Mathematics of Currency and Exchange: Arithmetic at the end of the Thirteenth Century

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Abstract

In Western Europe, a sophisticated banking system for the purposes of international trade had evolved by the end of the thirteenth century. It was based upon the 'bill of exchange', which enabled an exporter of goods to receive payment in his own currency, by means of a balancing payment made in the currency of the importer. This paper discusses the arithmetical tools that were available for use in accounting for transactions made in different currencies. It is argued that algorithmic methods based on the Hindu-Arabic numerals were used at the higher levels of banking, in order to prepare tables of foreign exchange such as those collected by the Florentine banker, Francesco Pegolotti. On the other hand, the clerks who were responsible for routine book-keeping would have used a simple abacus and counters, and recorded their transactions in Roman numerals.

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1. Money and trade

Figure 1-1 shows the kind of money that would have been familiar to the Vikings in Britain in the early part of the tenth century. It is part of a hoard found at Cuerdale in Lancashire, and was probably intended for paying the troops, or for buying provisions.

Fig.1-1: Money in tenth century Britain

Economists have devoted much effort to the theoretical analysis of money (Davies 1994). Since money fulfils many different functions, they distinguish many different kinds of money . For the purpose of this article, the two functions highlighted by Jevons (1872) will suffice. They are:

- a medium of exchange – represented by *money-objects*, such as coins, hacksilver, gold dust, cowrie shells; and
- a measure of value – represented by *accounting units*, such as pounds, shillings and pence.

The items in the Cuerdale hoard are *money-objects*, mainly small pieces of silver (now known as *hacksilver*), but with some of the special money-objects that we call coins. These coins represented the standard accounting unit at that time, the penny.

Jevons argued that the need for money stemmed from the inconvenience of *barter*. In order to illustrate this point, Salzman (1931) gave an example taken from an eleventh century manuscript (Fig.1-2), to which he gave the caption ‘A Simple Bargain’.

Fig.1-2: A Simple Bargain ?

Closer inspection of the picture reveals that the transaction is by no means simple. At first sight, it appears a chicken is being exchanged for another object, which looks rather like an old sock. However, common-sense suggests that the 'sock' must be a more valuable item, such as a joint of meat. More confusing is the fact that the chicken-man has a coin in his hand. Is he offering the coin as part of the bargain, or has he received it from the sock-man? Is the sock-man's object in fact a bag of coins? As far as I can see, the original manuscript throws no light on these questions. By the end of the Middle Ages barter had become a very complex operation, as explained by John Mason in his recent paper (2007).

Coins have been used as money-objects since around 600 BC. Originally they were lumps of gold or silver, and in medieval times that was effectively still the situation. The intrinsic value of a coin was determined by the amount of precious metal that it contained, and ultimately that had to be determined by making two measurements: weighing to determine the mass, and assaying to determine the fineness, that is, the proportions of precious metal and base metal. The authorities who issued coins would put their stamp on them in an attempt to guarantee their value as a medium of exchange. But the guarantee was far from being absolute, and frequently coins were not accepted at their face value, particularly in large transactions. The process of assaying was complicated, and could be used only in the most important situations, such as payments to the king from his local representatives. In trade, the fineness of a coin could be estimated by using a touchstone, but that was not an exact method. On the other hand, weighing coins, either individually or in bulk, was both practical and exact, and it was frequently employed in commercial transactions. Special weights were used to ensure that coins were of the correct value, and many of these weights have survived in England (Biggs 1990). Fig.1-3, taken from a thirteenth century Islamic manuscript, shows a pile of silver coins being weighed to establish their value as the price for a group of slaves.

Fig.1-3: Weighing coins in trade [Bib. Nat., Paris]

Gold coins were common in Islam from the ninth century, onwards, and they were minted in Christian Europe from the twelfth century onwards. In Figure 1-4 the first coin shown (1) is a *dinar* from Alexandria, minted 1196-1218. This was the standard style of the Islamic dinar for several centuries. Coin (2) is a *morabetino* of Alfonso of Castile (1158-1214), clearly imitating the style of the dinar. Coin (3) is an *augustale* of Frederick II as King of Sicily (1197-1250), and coin (4) is a *florin*, minted in Florence from 1252 onwards.

