A Matrix Method for Flow Polynomials

Norman Biggs

Department of Mathematics
London School of Economics
Houghton Street
London WC2A 2AE
U.K.
n.l.biggs@lse.ac.uk

July 2008
CDAM Research Reports Series LSE-CDAM 2008-08

Abstract

A method for calculating flow polynomials based on a transfer matrix is described. It is analogous to the method used for chromatic polynomials, although there is as yet no parallel development of the theory. The new method is applied to a family of bracelets, and the limit curves for the flow roots are obtained. There is an unexplained similarity between these calculations and the corresponding ones for the chromatic polynomials of the same family.

A Matrix Method for Flow Polynomials

1. Introduction

The *flow polynomial* $F(G;k)$ gives the number of proper k-flows on a graph G. It is analogous to the *chromatic polynomial* $C(G;k)$, which gives the number of proper k-colourings of G. In fact, both these polynomials are partial evaluations of a two-variable polynomial [**13**], now known as the *Tutte polynomial* $T(G;x,y)$:

$$F(G;k) = (-1)^{e-v+1}T(G;0,1-k), \qquad C(G;k) = (-1)^{v-1}T(G;1-k,0).$$

where v and e are, respectively, the numbers of vertices and edges of G.

Many hard problems can be expressed in terms of these polynomials, and for that reason they have been extensively studied by graph theorists. In a different form, the polynomials arise in statistical physics, and in that context their complex roots are of particular interest [**7,9,12**].

There has been some progress on the theory of chromatic polynomials and their complex roots in the situation when the graph has a type of cyclic symmetry. A *bracelet* is a graph G_n formed from n copies of a graph B by joining each copy to the next (in cyclic order) with a constant set of linking edges L. In this case it is possible to define a *transfer matrix* $T_L(k)$, with rows and columns indexed by the proper k-colourings of B, such that $C(G_n;k)$ is equal to the trace of $T_L(k)^n$, for all n. This means that the chromatic polynomials can be expressed in terms of the spectrum of $T_L(k)$. The fact that $T_L(k)$ commutes with the obvious representation of the symmetric group Sym_k on the set of proper k-colourings leads to an extensive theory, and a standard form for $C(G_n;k)$ [**4**]. An added bonus is that the standard form is well adapted to study of the complex roots, using the theory of equimodular curves [**2**].

In this paper a similar construction for the flow polynomial is described, based on a new type of transfer matrix, and it is used to calculate $F(G_n;k)$ for an interesting family of bracelets. Remarkably, the results are very similar to those for $C(G_n;k)$, although there is (as yet) no general theory to explain why. For example, there is no obvious action of Sym_k on the set of proper k-flows, and thus no standard form of spectral decomposition for the transfer matrix. (For other approaches to the transfer matrix, see [**10**] and the references given therein.)

The family of bracelets G_n that we use to illustrate our method have $B = K_3$ and a specific linking set L. For odd values of n, G_n is isomorphic to a graph known as a *Quartic Möbius Ladder* QML_{3n}, which turns up in several contexts, for example in recent work on matroids by Mayhew, Royle, and Whittle [**11**]. The graph QML_{2m+1} is constructed by taking an odd cycle C_{2m+1} and adding diagonals joining each vertex to the two opposite vertices. Explicitly, if the vertices are the integers mod $2m+1$, each vertex x is joined to $x \pm 1$ and $x \pm m$.

Theorem 1.1 Let G_n be the bracelet formed by n copies of K_3 with linking set $L = \{12, 31, 32\}$. When n is an odd number, G_n is isomorphic to QML_{3n}.

Proof Let $n = 2r+1$. As defined above, QML_{3n} is the graph with vertices the integers mod $6r+3$ (denoted by $0, 1, \ldots, 6r+2$), and edges joining each vertex x to $x \pm 1$ and $x \pm (3r+1)$. Denote the vertices of G_{2r+1} by the $2r+1$ triples i_1, i_2, i_3, $(0 \leq i \leq 2r)$, each such triple forming a copy of K_3. Then it is easy to check that the following is a graph isomorphism.

$$i_1 \mapsto 3i(r+1), \quad i_2 \mapsto 3(ir+i+r)+1, \quad i_3 \mapsto 3(ir+i+r)+2, \quad (i = 0, 1, \ldots, 2r).$$

□

2. Definitions

The terminology for flows is not standardized [**6**, **8**]. For our purposes it will suffice to use definitions that apply to a graph G with vertex-set V and edge-set E, and no loops or multiple edges, so that each edge $e \in E$ is an unordered pair of distinct vertices $e = \{u, v\}$. We shall choose one of the ordered pairs uv, vu, as an *oriented* edge, and denote the set of oriented edges by E also.

Let K be an abelian group of order k. A function f from the set of oriented edges to K defines a unique function, also denoted by f, on the ordered pairs xy (where either $xy \in E$ or $yx \in E$), by the rule $f(yx) = -f(xy)$. We shall be concerned with functions satisfying two conditions:

$$\text{for all } uv \in E \qquad f(uv) \neq 0;$$

$$\text{for all } v \in V \qquad \sum_u f(uv) = 0;$$

where the sum is taken over all u for which uv or vu is in E. The first condition is expressed by saying that f is *nowhere-zero*, and the second is known as the *conservation law.* When $f : E \to K$ satisfies these conditions, we shall say that it is a *proper* K-flow on G.

