

Pricing of contingent claims in a two-dimensional model with random dividends*

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We study a model where two assets are paying dividends with rates changing from one fixed value to another when any credit event occurs. The credit events are associated with the first times when the asset values fall to some given constant levels. The behavior of asset values is described by exponential diffusion processes with random drift rates and independent driving Brownian motions. We obtain closed form expressions for the ex-dividend prices of certain barrier-type contingent claims with structure similar to first- and second-to-default options in credit risk theory.

1 Introduction

One of the known advantages of structural modeling is the explicit form of the credit events that are associated with the passage times of the related asset value processes on some given

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lower barriers. Initiated by Merton [10] and Black and Cox [2], the passage time approach for credit risk theory was further developed and, as a result, a number of subsequent generalizations was obtained (see, e.g., Bielecki and Rutkowski [1; Chapters II and III] or Schönbucher [14; Chapter IX] for an extensive overview). In the present paper, we study a first passage time model for two assets paying dividends, which rates are random and change in time from one fixed value to another. The times of change of dividend rates are assumed to be the first times, when the asset values reach some given lower constant barriers. Such a model corresponds to the situation where the fall of one of the asset values impacts the change of dividend rates not only of the same asset but of the other ones as well. This may happen, for example, with one firm having several branches, where the change of the dividend rate of one of the branches makes an influence on the dividend policy not of the same branch but of the other ones as well. The obtained structure of dependent credit events can also be a contribution to the wide range of first passage time models with dependent defaults (see, e.g., Zhou [17], Giesecke [5], Overbeck and Schmidt [11], Valužis [16] and also [1; Chapter X] or [14; Chapter X] for further references). Note that some other models with random dividends were earlier considered in the literature (see, e.g., Geske [4]).

The purpose of the present paper is to derive the ex-dividend prices of certain barrier-type contingent claims, which structure is similar to the so-called first- and second-to-default options, in the models where the information is generated by both asset values and by the value of one of the assets only. The risk-neutral dynamics of the asset values is modeled by geometric Brownian motions with random drift rates changing their fixed values from one to another at the first times when the value processes fall to some constant levels during the allowed infinite time horizon. For simplicity of exposition, we restrict our consideration to a two-dimensional case and assume that the driving Brownian motions are independent. The prices of the claims are expressed through the transition density of the joint marginal distribution of linearly drifted Brownian motion with its running minimum and the density of its first passage time on a constant level. The consideration of dependent driving Brownian motions would lead to more complicated and less explicit formulas (see, e.g., Iyengar [7], He et al. [6] or Patras [12]). The results of the paper can be naturally extended to the case of several assets, which value processes are driven by independent Brownian motions with random drift rates.

The paper is organized as follows. In Section 2, we introduce the described above two-dimensional structural model. We also recall expressions for the joint transition density of a linearly drifted Brownian motion with its running minimum and for the density of the first time when it hits a constant level. These explicit expressions are used for the derivation of subsequent formulas. In Sections 3 and 4, we evaluate the ex-dividend prices of contingent claims having the structure similar to first- and second-to-default options with respect to the filtration generated by the both asset value processes (full information) and with respect to the filtration generated by one of the processes only (partial information), respectively. The main results of the paper are stated in Theorems 3.1, 3.2, 4.1 and 4.2.

2 The model

In this section we introduce a model with two asset processes paying random dividends and solve the problem of pricing derivatives having a structure similar to first- and second-to-default options under full and partial information.

2.1. For a precise formulation of the problem, let us consider a probability space (Ω, \mathcal{G}, P) with two *independent* standard Brownian motions $W^i = (W_t^i)_{t \geq 0}$, $i = 1, 2$. Suppose that there exist two processes $X^i = (X_t^i)_{t \geq 0}$, $i = 1, 2$, given by:

$$X_t^i = x_i \exp \left(\left(r - \frac{\sigma_i^2}{2} - \delta_{i,0} \right) t - (\delta_{i,1} - \delta_{i,0}) (t - \tau_1)^+ - (\delta_{i,2} - \delta_{i,0}) (t - \tau_2)^+ + \sigma_i W_t^i \right) \quad (2.1)$$

where $(t - \tau_i)^+ = \max\{t - \tau_i, 0\}$, $r \geq 0$ and σ_i , $\delta_{i,\ell}$, x_i are some given strictly positive constants for every $i = 1, 2$ and $\ell = 0, 1, 2$. Assume that the processes X^i , $i = 1, 2$, describe the risk-neutral dynamics of the values of some assets paying dividends, and τ_i , $i = 1, 2$, are random times to be specified below, at which some credit events occur resulting the changes of dividend rates. In more details, for every $i = 1, 2$ fixed, the asset number i pays dividends at the rate $\delta_{i,0}$ until the time $\tau_1 \wedge \tau_2$, at which the first credit event occurs and the dividend rate is changed into $\delta_{i,\ell}$, where $\ell = 1$ if $\tau_1 \wedge \tau_2 = \tau_1$ and $\ell = 2$ if $\tau_1 \wedge \tau_2 = \tau_2$. Then, the asset i pays dividends with the rate $\delta_{i,\ell}$ until the time $\tau_1 \vee \tau_2$, at which the second credit event occurs and the dividend rate is changed into $\delta_{i,3} = \delta_{i,1} + \delta_{i,2} - \delta_{i,0}$. After the both credit events occur, the asset i pays dividends with the rate $\delta_{i,3}$. Here r is the interest rate of a riskless banking

account and σ_i is the volatility coefficient.

Following the structural approach, let us define the random time τ_i by:

$$\tau_i = \inf\{t \geq 0 \mid X_t^i \leq b_i\} \quad (2.2)$$

where $b_i > 0$ is a given constant. By construction, τ_i is a stopping time with respect to the natural filtration $\mathcal{G}_t = \sigma(X_s^1, X_s^2 \mid 0 \leq s \leq t)$ generated by X^i , $i = 1, 2$. Then, the existence of such a pair of processes (X^1, X^2) can be easily deduced from the classical diffusion model with dividends by means of standard change-of-measure arguments.

2.2. The purpose of the present paper is to determine the rational (ex-dividend) prices of contingent claims having the following payoff structure. In the first claim, the amount $C(X_T^1, X_T^2)$ is paid at the maturity T if and only if no credit event occurs before the maturity, and the amount $D_{1,i}(\tau_i, X_{\tau_i}^1, X_{\tau_i}^2)$ is paid at the time τ_i if the time of first credit event is τ_i and it occurs before the maturity, for every $i = 1, 2$. In the second claim, the amount $C_i(X_T^1, X_T^2)$ is paid at the maturity T if and only if the two credit events occur before the maturity and the time of the second credit event is τ_i , and the amount $D_{2,i}(\tau_i, X_{\tau_i}^1, X_{\tau_i}^2)$ is paid at time τ_i if the time of the second credit event is τ_i and it occurs before the maturity, for every $i = 1, 2$. Without loss of generality, we further assume that the payoffs are already discounted by the banking account, that is equivalent to put r equal to zero.