Fig.1-4: Some gold coins (not to scale)

The morabetino and the augustale were, in a sense, experiments, but the florin was the first of many gold coins that were minted in large numbers in the later 13th and 14th centuries. Such coins facilitated trade by enabling large payments to be made with relatively small numbers of money-objects, but unfortunately the coins produced in various places were different in mass and fineness. For this reason, Francesco Balducci Pegolotti, a member of the Florentine banking house of the Bardi, compiled a notebook that listed (among many other things) the fineness of the gold coins that might be encountered by a banker at the end of the 13th century. Part of this list, taken from the printed version (Evans 1936), is shown in Fig.1-5. (We shall say more about Pegolotti's notebook later.)

LEGHE DI MONETE D'ORO

Fiorini d'oro sono a carati 24 d'oro fine per oncia.
Ducati d'oro a carati 24.
Gienovini d'oro vecchi a carati 23 1/4.
Gienovini d'oro nuovi a carati 24.
Lucchesi d'oro a cavallo, a carati 23 7/8.
Lucchesi d'oro a piede, a carati 23 3/4.
Carlini d'oro a carati 23 e 7/8.
Ragonesi d'oro di Cicilia a carati 23 e 7/8.
Romanini d'oro a carati 23 e 3/4.
Parigini d'oro a carati 23 e 3/4.
Dobbre da Rimirra d'oro a carati 23 3/4.
Dobbre d'oro di Morrocco a carati 23 3/4.
Castellani d'oro a carati 23 3/4.
Anfusini[3] d'oro vecchi a carati 20 1/2.
Anfusini[3] d'oro nuovi a carati 20 1/2.
Bisanti vecchi d'oro d'Allessandria piue nuovi a carati 23.
Bisanti vecchi d'oro d'Allessandria meno nuovi a carati 23 1/2.
Bisanti saracinati d'oro a carati 15.
Pezzi di bisanti d'oro a carati 12.
Pezzi di Tripoli d'oro a carati 11.
Oro di terì[1] a carati 16 e 2/3.
Agostantini d'oro a carati 20 e 1/2.
Casanini d'oro a carati 23 1/8.
Tanghi d'oro a carati 23 7/8.

Fig.1-5: A (small) extract from Pegolotti's notebook

Quite apart from the problem of variation in value, the use of gold coins for higher-value transactions did not avoid the practical difficulties attendant on international trade. At one time, a shipload of wool sent from London to Florence had to be balanced directly by barter (sending a shipload of other goods in return) or currency (sending coins or other money-objects). Both mechanisms were plainly insecure. For this reason an alternative mechanism, the *bill of exchange*, developed in Christian Europe in the 13th century.

A simplified description of the mechanism is shown in Fig. 1-6, which is adapted from the definitive account given by Spufford (1986). There are four parties:

the *remitter*, a merchant in Florence who imports the wool;
the *payee*, an agent in London of the remitter, who sends the wool to Florence;
the *drawer*, a banker in Florence, who receives payment from the remitter;
the *payer*, an agent in London of the drawer, who pays the payee.

The various steps in the procedure are shown in the diagram. For our purposes, the significant point is that nontrivial arithmetical calculations would be required in order to ensure that the outcome was equitable. In particular, it was necessary to compute carefully the amount of the payment in Florence that would ensure that the payee received the right amount in London, remembering that the two payments would be made in different currencies.

In the third part of this article, we shall look more closely at the calculations involved. But first we shall discuss the arithmetical tools that were available at the relevant time (c.1300), a subject on which much has been written, not all of it enlightening.