It is well known [**6**, **8**] that the number of proper K-flows depends only on the order k of the group K, and this number is a polynomial function of k that we refer to as the *flow polynomial.* For definiteness, we shall take flow-values in the group $\mathbb{Z}_k$, the integers mod k, written $0, 1, 2, \ldots, k-1$, and denote $\mathbb{Z}_k \setminus \{0\}$ by $\mathbb{Z}_k^*$.

In the case of a bracelet G_n, the edges of each copy of B are assumed to have the same fixed orientation, and the linking edges are considered to have the orientation that leads from one copy of B to the next one. Let F denote the set of functions $f : L \to \mathbb{Z}_k^*$. Given $f, f' \in F$, say that a function h from the oriented edges of B to $\mathbb{Z}_k^*$ is *compatible* with f and f' if the following holds: if f corresponds to a 'flow' into B, and f' corresponds to a 'flow' out of B, then h combines with f and f' to define a function that satisfies the conservation law at the vertices of B. Note that although the functions f,f', and h are required to be nowhere-zero, they are not required to satisfy the conservation law individually.

The following definition generalizes one proposed in [**5**]. Let T be the matrix with rows and columns indexed by the elements of F and entries given by

$$T_{ff'} \;=\; \text{the number of functions } h \text{ that are compatible with } f \text{ and } f'.$$

Note that the terms $T_{ff'}$ will normally be functions of k.

Theorem 2.1 The number of proper k-flows on G_n is equal to the trace of T^n.

Proof By definition

$$\operatorname{tr} T^n \;=\; \sum_{f_1} (T^n)_{f_1 f_1} \;=\; \sum_{f_1} \sum_{f_2, f_3, \ldots, f_n} T_{f_1 f_2} T_{f_2 f_3} \cdots T_{f_n f_1}.$$

Each term of the last sum is equal to the number of n-tuples of functions $(h_1, h_2, \ldots, h_n)$ such that h_i is compatible with f_i and f_{i+1} for $i = 1, 2, \ldots, n$ (setting $f_{n+1} = f_1$). This is precisely the number of ways of extending the 'flow' f_1 on one copy of L to a proper flow on the entire bracelet G_n. Hence the result.

□

For $i \in \mathbb{Z}_k$, let

$$F_i \;=\; \{f \in F \mid \sum_{l \in L} f(l) = i\}.$$

Clearly $T_{ff'} = 0$ unless f and f' are in the same set F_i. The set F is the disjoint union of $F_0, F_1, \ldots, F_{k-1}$, and so T is a block diagonal matrix

$$\operatorname{diag}(T_0, T_1, \ldots, T_{k-1}),$$

where each T_i is a square matrix of size $|F_i|$.

Lemma 2.2 Given L, let $n_L(k)$ be the polynomial function of k defined by

$$n_L(k) = k^{-1}\Big((k-1)^{|L|} - (-1)^{|L|}\Big).$$

Then

$$|F_i| \;=\; \begin{cases} n_L(k) & \text{if } i \neq 0, \\ n_L(k) + (-1)^{|L|} & \text{if } i = 0. \end{cases}$$

Proof We give a proof by induction on $|L|$ because it involves a decomposition of F_i that will be useful later. The result is easily verified when $|L| = 1$. Suppose that $|L| \geq 2$. Fix $m \in L$, let $L' = L \setminus \{m\}$, and let F' denote the set of functions $f' : L' \to \mathbb{Z}_k^*$.

Let $F_i(s)$ denote the subset of F_i containing the functions f such that $f(m) = s$. The function $f \mapsto f|_{L'}$ is a bijection from $F_i(s)$ to F'_{i-s}. Hence, assuming the result for L', we have

$$|F_0| = \sum_{s\neq 0} |F'_{-s}| = (k-1)n_{L'}(k) = n_L(k) + 1.$$

When $i \neq 0$,

$$|F_i| = \sum_{s\neq 0} |F'_{i-s}| = |F'_0| + \sum_{s\neq 0,i} |F'_{i-s}|$$
$$= n_{L'}(k) + 1 + (k-2)n_{L'}(k) = n_L(k).$$

□

3. The transfer matrix for the family G_n

In the case $L = \{12, 31, 32\}$, for a given function $f : L \to \mathbb{Z}_k^*$ we write $f = (a, b, c)$, where $f(12) = a, f(31) = b, f(32) = c$. Thus F_i denotes the set of such functions with $a + b + c = i$ $(i = 0, 1, \ldots k-1)$, and the transfer matrix T is a block diagonal matrix $\mathrm{diag}(T_0, T_1, \ldots, T_{k-1})$, where the rows and columns of T_i are indexed by the elements of F_i. Since $|L| = 3$ here, the size of T_0 is $|F_0| = k^2 - 3k + 2$, and the size of T_i $(i \neq 0)$ is $|F_i| = k^2 - 3k + 3$.

Lemma 3.1 Let $E = \{12, 23, 31\}$ be the set of oriented edges of K_3. The function $h : E \to \mathbb{Z}_k^*$ is compatible with $f = (a, b, c)$ and $f' = (a', b', c')$ in F_i if and only if it has the form

$$h(12) = x, \quad h(23) = x + i - b, \quad h(31) = x + a' - b,$$

and all terms are non-zero.