Note that the contingent claims described above have the structure similar to *first-* and *second-to-default* options in the credit risk theory. The ex-dividend price processes V_t^i , $i = 1, 2$, of such claims are given by:

$$V_t^1 = E \left[C(X_T^1, X_T^2) I(T < \tau_1 \wedge \tau_2) + \sum_{i=1}^2 D_{1,i}(\tau_i, X_{\tau_i}^i, X_{\tau_i}^{3-i}) I(\tau_i \leq T, \tau_i < \tau_{3-i}) \mid \mathcal{G}_t \right] \quad (2.3)$$

and

$$V_t^2 = E \left[\sum_{i=1}^2 (D_{2,i}(\tau_i, X_{\tau_i}^i, X_{\tau_i}^{3-i}) + C_i(X_T^i, X_T^{3-i})) I(\tau_{3-i} < \tau_i < T) \mid \mathcal{G}_t \right] \quad (2.4)$$

for any $0 \leq t \leq T$, respectively (see, e.g., [1; Chapter X] or [14; Chapter X]). Here the expectations are taken with respect to the martingale measure and $I(\cdot)$ denotes the indicator function. We further assume that $C(x_1, x_2)$, $C_i(x_i, x_{3-i})$ and $D_{k,i}(t, x_i, x_{3-i})$, $k, i = 1, 2$, are nonnegative functions such that the integrals appearing below are well defined. For example,

for the *European basket call* or *put* option with the strike price $K > 0$ we may set $C(x_1, x_2)$ and $C_i(x_i, x_{3-i})$ being equal to $(\alpha_1 x_1 + \alpha_2 x_2 - K)^+$ or $(K - \alpha_1 x_1 - \alpha_2 x_2)^+$ with some $\alpha_i \geq 0$, $i = 1, 2$, respectively. We may also assume linear recovery by setting $D_{k,i}(t, x_i, x_{3-i}) = \gamma + \beta_1 x_1 + \beta_2 x_2$ with some $\gamma \geq 0$ and $\beta_i \geq 0$ for $k, i = 1, 2$.

Moreover, we shall also determine the prices of the claims under the assumption that the information available on the market is generated by one of the assets only. In that case, the rational prices $V_t^{i,j}$, $i = 1, 2$, of the claims are given by:

$$V_t^{1,j} = E \left[C(X_T^1, X_T^2) I(T < \tau_1 \wedge \tau_2) + \sum_{i=1}^2 D_{1,i}(\tau_i, X_{\tau_i}^i, X_{\tau_i}^{3-i}) I(\tau_i \leq T, \tau_i < \tau_{3-i}) \middle| \mathcal{G}_t^j \right] \quad (2.5)$$

and

$$V_t^{2,j} = E \left[\sum_{i=1}^2 (D_{2,i}(\tau_i, X_{\tau_i}^i, X_{\tau_i}^{3-i}) + C_i(X_T^i, X_T^{3-i})) I(\tau_{3-i} < \tau_i < T) \middle| \mathcal{G}_t^j \right] \quad (2.6)$$

for any $0 \leq t \leq T$, respectively. Here $\mathcal{G}_t^j = \sigma(X_s^j | 0 \leq s \leq t)$ is the natural filtration of the process X^j for every $j = 1, 2$.

2.3. For the process X^i let us introduce the corresponding *running minimum* process $M^i = (M_t^i)_{t \geq 0}$ given by:

$$M_t^i = \min_{0 \leq s \leq t} X_s^i \wedge m_i \quad (2.7)$$

for any $x_i \geq m_i > b_i > 0$ fixed. Then, from the structure of (2.2) it is seen that the default time τ_i takes the form:

$$\tau_i = \inf\{t \geq 0 \mid M_t^i \leq b_i\} \quad (2.8)$$

and in order to obtain the initial values of (2.3)-(2.4) and (2.5)-(2.6), we shall put $x_i = m_i$ for every $i = 1, 2$.

It thus follows from (2.8) that the event $\{\tau_i > t\}$ can be expressed as $\{M_t^i > b_i\}$ for any $t \geq 0$, so that the process X^i admits the representation:

$$dX_t^i = -X_t^i (\delta_{i,0} + (\delta_{i,1} - \delta_{i,0}) I(M_t^1 \leq b_1) + (\delta_{i,2} - \delta_{i,0}) I(M_t^2 \leq b_2)) dt + X_t^i \sigma_i dW_t^i \quad (2.9)$$

with $X_0^i = x_i$. Therefore, we may conclude that $(X^1, M^1, X^2, M^2) = (X_t^1, M_t^1, X_t^2, M_t^2)_{t \geq 0}$ is a (time-homogeneous) strong Markov process with respect to the filtration $(\mathcal{G}_t)_{t \geq 0}$. In the sequel, we also use the notation $\tau_i = \tau_i(x_1, m_1, x_2, m_2)$ for $x_i \geq m_i > b_i > 0$ and every $i = 1, 2$.

Note that, by means of standard arguments of filtering theory (see, e.g., [9; Chapter IX]), it is shown that the process X^i admits the following representation:

$$dX_t^i = -X_t^i (\delta_{i,0} + (\delta_{i,i} - \delta_{i,0}) I(M_t^i \leq b_i) + (\delta_{i,3-i} - \delta_{i,0}) P[M_t^{3-i} \leq b_{3-i} | \mathcal{G}_t^i]) dt + X_t^i \sigma_i d\bar{W}_t^i \quad (2.10)$$

with $X_0^i = x_i$, where the innovation process $\bar{W}^i = (\bar{W}_t^i)_{t \geq 0}$ defined by:

$$\bar{W}_t^i = W_t^i - \frac{\delta_{i,3-i} - \delta_{i,0}}{\sigma_i} \int_0^t (I(M_s^{3-i} \leq b_{3-i}) - P[M_s^{3-i} \leq b_{3-i} | \mathcal{G}_s^i]) ds \quad (2.11)$$

is a standard Brownian motion with respect to the filtration $(\mathcal{G}_t^i)_{t \geq 0}$ according to P. Lévy's characterization theorem (see, e.g., [13; Chapter IV, Theorem 3.6]).