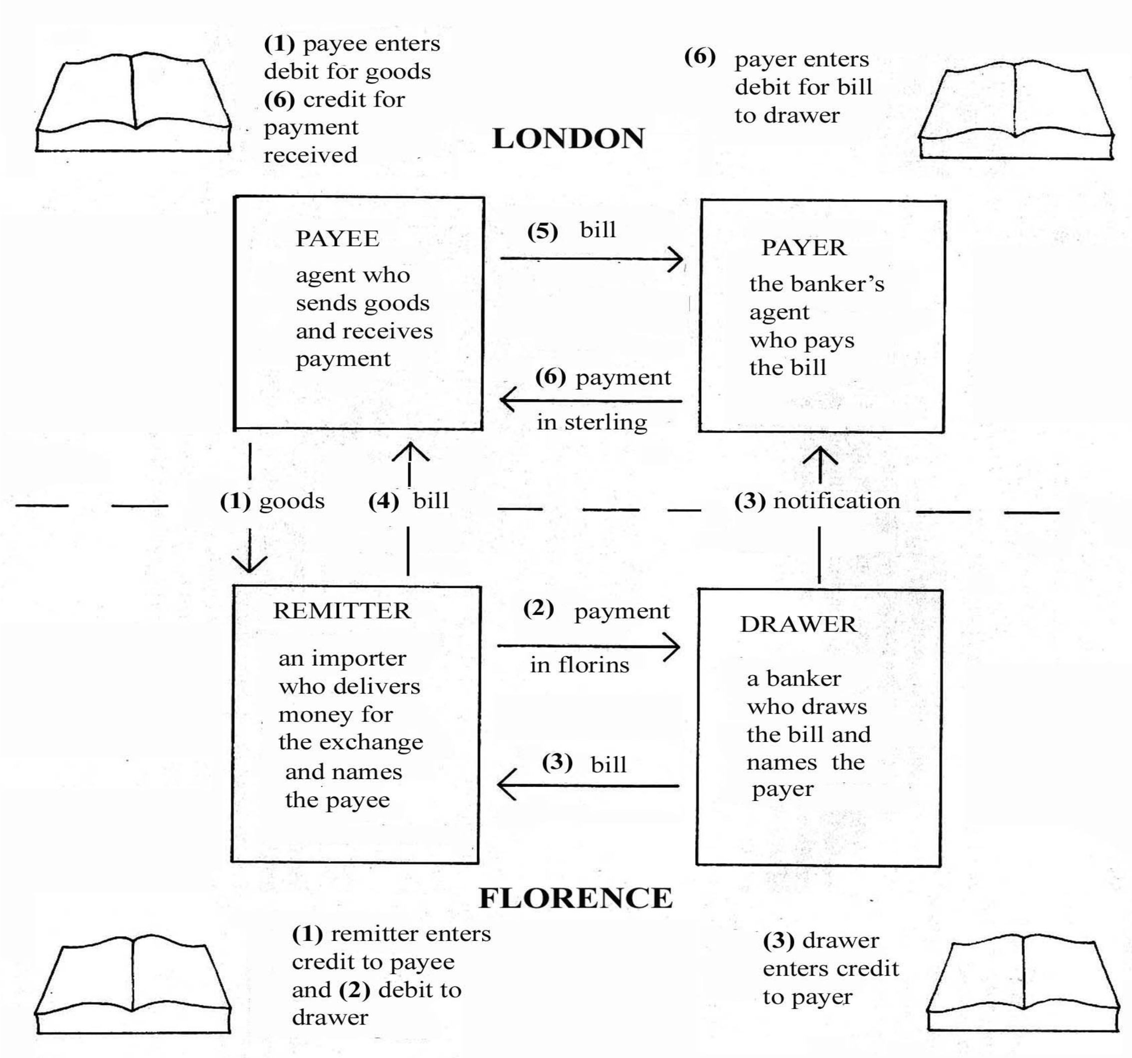

Fig.1-6: The Bill of Exchange (adapted from Spufford)

2. The tools of arithmetic

It is a truism that the Roman numerals, I, V, X, C and so on, are very little use for performing arithmetical calculations. The Romans themselves knew this, but nevertheless they managed to control a vast empire for several centuries, and this must have required a certain level of arithmetical expertise. In fact, the Romans used their numerals only to *record* numbers. In order to carry out arithmetical operations they used *calculi*, small pebbles or counters, that have given their name to the science of *calculation*. The calculi were arranged on a grid (*abacus*), which might have been scratched on the ground, drawn on a piece of cloth, or marked on a wooden board. The basic arithmetical operations were performed by moving the counters in certain ways: addition and subtraction were easy, but multiplication and division required more skill.