Proof Suppose the values of h are $x, x+\theta, x+\phi$. The conservation law at vertices 1 and 2 implies that

$$b + x + \phi = x + a', \quad \text{so that} \quad \phi = a' - b,$$

$$x + a + c = x + \theta, \quad \text{so that} \quad \theta = i - b.$$

□

Lemma 3.2 Suppose that f and f' are in F_i. Then, if $b = i$,

$$(T_i)_{ff'} = \begin{cases} k-1 & \text{if } a' = i; \\ k-2 & \text{if } a' \neq i; \end{cases}$$

and if $b \neq i$

$$(T_i)_{ff'} = \begin{cases} k-2 & \text{if } a' = b \text{ or } a' = i; \\ k-3 & \text{if } a' \neq b \text{ and } a' \neq i. \end{cases}$$

Proof Suppose $b = i$. Then the compatible functions h are of the form $(x, x, x+a'-i)$ for $x \in \mathbb{Z}_k^*$, and the terms are nonzero except when x is $i - a'$. If $a' = i$ there are $k-1$ such values of x, and if $a' \neq i$ there are $k-2$.

Suppose $b \neq i$. Then the compatible functions h are of the form

$$(x,\ x + i - b,\ x + a' - b)$$

for $x \in \mathbb{Z}_k^*$. The second term is nonzero except when x is $b - i$. If $a' = b$ or $a' = i$ the third term is equal to the first or second term, and so there are $k-2$ nonzero possibilities. If $a' \neq b$ and $a' \neq i$ all three terms are distinct and there are $k-3$ possibilities. □

4. Calculation of the eigenvalues

Lemma 4.1 The rank of each matrix T_i $(i = 0, 1, \ldots, k-1)$ is at most $k-1$.

Proof Let $F_i(s)$ denote the set of $f \in F_i$ for which $b = s$. According to Lemma 3.2, for each such f the values of $(T_i)_{ff'}$ are the same. In other words, all rows of T_i corresponding to $f \in F_i(s)$ are the same. Consequently T_i has at most $k-1$ distinct rows, one for each $s \neq 0$. □

It follows that the matrices T_i have many eigenvalues equal to 0, and these do not contribute to the trace of T^n. In order to determine the nonzero eigenvalues, we construct the corresponding eigenvectors. For each $i \in \mathbb{Z}_k$ define column vectors u^i and y_s^i $(s \in \mathbb{Z}_k^*)$, indexed by the functions $f \in F_i$, as follows:

$$(u^i)_f = 1 \text{ (for all } f \in F_i), \qquad (y_s^i)_f = \begin{cases} 1 & \text{if } f \in F_i(s); \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.2 The nonzero eigenvalues of T_0 are

$$(k-2)^3 \quad \text{with multiplicity } 1,$$

$$1 \quad \text{with multiplicity } \frac{1}{2}(k-1) \quad (k \text{ odd}),$$

$$-1 \quad \text{with multiplicity } \frac{1}{2}(k-3) \quad (k \text{ odd}).$$

(The last two multiplicities are slightly different when k is even, but for our purposes we do not need them explicitly.)

Proof The expression for T_0 takes a particularly simple form, since when $i = 0$ neither b nor a' can be equal to i. It follows from Lemma 3.2 that the entries of T_0 are given by

$$(T_0)_{ff'} = \begin{cases} k-2 & \text{if } a' = b; \\ k-3 & \text{if } a' \neq b. \end{cases}$$

We have

$$(T_0 y_s^0)_f = \sum_{f' \in F_0} (T_0)_{ff'} (y_s^0)_{f'} = \sum_{f' \in F_0(s)} (T_0)_{ff'},$$

where the last sum has $|F_0(s)| = k-2$ terms, corresponding to the functions $f' = (a', s, -(a'+s))$ with $a' \neq 0, -s$.

If $f \in F_0(-s)$, that is $b = -s$, then necessarily $a' \neq b$ and so all $k-2$ terms are equal to $k-3$, and the sum is $k^2 - 5k + 6$. If $b \neq -s$, then one term in the sum has $a' = b$ and is equal to $k-2$, while the remaining $k-3$ terms are equal to $k-3$, so the sum is $(k-2) + (k-3)^2 = k^2 - 5k + 7$. Hence

$$T_0 y_s^0 = (k^2 - 5k + 7)u^0 - y_{-s}^0.$$

Since $u^0 = y_1^0 + y_2^0 + \cdots + y_{k-1}^0$,

$$T_0 u^0 = (k-1)(k^2 - 5k + 7) - (y_{k-1}^0 + y_{k-2}^0 + \cdots + y_1^0) = (k-2)^3 u^0.$$

Suppose k is odd. The formula for $T_0 y_s^0$ implies that, for $s = 1, 2, \ldots, \frac{1}{2}(k-1)$ the vectors $y_s^0 - y_{-s}^0$ are eigenvectors with eigenvalue 1, and for $s = 1, 2, \ldots, \frac{1}{2}(k-3)$ the vectors $y_s^0 + y_{-s}^0 - y_{s+1}^0 - y_{-(s+1)}^0$ are eigenvectors with eigenvalue -1. □

Theorem 4.3 For each $i \neq 0$ the nonzero eigenvalues of T_i are the eigenvalues of

$$\begin{pmatrix} (k-2)(k^2-4k+6) & k^2-4k+5 \\ k^2-4k+5 & k-2 \end{pmatrix}$$

(once each), 1, and -1 (both with multiplicity $\frac{1}{2}(k-3)$, provided k is odd).