2.4. Let us introduce the process $X^{i,j} = (X_t^{i,j})_{t \geq 0}$ defined by:

$$X_t^{i,j} = x_i \exp \left(- \left(\delta_{i,j} + \frac{\sigma_i^2}{2} \right) t + \sigma_i W_t^i \right) \quad (2.12)$$

and its running minimum process $M^{i,j} = (M_t^{i,j})_{t \geq 0}$ given by:

$$M_t^{i,j} = \min_{0 \leq s \leq t} X_s^{i,j} \wedge m_i \quad (2.13)$$

for any $x_i \geq m_i > b_i > 0$ and every $i = 1, 2$ and $j = 0, 1, 2, 3$. Observe that from (2.1) and (2.12) it is seen that $X_t^i = X_t^{i,0}$ holds for all $0 \leq t \leq \tau_1 \wedge \tau_2$, since we have put $r = 0$. It is known (see, e.g., [13], [8] or [3]) that the transition density $g_{i,j}$ of the Markov process $(X^{i,j}, M^{i,j})$ defined by:

$$P_{x_i, m_i}[X_t^{i,j} \in dz, M_t^{i,j} \in dy] = g_{i,j}(x_i, m_i; t, z, y) dz dy \quad (2.14)$$

admits the representation:

$$g_{i,j}(x_i, m_i; t, z, y) = \frac{2}{\sigma_i^3 \sqrt{2\pi t^3}} \frac{\ln(y^2/(x_i z))}{zy} \exp \left(- \frac{\ln^2(y^2/(x_i z))}{2\sigma_i^2 t} + \frac{\rho_{i,j}}{\sigma_i} \ln(z/x_i) - \frac{\rho_{i,j}^2 t}{2} \right) \quad (2.15)$$

for all $t > 0$ and $z \geq y$ with $m_i \geq y > 0$, and equals zero otherwise. Here P_{x_i, m_i} denotes probability under the assumption that $(X^{i,j}, M^{i,j})$ starts at (x_i, m_i) , and we set $\rho_{i,j} = -\delta_{i,j}/\sigma_i - \sigma_i/2$.

Let us also define the corresponding hitting time $\tau_{i,j}$ having the form:

$$\tau_{i,j} = \inf\{t \geq 0 \mid X_t^{i,j} \leq b_i\} \quad (2.16)$$

for every $i = 1, 2$ and $j = 0, 1, 2$. It is known that the density $h_{i,j}$ of $\tau_{i,j}$ defined by:

$$P_{x_i, m_i}[\tau_{i,j} \in dt] = h_{i,j}(x_i; t) dt \quad (2.17)$$

admits the representation:

$$h_{i,j}(x_i; t) = \frac{\ln(x_i/b_i)}{\sigma_i \sqrt{2\pi t^3}} \exp\left(-\frac{(\ln(x_i/b_i) + \rho_{i,j}\sigma_i t)^2}{2\sigma_i^2 t}\right) \quad (2.18)$$

for all $t > 0$ and $x_i \geq m_i > b_i > 0$.

3 Case of full information

In this section we compute conditional expectations (2.3) and (2.4).

3.1. Let us begin by computing the terms for the *first-to-default* (2.3). For this, applying the Markov property of the process (X^1, M^1, X^2, M^2) , we get:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2}[C(X_T^1, X_T^2) I(T < \tau_1 \wedge \tau_2) | \mathcal{G}_t] \\ &= I(t < \tau_1 \wedge \tau_2) E_{x_1, m_1, x_2, m_2}[C(X_T^1, X_T^2) I(T < \tau_1 \wedge \tau_2) | \mathcal{G}_t] \\ &= I(t < \tau_1 \wedge \tau_2) E_{X_t^1, M_t^1, X_t^2, M_t^2}[C(X_{T'}^1, X_{T'}^2) I(T' < \tau'_1 \wedge \tau'_2)] \end{aligned} \quad (3.1)$$

where we set $T' = T - t$ for each $0 \leq t \leq T$. Here E_{x_1, m_1, x_2, m_2} denotes expectation under the assumption that the process (X^1, M^1, X^2, M^2) starts at (x_1, m_1, x_2, m_2) with some $x_i \geq m_i > b_i > 0$, $i = 1, 2$. In case $\tau_i > t$, we put $\tau'_i = \tau_i(X_t^1, M_t^1, X_t^2, M_t^2)$. Then, using the fact that the event $\{\tau_i > t\}$ can be represented in the form $\{M_t^i > b_i\}$ for any $t \geq 0$, we have:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2}[C(X_{T'}^1, X_{T'}^2) I(T' < \tau'_1 \wedge \tau'_2)] \\ &= E_{x_1, m_1, x_2, m_2}[C(X_{T'}^1, X_{T'}^2) I(M_{T'}^1 > b_1, M_{T'}^2 > b_2)] \\ &= E_{x_1, m_1, x_2, m_2}[C(X_{T'}^{1,0}, X_{T'}^{2,0}) I(M_{T'}^{1,0} > b_1, M_{T'}^{2,0} > b_2)] \end{aligned} \quad (3.2)$$

where $\tau'_i = \tau_i(x_1, m_1, x_2, m_2)$ and the processes $(X^{i,0}, M^{i,0})$, $i = 1, 2$, are defined in (2.12)-(2.13) above. Hence, from (3.1) and (3.2) we obtain:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2}[C(X_T^1, X_T^2) I(T < \tau_1 \wedge \tau_2) | \mathcal{G}_t] \\ &= I(t < \tau_1 \wedge \tau_2) \int_{b_1}^{\infty} \int_{b_1}^{\infty} \int_{b_2}^{\infty} \int_{b_2}^{\infty} C(x'_1, x'_2) \prod_{\ell=1}^2 g_{\ell,0}(X_t^\ell, M_t^\ell; T-t, x'_\ell, m'_\ell) dx'_\ell dm'_\ell \end{aligned} \quad (3.3)$$

where the functions $g_{i,0}$, $i = 1, 2$, are given in (2.15) above.

