)$. □

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