Sadly we know very few details about the abacus in Roman times and for many centuries thereafter. The grid itself was ephemeral (even a wooden board would not survive for long), and the counters were fairly anonymous (and, one suspects, they have often been wrongly classified by archaeologists as 'gaming pieces' or some other convenient misnomer). From the Anglo-Saxon and Viking periods in Britain there is only a trace of documentary evidence about the abacus (see Murray (1978) p.164 and note 11, p.454). There is a written account of the method of 'finger-reckoning' in a 10th century copy of Bede's *De Tempore Ratione* (Fig.2-1), but it is hard to believe that this clumsy procedure was the only means by which the government and the owners of great estates managed their affairs.

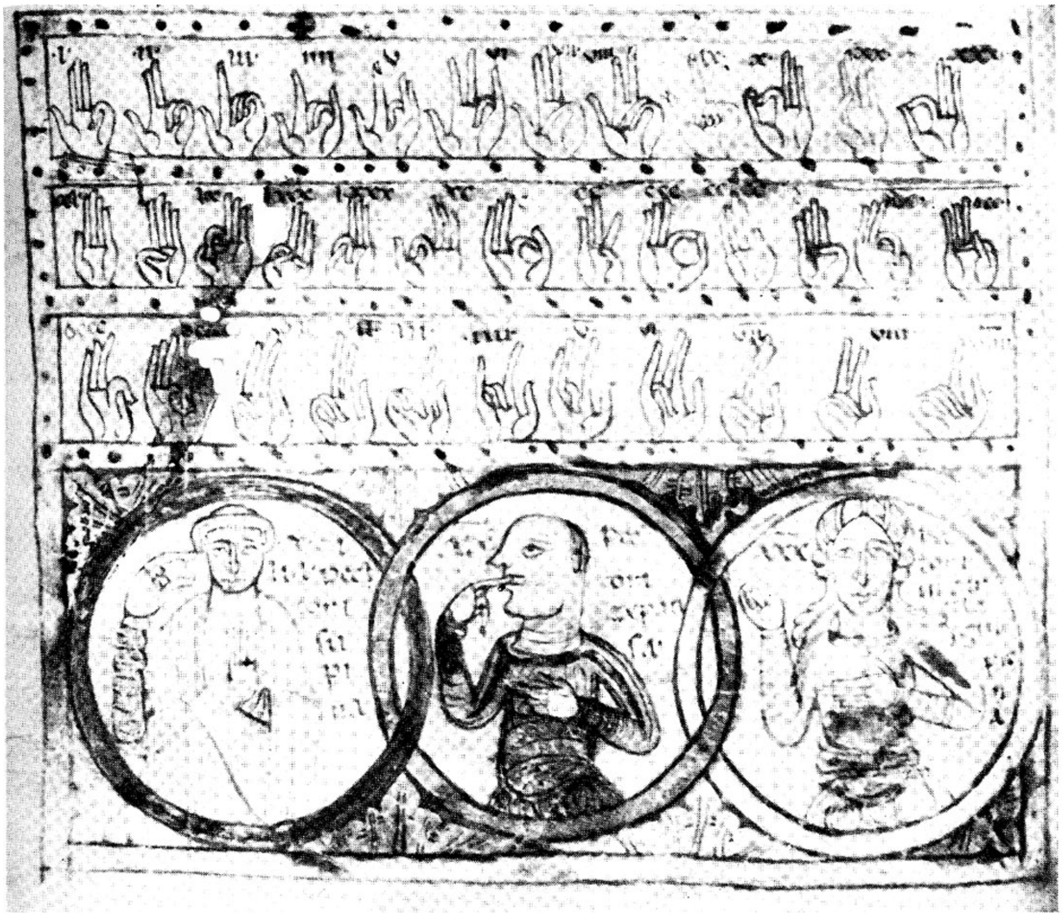

Fig.2-1: Anglo-Saxon finger-reckoning, tenth century
[BL ms Royal BA XI fo.33v]

It is not until the turn of the millenium that we begin to find hard evidence about the abacus, and then the picture is confused, because several innovations were taking place. The design of the abacus was improved and, concurrently, the Hindu-Arabic numerals were incorporated into the procedure. These two innovations made the abacus into an essential tool of government and administration at the highest levels. The final innovation was the replacement of the abacus itself by procedures that could be carried out 'in writing', rather than with counters, and here the new numerals were a crucial factor.