Proof We have

$$(T_i y_s^i)_f = \sum_{f' \in F_i} (T_i)_{ff'} (y_s^i)_{f'} = \sum_{f' \in F_i(s)} (T_i)_{ff'}.$$

Suppose $s = i$. Then the last sum has $|F_i(i)| = k-1$ terms, corresponding to the functions $f' = (a', i, -a')$ with $a' \neq 0$. If $b = i$ the term with $a' = i$ is the only one equal to $k-1$, and the other $k-2$ terms are equal to $k-2$. Hence in this case the sum is $k^2 - 3k + 3$. If $b \neq i$ the two terms with $a' = b$ and $a' = i$ are equal to $k-2$, and the remaining $k-3$ terms are equal to $k-3$. Hence in this case the sum is $k^2 - 4k + 5$. It follows that

$$T_i y_i^i = (k^2 - 4k + 5)u^i + (k-2)y_i^i.$$

Suppose $s \neq i$. Then the last sum has $|F_i(s)| = k-2$ terms, corresponding to the functions $f' = (a', s, i - (a' + s))$ with $a' \neq 0, i - s$.

If $b = i$ the term with $a' = i$ is the only one equal to $k-1$, and the other $k-3$ terms are equal to $k-2$. Hence in this case the sum is $k^2 - 4k + 5$. If $b = i - s$ the term with $a' = i - s$ is equal $k-2$ and the remaining $k-3$ terms are equal to $k-3$, so the sum is $k^2 - 5k + 7$. If $b \neq i, i - s$ the two terms with $a' = b$ and $a' = i$ are equal to $k-2$, and the remaining $k-4$ terms are equal to $k-3$. Hence in this case the sum is $k^2 - 5k + 8$.

It follows that

$$T_i y_s^i = (k^2 - 5k + 8)u^i + (k-3)y_i^i - y_{i-s}^i \quad (s \neq i).$$

Since $u^i = y_1^i + y_2^i + \cdots + y_{k-1}^i$,

$$\begin{aligned} T_i u^i = (k-2)(k^2 - 5k + 8)u^i + (k-2)(k-3)y_i^i - (u^i - y_i^i) \\ +(k^2 - 4k + 5)u^i + (k-2)y_i^i \\ = (k-2)(k^2 - 4k + 6)u^i + (k^2 - 4k + 5)y_i^i. \end{aligned}$$

Hence the action of T_i on the space spanned by u^i and y_i^i is represented by the matrix given in the statement of the theorem, and two of its eigenvalues are the eigenvalues of this matrix.

The formula for $T_i y_s^i$ implies that vectors of the form $y_s^i - y_{i-s}^i$ are eigenvectors with eigenvalue 1. When k is odd there are $\frac{1}{2}(k-3)$ linearly independent ones, corresponding to s taking the values $1, 2, \ldots, \frac{1}{2}(k-1)$, excepting either $s = i$ or $s = -i$, whichever is in the range. Similarly vectors of the form

$y^i_s + y^i_{i-s} - y^i_{s+1} - y^i_{i-s-1}$ are eigenvectors with eigenvalue -1, and when k is odd there are $\frac{1}{2}(k-3)$ linearly independent ones. □

5. The flow polynomial and its roots

We have shown that the nonzero eigenvalues of T_0 are $(k-2)^3$, 1, and -1 (with appropriate multiplicities), and the nonzero eigenvalues of each of the $k-1$ matrices T_i $(i \neq 0)$ are the eigenvalues of a 2×2 matrix, and 1, -1 (with appropriate multiplicities).

When k is odd, the total multiplicities of 1 and -1 are, respectively,

$$\frac{1}{2}(k-1) + (k-1) \times \frac{1}{2}(k-3) = \frac{1}{2}(k^2 - 3k + 2) \quad \text{and}$$

$$\frac{1}{2}(k-3) + (k-1) \times \frac{1}{2}(k-3) = \frac{1}{2}(k^2 - 3k).$$

Since the flow polynomial is determined by its form when k is odd, we have the required result.

Theorem 5.1 The flow polynomial of the bracelet G_n is

$$(k-2)^{3n} + (k-1)\,\mathrm{tr}M(k)^n + \epsilon_n(k),$$

where

$$M(k) = \begin{pmatrix} (k-2)(k^2-4k+6) & k^2-4k+5 \\ k^2-4k+5 & k-2 \end{pmatrix},$$

$$\epsilon_n(k) = \frac{1}{2}(k^2 - 3k + 2)1^n + \frac{1}{2}(k^2 - 3k)(-1)^n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ k^2 - 3k + 1 & \text{if } n \text{ is even.} \end{cases}$$

□

The following discussion applies when n is odd, so that G_n is the Quartic Möbius Ladder QML_{3n}. The even case is essentially the same.

Let $w = k-2$, and regard w as a complex variable. Then the flow polynomial of QML_{3n} (n odd) can be written as

$$(w^3)^n + (w+1)(\lambda_1(w)^n + \lambda_2(w)^n) + 1,$$

where $\lambda_1(w)$ and $\lambda_2(w)$ are the roots of the characteristic equation of $M(w)$, which is:

$$\lambda^2 - (w^3 + 3w)\lambda - 1 = 0.$$

Technically, the characteristic equation determines an algebraic function that is defined on a suitable Riemann surface, and hence $\lambda(w)$, $\lambda_2(w)$ cannot be defined consistently throughout the complex plane. A naive viewpoint will suffice for most of our discussion, but we shall need to address this problem at the final stage of the analysis.