In a similar way, using the Markov property of the process (X^1, M^1, X^2, M^2) , we get:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [D_{1,i}(\tau_i, X_{\tau_i}^i, X_{\tau_i}^{3-i}) I(\tau_i \leq T, \tau_i < \tau_{3-i}) | \mathcal{G}_t] \\
&= D_{1,i}(\tau_i, b_i, X_{\tau_i}^{3-i}) I(\tau_i \leq t, \tau_i < \tau_{3-i}) \\
&\quad + E_{x_1, m_1, x_2, m_2} [D_{1,i}(\tau_i, b_i, X_{\tau_i}^{3-i}) I(t < \tau_i \leq T, \tau_i < \tau_{3-i}) | \mathcal{G}_t] \\
&= D_{1,i}(\tau_i, b_i, X_{\tau_i}^{3-i}) I(\tau_i \leq t, \tau_i < \tau_{3-i}) \\
&\quad + I(t < \tau_1 \wedge \tau_2) E_{X_t^1, M_t^1, X_t^2, M_t^2} [D_{1,i}(t + \tau'_i, b_i, X_{\tau'_i}^{3-i}) I(\tau'_i \leq T, \tau'_i < \tau'_{3-i})]
\end{aligned} \tag{3.4}$$

for $0 \leq t \leq T$ and

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [D_{1,i}(t + \tau'_i, b_i, X_{\tau'_i}^{3-i}) I(\tau'_i \leq T', \tau'_i < \tau'_{3-i})] \\
&= E_{x_1, m_1, x_2, m_2} [D_{1,i}(t + \tau'_i, b_i, X_{\tau'_i}^{3-i}) I(M_{\tau'_i}^{3-i} > b_{3-i}) I(\tau'_i \leq T')] \\
&= E_{x_1, m_1, x_2, m_2} [D_{1,i}(t + \tau_{i,0}, b_i, X_{\tau_{i,0}}^{3-i,0}) I(M_{\tau_{i,0}}^{3-i,0} > b_{3-i}) I(\tau_{i,0} \leq T')]
\end{aligned} \tag{3.5}$$

for $x_i \geq m_i > b_i > 0$, where $\tau'_i = \tau_i(x_1, m_1, x_2, m_2)$ and the processes $(X^{3-i,0}, M^{3-i,0})$ as well as the hitting times $\tau_{i,0}$, $i = 1, 2$, are defined in (2.12)-(2.13) and (2.16) above. Therefore, taking into account the independence of $\tau_{i,0}$ and $(X^{3-i,0}, M^{3-i,0})$, from (3.4) and (3.5) we conclude that:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [D_{1,i}(\tau_i, X_{\tau_i}^i, X_{\tau_i}^{3-i}) I(t < \tau_i \leq T, \tau_i < \tau_{3-i}) | \mathcal{G}_t] \\
&= I(t < \tau_1 \wedge \tau_2) \int_0^{T'} \int_{b_{3-i}}^\infty \int_{b_{3-i}}^\infty D_{1,i}(t + u, b_i, x'_{3-i}) h_{i,0}(X_t^i; u) \\
&\quad \times g_{3-i,0}(X_t^{3-i}, M_t^{3-i}; u, x'_{3-i}, m'_{3-i}) du dx'_{3-i} dm'_{3-i}
\end{aligned} \tag{3.6}$$

where the functions $g_{i,0}$ and $h_{i,0}$, $i = 1, 2$, are given in (2.15) and (2.18) above.

Summarizing the facts proved above let us now formulate the following assertion.

Theorem 3.1. *The ex-dividend price of the first-to-default option (2.3) in the model with full information is given by the sum of the terms (3.3) and (3.6).*

3.2. Let us continue by computing the terms for the *second-to-default* (2.4). For this,

applying the Markov property of the process (X^1, M^1, X^2, M^2) , we get:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [(D_{2,i}(\tau_i, X_{\tau_i}^i, X_{\tau_i}^{3-i}) + C_i(X_T^i, X_T^{3-i})) I(\tau_{3-i} < \tau_i \leq T) | \mathcal{G}_t] \\
&= E_{x_1, m_1, x_2, m_2} [(D_{2,i}(\tau_i, b_i, X_{\tau_i}^{3-i}) + C_i(X_T^i, X_T^{3-i})) I(\tau_{3-i} < \tau_i \leq t) | \mathcal{G}_t] \\
&\quad + E_{x_1, m_1, x_2, m_2} [(D_{2,i}(\tau_i, b_i, X_{\tau_i}^{3-i}) + C_i(X_T^i, X_T^{3-i})) I(\tau_{3-i} \leq t < \tau_i \leq T) | \mathcal{G}_t] \\
&\quad + E_{x_1, m_1, x_2, m_2} [(D_{2,i}(\tau_i, b_i, X_{\tau_i}^{3-i}) + C_i(X_T^i, X_T^{3-i})) I(t < \tau_{3-i} < \tau_i \leq T) | \mathcal{G}_t] \\
&= I(\tau_{3-i} < \tau_i \leq t) (D_{2,i}(\tau_i, b_i, X_{\tau_i}^{3-i}) + E_{X_t^1, M_t^1, X_t^2, M_t^2} [C_i(X_{T'}^i, X_{T'}^{3-i})]) \\
&\quad + I(\tau_{3-i} \leq t < \tau_i) E_{X_t^1, M_t^1, X_t^2, M_t^2} [(D_{2,i}(t + \tau'_i, b_i, X_{\tau'_i}^{3-i}) + C_i(X_{T'}^i, X_{T'}^{3-i})) I(\tau'_i \leq T')] \\
&\quad + I(t < \tau_1 \wedge \tau_2) E_{X_t^1, M_t^1, X_t^2, M_t^2} [(D_{2,i}(t + \tau'_i, b_i, X_{\tau'_i}^{3-i}) + C_i(X_{T'}^i, X_{T'}^{3-i})) I(\tau'_{3-i} < \tau'_i \leq T')]
\end{aligned} \tag{3.7}$$

for all $0 \leq t \leq T$, and continue by computing each of the terms separately.

Firstly, we see that:

$$E_{x_1, m_1, x_2, m_2} [C_i(X_{T'}^i, X_{T'}^{3-i})] = E_{x_1, m_1, x_2, m_2} [C_i(X_{T'}^{i,3}, X_{T'}^{3-i,3})] \tag{3.8}$$

for $x_i \geq m_i$ with $b_i \geq m_i > 0$, where the processes $(X^{i,3}, M^{i,3})$, $i = 1, 2$, are defined in (2.12)-(2.13) above. Then, using the independence of $(X^{i,3}, M^{i,3})$ and $(X^{3-i,3}, M^{3-i,3})$, we have:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [C_i(X_T^i, X_T^{3-i}) I(\tau_{3-i} < \tau_i \leq t) | \mathcal{G}_t] \\
&= I(\tau_{3-i} < \tau_i \leq t) \int_0^\infty \int_0^{b_i} \int_0^\infty \int_0^{b_{3-i}} C_i(x'_i, x'_{3-i}) \prod_{\ell=i}^{3-i} g_{\ell,3}(X_t^\ell, M_t^\ell; T-t, x'_\ell, m'_\ell) dx'_\ell dm'_\ell
\end{aligned} \tag{3.9}$$

where the functions $g_{i,3}$, $i = 1, 2$, are defined in (2.15) above.