The name of Gerbert of Aurillac is usually linked with the introduction of the improved abacus into Europe. At various times he resided in Barcelona, Magdeburg, Rome and Rheims, and in 999 he became Pope Sylvester II. Whether he himself made the innovations, or simply described them, is not clear. But it is clear that he made the abacus into an important tool of administration in all its forms. For details of Gerbert's mathematical career the reader is referred to Murray (1978).

In England, we have a manuscript from Ramsey Abbey, dating from about 1110, that apparently depicts the abacus as it was known to Gerbert (Fig.2-2). An excellent account of this manuscript was given by Yeldham (1926).

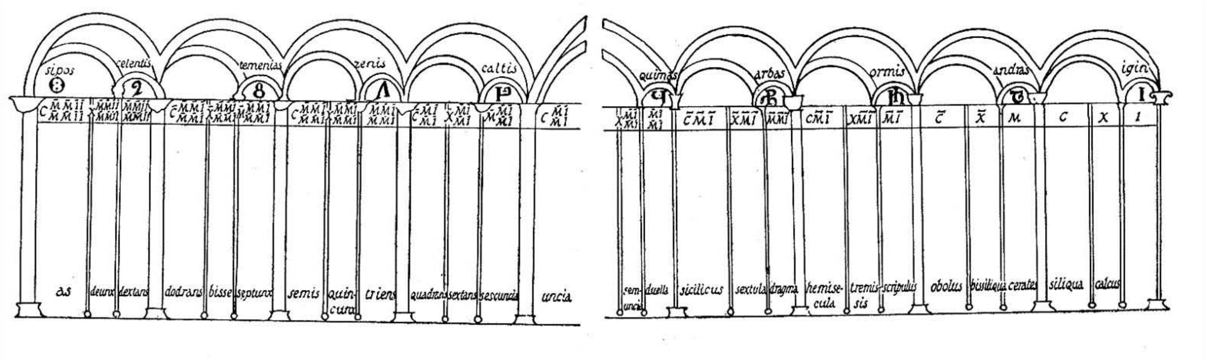

Fig.2-2: Gerbert's abacus [St John's College, Oxford, ms 17]

The table is set out in nine blocks of three columns each. Each block represents a certain range, and within each block the columns correspond to the hundreds (c), tens (x), and units (i) of that range. The arrangement of the columns shows that the principle of 'place-value' was an integral part of the system, so that very large numbers could be accommodated, although Roman numerals were used to record them. On the other hand, among the arches we find the Hindu-Arabic numerals, both the symbols and their names. Even more significant is fact, clearly stated in the accompanying text, that the counters intended for use on this grid were also inscribed with the Hindu-Arabic numerals. This feature serves to distinguish what we shall call *Gerbert's abacus* from the simple abacus with plain counters. The presence at the foot of the columns of the Roman words for the fractions suggests that division was among the operations for which Gerbert's abacus could be used, and this too is confirmed by the text.

Another piece of English evidence for the incorporation of the Hindu-Arabic numerals comes from a document written about 1130, and described in detail by Poole (1912). The author, Turchill, is believed to have been a clerk in the royal household, and his manual of instruction for users of the abacus refers quite clearly to counters that were inscribed with the Hindu-Arabic numerals. There is good agreement between the Ramsey and Turchill manuscripts, but more work on these and other manuscripts that appear to show the Hindu-Arabic numerals being used in conjunction with an abacus is desirable.

Tentatively, the conclusion is that the sophisticated Gerbert's abacus was known and used at the higher levels of administration in the twelfth century. This would have included the royal household and probably a few of the large monastic houses that controlled great estates. A simpler form of the abacus, with un-numbered counters, was used by merchants and others who needed to carry out addition and multiplication on a regular basis. Indeed, we know for certain that the simple abacus remained in use into the post-medieval period.

Given the existence of Gerbert's abacus, in which arithmetic was done by the manipulation of numbered counters in columns, it was a short step to arithmetic in which the numerals themselves were manipulated: the so-called *pen-reckoning*. Often this would have been done by chalking the numbers on a slate (or by a similarly ephemeral means), and so early examples are quite scarce. But some 14th century examples are known (Fig.2-3).