The theorem of Beraha, Kahane and Weiss [**1**] tells us that as $n \to \infty$ the roots of the flow polynomial cluster around parts of the *equimodular curves*, comprising the points w such that any two of the values w^3, $\lambda_1(w)$, $\lambda_2(w)$, 1 are equal in modulus. The theory developed in [**2**] provides implicit equations for the equimodular curves.

Theorem 5.2 (a) The eigenvalue w^3 is equal in modulus to one of $\lambda_1(w), \lambda_2(w)$ if and only if there is a real number $t \in [0, 4]$ such that

$$(4-t)w^{12} + 3(4-t)w^{10} + 9w^8 - (t-4)(t-1)w^6 + 3(t-2)w^4 + 1 = 0.$$

(b) The eigenvalues $\lambda_1(w), \lambda_2(w)$ are equal in modulus if and only if there is a real number $t \in [0, 4]$ such that

$$w^6 + 6w^4 + 9w^2 + t = 0.$$

Proof (a) The three values are the eigenvalues of the 3×3 matrix that reduces to a 1×1 block $[w^3]$ and a 2×2 block $M(w)$. The characteristic equations of the blocks are

$$\lambda + b_1 = 0, \qquad \lambda^2 + c_1\lambda + c_2 = 0,$$

$$\text{where} \quad b_1 = -w^3, \quad c_1 = -(w^3 + 3w), \quad c_2 = -1.$$

According to [**2**, Section 6], substituting the coefficients in the 'generic' expression

$$r_{1,2}(t, b_1, c_1, c_2) = (t^2 - 4t + 2)b_1^2c_2 - (t-2)(b_1^3c_1 + b_1c_1c_2) + b_1^4 + b_1^2c_1^2 + c_2^2,$$

produces an expression that has the property stated.

(b) For an irreducible 2×2 matrix the relevant generic expression is

$$r_2(t, c_1, c_2) = c_1^2 - tc_2,$$

which, for $M(w)$ gives the equation stated above. □

The equation $f(t, w) = 0$, where $f(t, w)$ is

$$(4-t)w^{12} + 3(4-t)w^{10} + 9w^8 - (t-4)(t-1)w^6 + 3(t-2)w^4 + 1$$

defines a curve $\mathcal{C}_f$ that can be plotted directly, but it is instructive to pursue the theoretical analysis a bit further. Since f is a polynomial of degree 12 in w, the curve $\mathcal{C}_f$ can be thought of as twelve images of the interval $[0, 4]$. These arcs will be smooth unless the Jacobian J_f of the transformation $w \mapsto f(t, w)$ vanishes. In fact, J_f is also the discriminant of f as a polynomial in w, which turns out to be

$$J_f = 2^{12}3^{12}\, t^{14}(t-4)^7(t^2-9t+27)^4.$$

This expression tells us several things about the arc-decomposition of $\mathcal{C}_f$. Since f reduces to a polynomial of degree 8 when $t = 4$, the end-points with $t = 4$ include a quadruple point at infinity. The finite end-points are the eight roots of

$$f(4, w) = (3w^4 + 1)^2 = 0,$$

that is, four double points at $w = (1/12)^{1/4}(\pm 1 \pm i)$, where the arcs join up in pairs.

The factorisation of J_f indicates that the points with $t = 0$ are particularly interesting. Since

$$f(0, w) = (2w^2 - 1)^2(w^2 + 1)^4$$

the twelve end-points with $t = 0$ are actually two double points at $w = \pm 1/\sqrt{2}$, and two quadruple points at $w = \pm i$. Also, the fact that J_f has a factor t^{14} indicates that $\mathcal{C}_f$ has singularities at some of these points, and it turns out that they are the points $\pm i$.

The Jacobian also indicates that there are no other singularities of $\mathcal{C}_f$, since the factor $(t^2 - 9t + 27)^4$ vanishes only for complex values of t. All these facts are evident from a plot of the curve $\mathcal{C}_f$ (Figure 1).

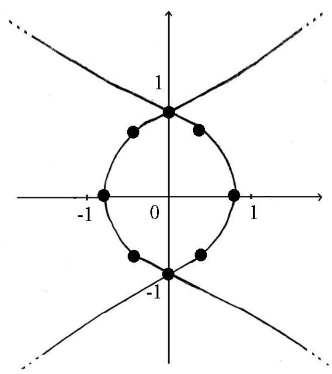

Fig.1: the curve $\mathcal{C}_f$, showing the twelve arcs

The Cartesian equation of $\mathcal{C}_f$ can be obtained by substituting $w = u + iv$ in $f(t, w)$, equating the real and imaginary parts to zero (remembering that t must be real), and eliminating t from these two equations. The eliminant has three factors, the one which defines $\mathcal{C}_f$ being

$$2(u^2 + v^2)^2(u^2 - v^2) + 3(u^2 + v^2)^2 = 1.$$

Note that the factor $u^2 - v^2$ in the leading term implies that the asymptotes are $u + v = 0$ and $u - v = 0$.

The curve $\mathcal{C}_g$ where $|\lambda_1(w)| = |\lambda_2(w)|$ is defined by the equation $g(t, w) = 0$ where

$$g(t, w) = w^6 + 6w^4 + 9w^2 + t = 0.$$

In this case there are six arcs with end-points given by

$$g(0, w) = w^2(w^2 + 3)^2 = 0, \qquad g(4, w) = (w^2 + 4)(w^2 + 1)^2 = 0.$$

So the six $t = 0$ points are three double points at $w = 0, \pm i\sqrt{3}$, and the six $t = 4$ points are double points at $w = \pm i$ and single points at $w = \pm 2i$. These four arcs combine to form the interval from $-2i$ to $2i$ on the imaginary axis.