Secondly, we observe that:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [D_{2,i}(t + \tau'_i, b_i, X_{\tau'_i}^{3-i}) I(\tau'_i \leq T')] \\
&= E_{x_1, m_1, x_2, m_2} [D_{2,i}(t + \tau_{i,3-i}, b_i, X_{\tau_{i,3-i}}^{3-i,3-i}) I(\tau_{i,3-i} \leq T')]
\end{aligned} \tag{3.10}$$

for $x_i \geq m_i > b_i > 0$ and $x_{3-i} \geq m_{3-i}$ with $b_{3-i} \geq m_{3-i} > 0$, where the processes $(X^{3-i,3-i}, M^{3-i,3-i})$ as well as the hitting times $\tau_{i,3-i}$, $i = 1, 2$, are defined in (2.12)-(2.13) and (2.16) above. Then, taking into account the independence of $\tau_{i,3-i}$ and $(X^{3-i,3-i}, M^{3-i,3-i})$,

we get:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [D_{2,i}(t + \tau_i, b_i, X_{\tau_i}^{3-i}) I(\tau_{3-i} \leq t < \tau_i \leq T) | \mathcal{G}_t] \\
&= I(\tau_{3-i} \leq t < \tau_i) \int_0^{T-t} \int_0^\infty \int_0^{b_{3-i}} D_{2,i}(t + u, b_i, x'_{3-i}) h_{i,3-i}(X_t^i; u) \\
&\quad \times g_{3-i,3-i}(X_t^{3-i}, M_t^{3-i}; u, x'_{3-i}, m'_{3-i}) du dx'_{3-i} dm'_{3-i}
\end{aligned} \tag{3.11}$$

where the functions $g_{3-i,3-i}$ and $h_{i,3-i}$, $i = 1, 2$, are defined in (2.15) and (2.18) above.

Thirdly, applying the strong Markov property of (X^1, M^1, X^2, M^2) , we get:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [C_i(X_{T'}^i, X_{T'}^{3-i}) I(\tau'_i \leq T')] \\
&= E_{x_1, m_1, x_2, m_2} [\widehat{C}_i(X_{\tau'_i}^i, M_{\tau'_i}^i, X_{\tau'_i}^{3-i}, M_{\tau'_i}^{3-i}; T' - \tau'_i) I(\tau'_i \leq T')] \\
&= E_{x_1, m_1, x_2, m_2} [\widehat{C}_i(b_i, b_i, X_{\tau_{i,3-i}}^{3-i,3-i}, M_{\tau_{i,3-i}}^{3-i,3-i}; T' - \tau_{i,3-i}) I(\tau_{i,3-i} \leq T')]
\end{aligned} \tag{3.12}$$

for $x_i \geq m_i > b_i > 0$ and $x_{3-i} \geq m_{3-i}$ with $b_{3-i} \geq m_{3-i} > 0$, where the functions \widehat{C}_i , $i = 1, 2$, are defined by:

$$\begin{aligned}
\widehat{C}_i(x_i, m_i, x_{3-i}, m_{3-i}; T' - u) &= E_{x_1, m_1, x_2, m_2} [C_i(X_{T'-u}^i, X_{T'-u}^{3-i})] \\
&= E_{x_1, m_1, x_2, m_2} [C_i(X_{T'-u}^{i,3}, X_{T'-u}^{3-i,3})]
\end{aligned} \tag{3.13}$$

for $x_\ell \geq m_\ell$ with $b_\ell \geq m_\ell > 0$, $\ell = 1, 2$, and any $0 \leq u \leq T'$ fixed. Thus, using the independence of $\tau_{i,3-i}$ and $(X^{3-i,3-i}, M^{3-i,3-i})$, from (3.12) we obtain:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [C_i(X_T^i, X_T^{3-i}) I(\tau_{3-i} \leq t < \tau_i \leq T)] \\
&= I(\tau_{3-i} \leq t < \tau_i) \int_0^{T-t} \int_0^\infty \int_0^{b_{3-i}} \widehat{C}_i(b_i, b_i, x'_{3-i}, m'_{3-i}; T - t - u) h_{i,3-i}(X_t^i; u) \\
&\quad \times g_{3-i,3-i}(X_t^{3-i}, M_t^{3-i}; u, x'_{3-i}, m'_{3-i}) du dx'_{3-i} dm'_{3-i}
\end{aligned} \tag{3.14}$$

where, by virtue of independence of $(X^{i,3}, M^{i,3})$ and $(X^{3-i,3}, M^{3-i,3})$, from (3.13) it follows that:

$$\begin{aligned}
& \widehat{C}_i(x'_i, m'_i, x'_{3-i}, m'_{3-i}; T - t - u) \\
&= \int_0^\infty \int_0^{b_i} \int_0^\infty \int_0^{b_{3-i}} C_i(x''_i, x''_{3-i}) \prod_{\ell=i}^{3-i} g_{\ell,3}(x'_\ell, m'_\ell; T - t - u, x''_\ell, m''_\ell) dx''_\ell dm''_\ell
\end{aligned} \tag{3.15}$$

and the functions $g_{i,\ell}$ and $h_{i,\ell}$, $i = 1, 2$, $\ell = 1, 2, 3$, are defined in (2.15) and (2.18) above.

Now, applying the strong Markov property of (X^1, M^1, X^2, M^2) , we get:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [D_{2,i}(t + \tau'_i, b_i, X_{\tau'_i}^{3-i}) I(\tau'_{3-i} < \tau'_i \leq T')] \tag{3.16} \\
&= E_{x_1, m_1, x_2, m_2} [D_{2,i}(t + \tau'_i, b_i, X_{\tau'_i}^{3-i}) I(M_{\tau'_i}^i > b_i) I(\tau'_i \leq T')] \\
&= E_{x_1, m_1, x_2, m_2} [\widehat{D}_{2,i}(t + \tau'_{3-i}, X_{\tau'_{3-i}}^i, M_{\tau'_{3-i}}^i, X_{\tau'_{3-i}}^{3-i}, M_{\tau'_{3-i}}^{3-i}; T' - \tau'_{3-i}) I(M_{\tau'_{3-i}}^i > b_i) I(\tau'_{3-i} \leq T')] \\
&= E_{x_1, m_1, x_2, m_2} [\widehat{D}_{2,i}(t + \tau_{3-i,0}, X_{\tau_{3-i,0}}^{i,0}, M_{\tau_{3-i,0}}^{i,0}, b_{3-i}, b_{3-i}; T' - \tau_{3-i,0}) I(M_{\tau_{3-i,0}}^{i,0} > b_i) I(\tau_{3-i,0} \leq T')]
\end{aligned}$$