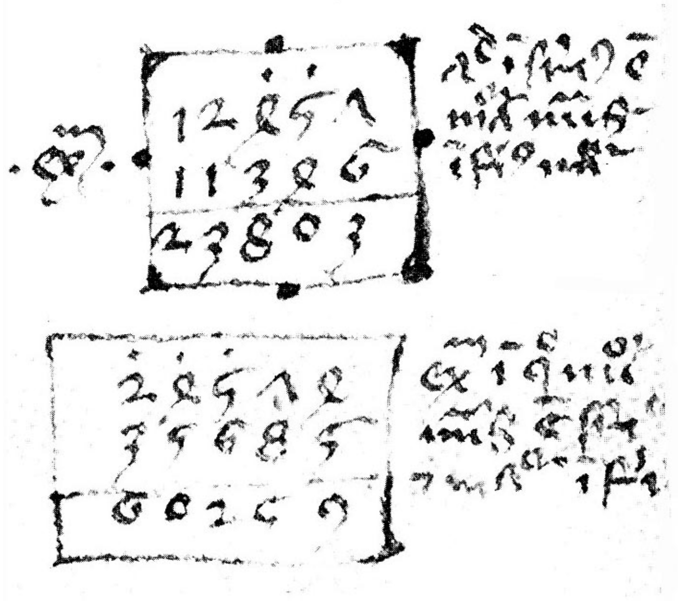

Fig.2-3: Pen-reckoning, probably from the 1320s [Bodleian, ms Ashmole 1522 fo. 18r]

Mathematically, the significant point is that *the procedures required to carry out arithmetical operations (such as long division) on Gerbert's abacus are very similar to those that are needed for the same operations in pen-reckoning.* This point is often overlooked, possibly because the terminology used in medieval documents was itself unclear.

The problem begins in the ninth century, with a mathematician whose name is now usually written in English as *Al-Khwarismi* (or a close approximation to it). Unfortunately very little is known about him, and even his real name is a matter of conjecture. His writings are believed to have been responsible for disseminating the Hindu system of numerals, and the associated methods of arithmetic, via Islam to Christian Europe. In the twelfth century, Al-Khwarismi's influence led to the use of the word *algorist* to denote someone who used the Hindu systems. As we have seen, it could refer either to the use of numbered counters, or to numbers written on a slate. Both systems employed the Hindu-Arabic numerals, and the methods of calculation were similar. In modern times the word *algorithm* has come to mean any definite procedure for solving a mathematical problem.

The confusion is compounded by the interpretation of the word *abacist* as the antithesis of an algorist, despite the fact that users of the abacus used algorithmic methods. More scope for confusion arises from the focus on the supposed significance of the Hindu-Arabic symbol for zero, which had no counterpart in the Roman system. Even on the simplest abacus, zero could be indicated by the absence of a counter. Consequently, the zero should be seen as an important part of the algorithmic process, rather a ground-breaking concept.

These remarks throw some light on an event that is relevant to our study of international trade. In 1299 the Florentine bankers guild formulated an edict, the meaning of which has been obscured by verbal confusion. It forbade the use of Hindu-Arabic numerals in account books 'or any part thereof where payments and receipts are recorded', and insisted that the numbers be written 'openly and at length, using letters'. Murray (1978) correctly points out that this was merely an official rule relating to public documents. It did not mean that the Florentine bankers were prevented from using the Hindu-Arabic numerals for the purposes of calculation; indeed they had probably been doing so for at least a century, and it would have impossible for them carry on their businesses in any other way. By 1345 Giovanni Villani could report that at least a thousand children were being taught *abaco* and *algorismo* in Florence (see XI, c.94 in the 1857 edition of the *Cronica*). Surely this implies the use of the Hindu-Arabic numerals, and probably the techniques of pen-reckoning as well.

Of course, the interpretation offered here is different from the one based on a cursory examination of the printed books of arithmetical instruction that began to appear in the 16th century. The frontispiece (Fig.2-4) to Reisch's *Margarita Philosophica* (1503) is a case in point.