Thus far we have determined the equimodular curves given by equality between the moduli of two of the first three of the values w^3, $\lambda_1(w)$, $\lambda_2(w)$, 1. The curves involving the value 1 are easily dealt with. The curve $|w^3| = 1$ is just the circle $|w| = 1$. For the curves $|\lambda_1(w)| = 1$ and $|\lambda_2(w)| = 1$ note that since $\lambda_1(w)\lambda_2(w) = -1$, it follows that $|\lambda_1(w)| = 1$ if and only if $|\lambda_1(w)| = |\lambda_2(w)|$. So this curve is identical with $\mathcal{C}_g$. The totality of the equimodular curves is thus as illustrated in Figure 2.

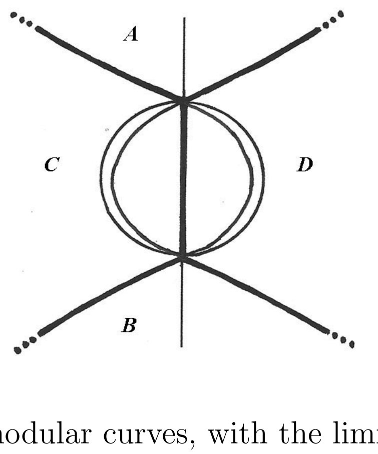

Fig.2: the equimodular curves, with the limit curve in bold

According to the Beraha-Kahane-Weiss theorem, the limit curve comprises only those parts of the equimodular curves that are *dominant*: that is, where the two values defining the curve are larger in modulus than the others. These parts are denoted by bold lines in Figure 2, and they divide the complex plane into four regions A, B, C, D as shown. It is easy to check that w^3 is the dominant eigenvalue in the regions A and B.

Some care is needed in describing the dominant eigenvalues in the regions C and D. However, since these regions are path-connected and contain no branch points of the algebraic function that represents $\lambda_1(w)$ and $\lambda_2(w)$, the monodromy principle can be applied. Let $w_1 = -1/\sqrt{2}$. If we choose the notation so that $\lambda_1(w_1) = 2\sqrt{2}$ and $\lambda_2(w_1) = -1/2\sqrt{2}$ then $\lambda_1(w)$ is uniquely defined in C and dominates in that region. Similarly, if $w_2 = 1/\sqrt{2}$ and we choose the notation so that $\lambda_1(w_2) = 1/2\sqrt{2}$ and $\lambda_2(w_2) = -2\sqrt{2}$ then $\lambda_2(w)$ is uniquely defined in D and dominates in that region.

Since the segments of the equimodular curve $\mathcal{C}_g$ from i to $2i$ and from $-i$ to $-2i$ lie in the regions A and B where w^3 dominates, they do not contribute to the limit curve. Similarly, the circle $|w| = 1$, and the parts of $\mathcal{C}_f$ for which the imaginary part of w lies between -1 and 1, lie in regions where the eigenvalues defining the curve are not dominant. Thus the limit curves are indeed as indicated in Figure 2.

6. The chromatic roots of the bracelets G_n

There are interesting, but largely unexplained, similarities between the results obtained above for the flow polynomials and the results for the chromatic polynomials of the same family.

There is a standard form for the chromatic polynomials of a family of bracelets with B a complete graph K_b and constant linking set L. It is a sum of terms, one for each partition π of a non-negative integer $\ell = |\pi|$ with $\ell \leq b$:

$$\sum_{\pi} m_\pi(k)\mathrm{tr}(T_{L,\pi}(k)^n),$$

where $m_\pi(k)$ is a polynomial in k with rational coefficients (independent of L), and $T_{L,\pi}(k)$ is a matrix whose entries are polynomials in k with integer coefficients. This formula is justified by theoretical results that depend ultimately on the fact that the transfer matrix centralizes an obvious action of the symmetric group Sym_k on the k-colourings of K_b [**4**].

When $b = 3$ an explicit recipe for calculating the relevant matrices $T_{L,\pi}$, for any linking set L, has been published [**3**, Section 6]. We briefly review this calculation for the case $L = \{12, 31, 32\}$. The first step is to list the set $\mathcal{M}(L)$ of ‘matchings’ that form a subset of L: $\mathcal{M}(L) = \{\emptyset, \{12\}, \{31\}, \{32\}, \{12, 31\}\}$. Each T_L matrix can then be expressed in terms of matrices U_M, where $M \in \mathcal{M}(L)$:

$$T_L = U_\emptyset - (U_{12} + U_{31} + U_{32}) + U_{12,31}.$$

Matchings with $|M| \leq |\pi|$ make zero contribution, and since $\mathcal{M}(L)$ contains no matchings of size three, we need only consider the partitions π of $0, 1$, and 2. According to the general theory we have $m_{[0]}(k) = 1$, and

$$m_{[1]}(z) = k - 1, \quad m_{[20]}(k) = \frac{1}{2}k(k-3), \quad m_{[11]}(k) = \frac{1}{2}(k-1)(k-2),$$

and it remains only to use the recipe given in [**3**] to calculate the corresponding matrices $T_{L,[0]}$, $T_{L,[1]}$, $T_{L,[20]}$, $T_{L,[11]}$.