for $x_i \geq m_i > b_i > 0$, where the functions $\widehat{D}_{2,i}$, $i = 1, 2$, are defined by:

$$\begin{aligned}
& \widehat{D}_{2,i}(t + v, x_i, m_i, x_{3-i}, m_{3-i}; T' - v) \tag{3.17} \\
&= E_{x_1, m_1, x_2, m_2} [D_{2,i}(t + v + \tau'_i, b_i, X_{\tau'_i}^{3-i}) I(\tau'_i \leq T' - v)] \\
&= E_{x_1, m_1, x_2, m_2} [D_{2,i}(t + v + \tau_{i,3-i}, b_i, X_{\tau_{i,3-i}}^{3-i,3-i}) I(\tau_{i,3-i} \leq T' - v)]
\end{aligned}$$

for $x_i \geq m_i > b_i > 0$ and $x_{3-i} \geq m_{3-i}$ with $b_{3-i} \geq m_{3-i} > 0$, and any $0 \leq v \leq T'$ fixed.

Hence, using the independence of $\tau_{3-i,0}$ and $(X^{i,0}, M^{i,0})$, from (3.16) we obtain:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [D_{2,i}(\tau_i, b_i, X_{\tau_i}^{3-i}) I(t < \tau_{3-i} < \tau_i \leq T) | \mathcal{G}_t] \tag{3.18} \\
&= I(t < \tau_1 \wedge \tau_2) \int_0^{T-t} \int_0^\infty \int_{b_i}^\infty \widehat{D}_{2,i}(t + v, x'_i, m'_i, b_{3-i}, b_{3-i}; T - t - v) h_{3-i,0}(X_t^{3-i}; v) \\
&\quad \times g_{i,0}(X_t^i, M_t^i; v, x'_i, m'_i) dv dx'_i dm'_i
\end{aligned}$$

where, by virtue of the independence of $\tau_{i,3-i}$ and $(X^{3-i,3-i}, M^{3-i,3-i})$, from (3.17) it follows that:

$$\begin{aligned}
& \widehat{D}_{2,i}(t + v, x'_i, m'_i, x'_{3-i}, m'_{3-i}; T - t - v) \tag{3.19} \\
&= \int_0^{T-t-v} \int_0^\infty \int_0^{b_{3-i}} D_{2,i}(t + v + u, b_i, x''_{3-i}) h_{i,3-i}(x'_i; u) \\
&\quad \times g_{3-i,3-i}(x'_{3-i}, m'_{3-i}; u, x''_{3-i}, m''_{3-i}) du dx''_{3-i} dm''_{3-i}
\end{aligned}$$

and the functions $g_{i,\ell}$ and $h_{i,\ell}$, $i = 1, 2$, $\ell = 0, 1, 2$, are defined in (2.15) and (2.18) above.

Finally, we see that:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [C_i(X_{T'}^i, X_{T'}^{3-i}) I(\tau'_{3-i} < \tau'_i \leq T')] \tag{3.20} \\
&= E_{x_1, m_1, x_2, m_2} [C_i(X_{T'}^i, X_{T'}^{3-i}) I(M_{\tau'_{3-i}}^i > b_i) I(\tau'_i \leq T')] \\
&= E_{x_1, m_1, x_2, m_2} [\tilde{C}_i(X_{\tau'_{3-i}}^i, M_{\tau'_{3-i}}^i, X_{\tau'_{3-i}}^{3-i}, M_{\tau'_{3-i}}^{3-i}; T' - \tau'_{3-i}) I(M_{\tau'_{3-i}}^i > b_i) I(\tau'_{3-i} \leq T')] \\
&= E_{x_1, m_1, x_2, m_2} [\tilde{C}_i(X_{\tau_{3-i,0}}^i, M_{\tau_{3-i,0}}^i, b_{3-i}, b_{3-i}; T' - \tau_{3-i,0}) I(M_{\tau_{3-i,0}}^i > b_i) I(\tau_{3-i,0} \leq T')]
\end{aligned}$$

for $x_i \geq m_i > b_i > 0$, where the functions \tilde{C}_i , $i = 1, 2$, are defined by:

$$\begin{aligned}
& \tilde{C}_i(x_i, m_i, x_{3-i}, m_{3-i}; T' - v) \tag{3.21} \\
&= E_{x_1, m_1, x_2, m_2} [\hat{C}_i(X_{\tau'_i}^i, M_{\tau'_i}^i, X_{\tau'_i}^{3-i}, M_{\tau'_i}^{3-i}; T' - v - \tau'_i) I(\tau'_i \leq T' - v)] \\
&= E_{x_1, m_1, x_2, m_2} [\hat{C}_i(b_i, b_i, X_{\tau_{i,3-i}}^{3-i,3-i}, M_{\tau_{i,3-i}}^{3-i,3-i}; T' - v - \tau_{i,3-i}) I(\tau_{i,3-i} \leq T' - v)]
\end{aligned}$$

for $x_i \geq m_i > b_i > 0$ and $x_{3-i} \geq m_{3-i}$ with $b_{3-i} \geq m_{3-i} > 0$, and any $0 \leq v \leq T'$ fixed, where the functions \hat{C}_i , $i = 1, 2$, are defined in (3.13) above. Hence, using the independence of $\tau_{3-i,0}$ and $(X^{i,0}, M^{i,0})$, from (3.20) we obtain:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [C_i(X_T^i, X_T^{3-i}) I(t < \tau_{3-i} < \tau_i \leq T) | \mathcal{G}_t] \tag{3.22} \\
&= I(t < \tau_1 \wedge \tau_2) \int_0^{T-t} \int_{b_i}^\infty \int_{b_i}^\infty \tilde{C}_i(x'_i, m'_i, b_{3-i}, b_{3-i}; T - t - v) h_{3-i,0}(X_t^{3-i}; v) \\
&\quad \times g_{i,0}(X_t^i, M_t^i; v, x'_i, m'_i) dv dx'_i dm'_i
\end{aligned}$$

where, by virtue of the independence of $\tau_{i,3-i}$ and $(X^{3-i,3-i}, M^{3-i,3-i})$, from (3.21) it follows that:

$$\begin{aligned}
& \tilde{C}_i(x'_i, m'_i, x'_{3-i}, m'_{3-i}; T - t - v) \tag{3.23} \\
&= \int_0^{T-t-v} \int_0^\infty \int_0^{b_{3-i}} \hat{C}_i(b_i, b_i, x''_{3-i}, m''_{3-i}; T - t - v - u) h_{i,3-i}(x'_i; u) \\
&\quad \times g_{3-i,3-i}(x'_{3-i}, m'_{3-i}; u, x''_{3-i}, m''_{3-i}) du dx''_{3-i} dm''_{3-i}
\end{aligned}$$

the functions \hat{C}_i admit the representation (3.15) and the functions $g_{i,\ell}$ and $h_{i,\ell}$, $i = 1, 2$, $\ell = 0, 1, 2$, are defined in (2.15) and (2.18) above.