Fig.2-4: A misleading picture

Here we see a smiling Boethius doing his algorism, while the glum Pythagoras struggles with his simple abacus. Any historical inference drawn from this little cameo is unlikely to stand up.

3. Arithmetic of foreign exchange

In order to assess the extent to which arithmetic was used in international trade, we shall focus on the exchange between London and Florence at the beginning of the fourteenth century. This was the time of the edict mentioned above; also, according to Grierson (1979), it was when Pegolotti was assembling the material for *La Pratica Mercatoria*. It must be remembered that no original version of Pegolotti's notebook is known, so we shall refer to the printed edition prepared by Evans (1936), which is itself based on a manuscript dated 1472. According to the editor, that manuscript uses both Roman numerals and Hindu-Arabic numerals, but all numerals are rendered in the latter form in the printed version, as shown in Fig.1-5 and Fig.3-1.

Another caveat is that a modern discussion will use notation that was not available to the 14th century practitioners. For example, the now-ubiquitous sign of equality (=) was not introduced until the 16th century. For this reason we must be careful to ensure that the modern viewpoint does not obscure the thought-processes of the medieval arithmeticians.

The general context for exchange transactions was that at each place (call it X) payments were made using a money-object M_X, and accounts were kept in X-units. The authorities in X decreed that M_X should be worth a certain number v_X of X-units, and this number was fixed, usually for many years. In London (L) and Florence (F) these quantities were as follows.

- In London accounts were kept in sterling pence (L-units). For large-scale commercial transactions the money-object M_L was a quantity of silver known as a mark, worth $v_L = 160$ pence.
- In Florence accounts were kept in denari piccoli (F-units). For large-scale commercial transactions the money-object M_F was a gold coin known as a florin, worth $v_F = 348$ denari.

Trade between London and Florence was conducted in terms of an *exchange rate* $e = e_{LF}$, which we shall define to be the number of M_F's (florins) that equate to one M_L (mark). Thus it is a consequence of the definition that e_{FL}, the number of marks that equate to one florin, is $1/e$. In practice the exchange rate varied, because it was affected by economic factors, and so it was important to be able to make calculations based on a range of values of e. Here is Pegolotti's table for the exchange between London and Florence.

A denari 33	sterl. il fior. viene il mar. lire 7,	sol. —,	den. 7 e 2/11 [3/11] a fior.	
A denari 33¼	starl. il fior. viene il mar. lire 6,	sol. 19,	den. 6 e 78/133	
A denari 33½	sterl. il fior. viene il mar. lire 6,	sol. 18,	den. 6 e 6/67	
A denari 33¾	sterl. il fior. viene il mar. lire 6,	sol. 17,	den. 4 e 8/45 [d. 5 7/9]	
A denari 34	sterl. il fior. viene il mar. lire 6,	sol. 16,	den. 5 e 11/17	
A denari 34¼	sterl. il fior. viene il mar. lire 6,	sol. 15,	den. 5 e 95/137	
A denari 34½	sterl. il fior. viene il mar. lire 6,	sol. 14,	den. 5 e 21/23	
A denari 34¾	sterl. il fior. viene il mar. lire 6,	sol. 13,	den. 6 e 42/139	
A denari 35	sterl. il fior. viene il mar. lire 6,	sol. 12,	den. 6 e 6/7	
A denari 35¼	sterl. il fior. viene il mar. lire 6,	sol. 11,	den. 7 e 81/141	
A denari 35½	sterl. il fior. viene il mar. lire 6,	sol. 10,	den. 8 e 32/71	
A denari 35¾	sterl. il fior. viene il mar. lire 6,	sol. 9,	den. 9 e 69/143	
A denari 36	sterl. il fior. viene il mar. lire 6,	sol. 8,	den. 10 e 2/3	
α			β	

Fig.3-1: Pegolotti's table for the exchange between London and Florence

There are (at least) two obvious complications.

1. The number of den[ari] is expressed in multiple-units called lire and sol[di].
2. Because books are kept in accounting units, while payments are made in money-objects, two types of money are involved.