For $\pi = [0]$ the U_M matrices are 1×1 matrices:

$$U_\emptyset = [k(k-1)(k-2)], \quad U_{12} = U_{31} = U_{32} = [(k-1)(k-2)], \quad U_{12,31} = [k-2].$$

Hence $T_{L,[0]} = [k^3 - 6k^2 + 12k - 8] = [(k-2)^3]$.

For $\pi = [1]$ the U_M matrices are 3×3 matrices: $U_\emptyset$ is the zero matrix and

$$U_{12} = \begin{pmatrix} 0 & (k-1)(k-2) & 0 \\ 0 & -(k-2) & 0 \\ 0 & -(k-2) & 0 \end{pmatrix}, \qquad U_{31} = \begin{pmatrix} -(k-2) & 0 & 0 \\ -(k-2) & 0 & 0 \\ (k-1)(k-2) & 0 & 0 \end{pmatrix},$$

$$U_{32} = \begin{pmatrix} 0 & -(k-2) & 0 \\ 0 & -(k-2) & 0 \\ 0 & (k-1)(k-2) & 0 \end{pmatrix}, \qquad U_{12,31} = \begin{pmatrix} 0 & k-2 & 0 \\ -1 & -1 & 0 \\ k-2 & 0 & 0 \end{pmatrix}.$$

Hence

$$T_{L,[1]} = -(U_{12} + U_{31} + U_{32}) + U_{12,31} = \begin{pmatrix} k-2 & -(k-2)(k-3) & 0 \\ k-3 & 2k-5 & 0 \\ -(k-2)^2 & k(k-2) & 0 \end{pmatrix}.$$

Note that the trace of $T^n_{L,[1]}$ is the same as the trace of the nth power of the 2×2 submatrix formed by the first two rows and columns.

Finally, for $\pi = [20]$ and $\pi = [11]$ there is only one non-zero matrix, $U_{12,31}$. In both cases it takes the form

$$U_{12,31} = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 0 & 0 \\ \gamma & 0 & 0 \end{pmatrix},$$

where $\alpha = -1$ when $\pi = [20]$ and $\alpha = 1$ when $\pi = [11]$. It follows that the trace of $T^n_{L,[20]}$ is $(-1)^n$ and the trace of $T^n_{L,[11]}$ is 1^n, so the corresponding terms in the standard formula are

$$\frac{1}{2}k(k-3)(-1)^n + \frac{1}{2}(k-1)(k-2)1^n = \begin{cases} k^2 - 3k + 1 & \text{if } n \text{ is even;} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 6.1 Let $M^*(k)$ be the matrix

$$\begin{pmatrix} k-2 & -(k-2)(k-3) \\ k-3 & 2k-5 \end{pmatrix}.$$

The chromatic polynomial of the bracelet G_n is

$$(k-2)^{3n} \ + \ (k-1)\mathrm{tr}M^*(k)^n \ + \ \eta_n(k),$$

where $\eta_n(k)$ is $k^2 - 3k + 1$ if n is even and 1 if n is odd. □

As usual, let $w = k - 2$ and consider w as a complex variable. When n is odd the chromatic polynomial can be written as

$$(w^3)^n + (w+1)(\mu_1(w)^n + \mu_2(w)^n) + 1,$$

where $\mu_1(w)$ and $\mu_2(w)$ are the roots of the characteristic equation of M^*:

$$\mu^2 - (3w-1)\mu + w^3 = 0.$$

Here too we must bear in mind the caveats mentioned in the previous section.

Applying the theory of equimodular curves as before, we conclude that two of the values $w^3, \mu_1(w), \mu_2(w)$ are equal in modulus if and only if there is a real number $t \in [0, 4]$ such that either $f^*(t, w) = 0$ or $g^*(t, w) = 0$, where

$$f^*(t, w) = w^6 - (3t-6)w^4 + (t^2 - 3t)w^3 + 9w^2 - 3tw + t$$

$$g^*(t, w) = tw^3 - 9w^2 + 6w - 1.$$

The first equation defines the curve $\mathcal{C}_{f^*}$ where $|w^3|$ is equal to one of $|\mu_1(w)|$, $|\mu_2(w)|$, and the second defines the curve $\mathcal{C}_{g^*}$ where $|\mu_1(w)| = |\mu_2(w)|$.

Since f^* is a polynomial of degree 6 in w, the curve $\mathcal{C}_{f^*}$ can be thought of as six images of the interval $[0, 4]$ (Figure 3). In this case the discriminant J_{f^*} turns out to be

$$J_{f^*} = 3^6 \, t^3(t-4)^7(t^2 + t + 7)^2.$$

The Cartesian equation of $\mathcal{C}_{f^*}$ is a *limaçon*:

$$u^4 - 3u^2 + 2u + 2u^2v^2 - 3v^2 + v^4 = 0.$$

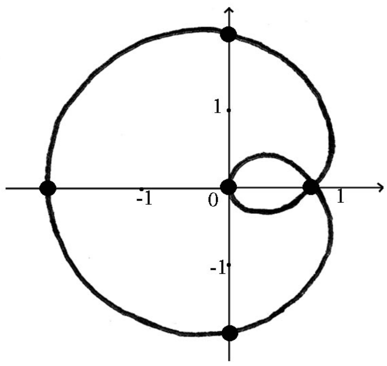

Fig.3: the curve $\mathcal{C}_{f^*}$, showing the six arcs

The curve $\mathcal{C}_{g^*}$ consists of three arcs that combine to form the interval $[1/4, \infty)$ on the real axis. This corresponds to the fact that the eigenvalues $\mu_1(w)$, $\mu_2(w)$ are complex conjugates when w belongs to this interval.