Therefore, summarizing the facts proved above we are now ready to formulate the following assertion.

Theorem 3.2. *The ex-dividend price of the second-to-default option (2.4) in the model with full information is given by the sum of (3.9), (3.11), (3.14), (3.18) and (3.22).*

4 Case of partial information

In this section we compute the conditional expectations (2.5) and (2.6).

4.1. Let us proceed by computing the terms for the *first-to-default* (2.5). For this, let $H(x_j, m_j, x_{3-j}, m_{3-j})$ be a nonnegative continuous function for any $j = 1, 2$ fixed. By virtue of independence of the processes $(X_t^{j,0}, M_t^{j,0})$ and $(X_t^{3-j,0}, M_t^{3-j,0})$ defined in (2.12)-(2.13), we get:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [H(X_t^j, M_t^j, X_t^{3-j}, M_t^{3-j}) I(t < \tau_j \wedge \tau_{3-j}) | \mathcal{G}_t^j] \\ &= I(t < \tau_j) E_{x_1, m_1, x_2, m_2} [H(X_t^{j,0}, M_t^{j,0}, X_t^{3-j,0}, M_t^{3-j,0}) I(M_t^{3-j,0} > b_{3-j}) | \mathcal{G}_t^j] \\ &= I(t < \tau_j) \int_{b_{3-j}}^{\infty} \int_{b_{3-j}}^{\infty} H(X_t^{j,0}, M_t^{j,0}, x'_{3-j}, m'_{3-j}) g_{3-j,0}(x_{3-j}, m_{3-j}; t, x'_{3-j}, m'_{3-j}) dx'_{3-j} dm'_{3-j} \end{aligned} \quad (4.1)$$

for all $0 \leq t \leq T$, where the functions $g_{3-j,0}$, $j = 1, 2$, are defined in (2.15) above.

Now, we see that:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, X_{\tau_j}^j, X_{\tau_j}^{3-j}) I(\tau_j \leq t, \tau_j < \tau_{3-j}) | \mathcal{G}_t^j] \\ &= I(\tau_j \leq t) E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(M_{\tau_j}^{3-j} > b_{3-j}) | \mathcal{G}_t^j] \\ &= I(\tau_j \leq t) E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_{j,0}, b_j, X_{\tau_{j,0}}^{3-j,0}) I(M_{\tau_{j,0}}^{3-j,0} > b_{3-j}) | \mathcal{G}_t^j] \end{aligned} \quad (4.2)$$

for $0 \leq t \leq T$, where the hitting times $\tau_{j,0}$, $j = 1, 2$, are defined in (2.16). Thus, using the independence of $\tau_{j,0}$ and $(X_t^{3-j,0}, M_t^{3-j,0})$, we have:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, X_{\tau_j}^j, X_{\tau_j}^{3-j}) I(\tau_j \leq t, \tau_j < \tau_{3-j}) | \mathcal{G}_t^j] \\ &= I(\tau_j \leq t) \int_{b_{3-j}}^{\infty} \int_{b_{3-j}}^{\infty} D_{1,j}(\tau_j, b_j, x'_{3-j}) g_{3-j,0}(x_{3-j}, m_{3-j}; \tau_j, x'_{3-j}, m'_{3-j}) dx'_{3-j} dm'_{3-j} \end{aligned} \quad (4.3)$$

where the functions $g_{3-j,0}$, $j = 1, 2$, are defined in (2.15) above.

Then, we observe that:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, X_{\tau_j}^j, X_{\tau_j}^{3-j}) I(\tau_j \leq t, \tau_j < \tau_{3-j}) | \mathcal{G}_t^{3-j}] \\ &= E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(\tau_j < \tau_{3-j} \leq t) | \mathcal{G}_t^{3-j}] \\ &\quad + E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(\tau_j \leq t < \tau_{3-j}) | \mathcal{G}_t^{3-j}] \end{aligned} \quad (4.4)$$

for $0 \leq t \leq T$ and $j = 1, 2$. Hence, by virtue of the independence of $\tau_{j,0}$ and $(X^{3-j,0}, M^{3-j,0})$ we obtain:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(\tau_j < \tau_{3-j} \leq t) | \mathcal{G}_t^{3-j}] \\ &= I(\tau_{3-j} \leq t) E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(M_{\tau_j}^{3-j} > b_{3-j}) I(\tau_j < \tau_{3-j}) | \mathcal{G}_t^{3-j}] \\ &= I(\tau_{3-j} \leq t) E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_{j,0}, b_j, X_{\tau_{j,0}}^{3-j,0}) I(M_{\tau_{j,0}}^{3-j,0} > b_{3-j}) I(\tau_{j,0} < \tau_{3-j}) | \mathcal{G}_t^{3-j}] \end{aligned} \quad (4.5)$$

and

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(\tau_j \leq t < \tau_{3-j}) | \mathcal{G}_t^{3-j}] \\ &= I(t < \tau_{3-j}) E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(M_{\tau_j}^{3-j} > b_{3-j}) I(\tau_j \leq t) | \mathcal{G}_t^{3-j}] \\ &= I(t < \tau_{3-j}) E_{x_1, m_1, x_2, m_2} [D_{j,1}(\tau_{j,0}, b_j, X_{\tau_{j,0}}^{3-j,0}) I(M_{\tau_{j,0}}^{3-j,0} > b_{3-j}) I(\tau_{j,0} \leq t) | \mathcal{G}_t^{3-j}] \end{aligned} \quad (4.6)$$

and thus, from (4.4) we conclude that:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(\tau_j \leq t, \tau_j < \tau_{3-j}) | \mathcal{G}_t^{3-j}] \\ &= \int_0^{\tau_{3-j} \wedge t} D_{1,j}(u, b_j, X_u^{3-j}) I(M_u^{3-j} > b_{3-j}) h_{j,0}(x_j; u) du \end{aligned} \quad (4.7)$$

where the functions $h_{j,0}$, $j = 1, 2$, are defined in (2.18) above.

Summarizing the facts proved above let us formulate the following assertion.