The first point is a simple matter of convenience. Because the denaro (d) represented a very small value, it was customary to express large amounts in terms of soldi ($s = 12d$) and lire ($\ell = 20s$). In fact the same multiples, 12 and 20, were used in many places, including London, where the multiple-units were called shillings and pounds.

The second point provides a strong clue as to the purpose of the table. Each line is a statement of the form:

$$\alpha \text{ L-units equates to } M_F \quad \Longrightarrow \quad M_L \text{ equates to } \beta \text{ F-units.}$$

For example, when $\alpha = 35$ sterling pence are worth one florin, one mark is worth $\beta = 1590\frac{6}{7}$ denari (that is, $6\ell\ 12s\ 6\frac{6}{7}d$).

The form of the table confirms that it was intended to determine the entries in the Florentine accounts that would correspond to the number of marks being paid in London. In the table, the 'exchange on London' is specified by the number α, rather than the exchange rate e as formally defined above. Saying that α pence make a florin is the same as saying that α/v_L marks make a florin, so $\alpha/v_L = e_{FL} = 1/e$. The corresponding number β has the property that goods valued at one mark in London must be accounted for as β denari in Florence. Thus $\beta = ev_F$. Eliminating e from the equations for α and β it follows that the table is effectively a list of values of the function given by

$$\beta = v_L v_F/\alpha.$$

Now we can see precisely which arithmetical operations are involved. A Florentine book-keeper would have needed to know how to account for goods valued at m marks, given that the current 'exchange on London' was α. The answer is simply βm, in denari, and so he must look up the correct value of β in the table and multiply it by m. On the other hand, in order to *compile* the table it would have been necessary to divide $v_L v_F$ by α, as in the formula displayed above.

This analysis seems to justify the claim that arithmetical expertise was hierarchical. The clerks who used the tables were only expected to multiply a sum of money β by an integer m, and this could have been done with a simple abacus and plain counters. On the other hand, the people who compiled the tables must have been proficient in long division, and that was most likely done using either Gerbert's numbered counters or pen-reckoning. Long division has always been the most complex of the four basic arithmetical operations, mainly because it requires a 'guess' for the trial quotient at each step. For a modern view of this problem, the discussion by Knuth (1981) is recommended.

Of course, there were many subsequent developments in foreign-exchange, and some of them required arithmetical skills. Concepts such as the *mint par*, *bullion points*, and *cyclic arbitrage* were matters of great practical significance right up to the twentieth century (Biggs 1997). And now, in the age of instantaneous electronic transfer of funds, other kinds of mathematical expertise have become invaluable.

4. Discussion

In exchange calculations small changes in the exchange rate can make a significant difference in the resulting payments, and therefore it is desirable for the rate to be quoted with a high degree of accuracy. Pegolotti's table for the exchange on London gives the values of α to the accuracy of a farthing (one-quarter of a penny) and, in some of his tables, the penny is divided into thirds as well as quarters. By the 15th century, some rates were being quoted in smaller units: Spufford (1986, p.115) gives examples in mites (1/24 of a penny). The device of resorting to smaller, fictional, units in order to express numbers more accurately was quite normal at this time, but the units were clumsy and irregular. There seems to have been no interest in extending the positional notation from the integers to the fractions, and consequently the modern decimal notation did not emerge until the sixteeenth century. This observation suggests that the concept of 'number' was largely taken for granted, and there was little progress towards the modern viewpoint, based on the construction of 'number systems' satisfying certain axioms. It had been known since antiquity that some numbers are *rational* – that is, expressible as ratios of whole numbers – and some are not. However, the concept of a number system as a continuum does not appear to figure in medieval thinking. In the world of commerce, the underlying calculations were often obscured by arbitrary terminology arising from traditional practices, and 'money' was a real commodity, not an abstract concept.

Although the development of commercial arithmetic did not lead to progress in the foundations of mathematics, it can be argued that it did create an environment in which mathematics was bound to flourish. Of course, arithmetic was also deployed for scientific purposes, most notably in astronomy. But there appears to be a direct link between the prosperity of the mercantile Italian city-states in the 13th and 14th centuries and the advances in algebra in the 16th century. The establishment of the 'abacus schools', and the production of textbooks for use in them, provided a solid foundation for progress.

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