The equimodular curves involving the value 1 are easily dealt with. The curve $|w^3| = 1$ is just the circle $|w| = 1$, and since $\mu_1(w)\mu_2(w) = w^3$, it follows that $|\mu_1(w)| = 1$ if and only if $|\mu_2(w)| = |w^3|$, and $|\mu_2(w)| = 1$ if and only if $|\mu_1(w)| = |w^3|$. Hence these curves are, respectively, the inner and outer loops of the curve $\mathcal{C}_{f^*}$ that we have already determined.

A similar analysis to that given in Section 5 leads to the conclusion that w^3 is the dominant eigenvalue outside the outer loop of $\mathcal{C}_{f^*}$. One of the μ's, say $\mu_1(w)$, dominates between the loops of $\mathcal{C}_{f^*}$, and 1 dominates inside the inner loop. The equimodular curves $\mathcal{C}_{g^*} = [1/4, \infty)$ and and $|w^3| = 1$ are not dominant, because they lie in regions where another value dominates. The final conclusion is that the limit curve is precisely $\mathcal{C}_{f^*}$.

Gordon Royle has calculated by direct means the flow polynomials and the chromatic polynomials of all Quartic Möbius Ladders with up to 59 vertices, including the graphs $QML_{6r+3} = G_{2r+1}$. His method is based on the standard deletion-contraction algorithm, and the results confirm that the limit curves are as in Figures 2 and 3.

7. An unexplained identity

The μ functions that occur in the chromatic polynomial of G_n are given by

$$\mu(w)^2 - (3w - 1)\mu(w) + w^3 = 0.$$

For $w \neq 0$ define $\lambda(w) = -w^3\mu(-w^{-2})$. The resulting equation for $\lambda(w)$ is

$$\lambda(w)^2 - (w^3 + 3w)\lambda(w) - 1 = 0.$$

This is the equation for the λ functions occurring in the flow polynomial, as found in Section 5.
Write $A = \lambda_1(w)^n + \lambda_2(w)^n$, so that the flow polynomial F_n satisfies

$$F_n(w) - (w^{3n} + 1) \ = \ (w+1)A.$$

The chromatic polynomial C_n evaluated at $-w^{-2}$ satisfies

$$C_n(-w^{-2}) - \Big((-w^{-2})^{3n} + 1\Big) \ = \ (-w^{-2} + 1)(\mu_1(-w^{-2})^n + \mu_2(-w^{-2})^n)$$

$$= (-1)^n w^{-(3n+2)}(w^2 - 1)A.$$

Hence, provided $w \neq 0$

$$(w-1)F_n(w) \ = \ (-1)^{n+1} w^{3n+2} C_n(-w^{-2}) + e_n(w),$$

where

$$e_n(w) = (-1)^{n+1} w^{3n+2} + (w-1)(w^{3n}+1) - w^{-3n+2}.$$

Thus for all n, the flow polynomial and the chromatic polynomial for G_n differ essentially only in six terms.

Acknowledgement I am grateful to Gordon Royle for asking me about the chromatic and flow polynomials of the Quartic Möbius Ladders, and for supplying the experimental data on their roots.

References

1 S. Beraha, J. Kahane, N.J. Weiss. Limits of zeros of recursively defined families of polynomials, *Studies in foundations and combinatorics, Adv. Math. Suppl. Studies* 1 (1978) 213-232.
2 N.L. Biggs. Equimodular curves, *Discrete Mathematics* 259 (2002) 37-57.
3 N.L. Biggs, M.H. Klin, P. Reinfeld. Algebraic methods for chromatic polynomials, *European J. Combinatorics* 25 (2004) 147-160.
4 N.L. Biggs. Specht modules and chromatic polynomials, *J. Combinatorial Theory B* 92 (2004) 359-377.
5 N.L. Biggs, G. Royle. The flow roots of the quartic plane ladders. *Isaac Newton Institute Preprint Series* (2008).
6 J.A. Bondy, U.S.R. Murty. *Graph Theory*. Springer (2008).
7 S-C. Chang, R. Shrock. Flow polynomials and their asymptotic limits for lattice strip graphs. *J. Statist. Physics* 112 (2003) 815-879.

8 R. Diestel. *Graph Theory.* Springer (1997).
9 J.L. Jacobsen, J.Salas. Transfer matrices and partition-function zeros for antiferromagnetic Potts models II, III. *J. Statist. Phys.* 104 (2001) 701-723; 112 (2003) 921-1017.
10 H. Lundow, K. Markström. Exact and approximate compression of transfer matrices for graph homomorphisms. *LMS J. Comput. Math.* 11 (2008) 1-14.
11 D. Mayhew, G. Royle, G. Whittle. The internally 4-connected binary matroids with no $M(K_{3,3})$ minor, submitted.
12 J. Salas, A. Sokal. Transfer matrices and partition-function zeros for antiferromagnetic Potts models. *J. Statist. Phys.* 104 (2001) 609-699.
13 W.T. Tutte. A contribution to the theory of chromatic polynomials. *Canad. J. Math.* 6 (1954) 80-91.