Theorem 4.1. *The ex-dividend price of first-to-default option (2.5) in the model with partial information is given by the sum of (4.1) and (4.3) or (4.7), where the function H is given appropriately by the corresponding value in (3.3) or (3.6), respectively.*

4.2. Let us conclude by computing the terms for the *second-to-default* (2.6). Firstly, taking into account the Markovian structure of the process (X^1, M^1, X^2, M^2) , we get:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [H(X_t^j, M_t^j, X_t^{3-j}, M_t^{3-j}) I(\tau_{3-j} \leq t < \tau_j) | \mathcal{G}_t^j] \\ &= I(t < \tau_j) \int_0^t \int_0^\infty \int_0^{b_{3-j}} H(X_t^j, M_t^j, x'_{3-j}, m'_{3-j}) h_{3-j,0}(x_{3-j}; u) \\ &\quad \times g_{3-j,3-j}(b_{3-j}, b_{3-j}; t - u, x'_{3-j}, m'_{3-j}) du dx'_{3-j} dm'_{3-j} \end{aligned} \quad (4.8)$$

and

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [H(X_t^j, M_t^j, X_t^{3-j}, M_t^{3-j}) I(\tau_{3-j} \leq t < \tau_j) | \mathcal{G}_t^{3-j}] \\ &= I(\tau_{3-j} \leq t) \int_{b_j}^\infty \int_{b_j}^\infty \widehat{H}(x'_j, m'_j, X_t^{3-j}, M_t^{3-j}; t - \tau_{3-j}) g_{j,0}(x_j, m_j; \tau_{3-j}, x'_j, m'_j) dx'_j dm'_j \end{aligned} \quad (4.9)$$

where the function \widehat{H} is defined by:

$$\begin{aligned} & \widehat{H}(x'_j, m'_j, x'_{3-j}, m'_{3-j}; t - v) \\ &= \int_{b_j}^{\infty} \int_{b_j}^{\infty} H(x''_j, m''_j, x'_{3-j}, m'_{3-j}) g_{j,3-j}(x'_j, m'_j; t - v, x''_j, m''_j) dx''_j dm''_j \end{aligned} \quad (4.10)$$

for $0 \leq t \leq T$ and the functions $g_{j,\ell}$, $j = 1, 2$, $\ell = 0, 1, 2$, are defined in (2.15) above.

Now, by virtue of Markovian structure of the process (X^1, M^1, X^2, M^2) , we see that:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [D_{2,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(\tau_{3-j} < \tau_j \leq t) | \mathcal{G}_t^j] \\ &= I(\tau_j \leq t) \int_0^{\tau_j} \int_0^{\infty} \int_0^{b_{3-j}} D_{2,j}(\tau_j, b_j, x'_{3-j}) h_{3-j,0}(x_{3-j}; u) \\ & \quad \times g_{3-j,3-j}(b_{3-j}, b_{3-j}; \tau_j - u, x'_{3-j}, m'_{3-j}) du dx'_{3-j} dm'_{3-j} \end{aligned} \quad (4.11)$$

and

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [D_{2,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(\tau_{3-j} < \tau_j \leq t) | \mathcal{G}_t^{3-j}] \\ &= I(\tau_{3-j} \leq t) \int_{\tau_{3-j}}^t \int_{b_j}^{\infty} \int_{b_j}^{\infty} D_{2,j}(u, b_j, X_u^{3-j}) h_{j,3-j}(x'_j; u - \tau_{3-j}) \\ & \quad \times g_{j,0}(x_j, m_j; \tau_{3-j}, x'_j, m'_j) du dx'_j dm'_j \end{aligned} \quad (4.12)$$

for $0 \leq t \leq T$, where the functions $g_{j,\ell}$ and $h_{j,\ell}$, $j = 1, 2$, $\ell = 0, 1, 2$, are defined in (2.15) and (2.18) above.

Finally, using again the Markovian structure of the process (X^1, M^1, X^2, M^2) , we obtain:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [H(X_t^j, M_t^j, X_t^{3-j}, M_t^{3-j}) I(\tau_{3-j} < \tau_j \leq t) | \mathcal{G}_t^j] \\ &= I(\tau_j \leq t) \int_0^{\tau_j} \int_0^{\infty} \int_0^{b_{3-j}} \widetilde{H}(X_t^j, M_t^j, x'_{3-j}, m'_{3-j}; t - \tau_j) h_{3-j,0}(x_{3-j}; v) \\ & \quad \times g_{3-j,3-j}(b_{3-j}, b_{3-j}; \tau_j - v, x'_{3-j}, m'_{3-j}) dv dx'_{3-j} dm'_{3-j} \end{aligned} \quad (4.13)$$

where the function \widetilde{H} is defined by:

$$\begin{aligned} & \widetilde{H}(x'_j, m'_j, x'_{3-j}, m'_{3-j}; t - v) \\ &= \int_0^{\infty} \int_0^{b_{3-j}} H(x'_j, m'_j, x''_{3-j}, m''_{3-j}) g_{3-j,3}(x'_j, m'_j; t - v, x''_{3-j}, m''_{3-j}) dx''_{3-j} dm''_{3-j} \end{aligned} \quad (4.14)$$

and

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [H(X_t^j, M_t^j, X_t^{3-j}, M_t^{3-j}) I(\tau_{3-j} < \tau_j \leq t) | \mathcal{G}_t^{3-j}] \\
&= I(\tau_{3-j} \leq t) \int_{\tau_{3-j}}^t \int_{b_j}^\infty \int_{b_j}^\infty \bar{H}(b_j, b_j, X_t^{3-j}, M_t^{3-j}; t-u) h_{j,3-j}(x'_j; u - \tau_{3-j}) \\
&\quad \times g_{j,0}(x_j, m_j; \tau_{3-j}, x'_j, m'_j) du dx'_j dm'_j
\end{aligned} \tag{4.15}$$

where the function \bar{H} is defined by:

$$\begin{aligned}
& \bar{H}(x'_j, m'_j, x'_{3-j}, m'_{3-j}; t-u) \\
&= \int_0^\infty \int_0^{b_j} H(x''_j, m''_j, x'_{3-j}, m'_{3-j}) g_{j,3}(x'_j, m'_j; t-u, x''_j, m''_j) dx''_j dm''_j
\end{aligned} \tag{4.16}$$

for $0 \leq t \leq T$ and the functions $g_{j,\ell}$ and $h_{j,\ell}$, $j = 1, 2$, $\ell = 0, 1, 2, 3$, are defined in (2.15) and (2.18) above.

Therefore, summarizing the facts proved above we are now ready to formulate the following assertion.

Theorem 4.2. *The ex-dividend price of first-to-default option (2.6) in the model with partial information is given by the sum of (4.1), (4.8), (4.9), (4.11), (4.12), (4.13) and (4.15), where the function H is given appropriately by the corresponding value in (3.9), (3.14) or (3.22), respectively.*

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