Chromatic Roots of the Quartic Möbius Ladders

Norman Biggs

Department of Mathematics
London School of Economics
Houghton Street
London WC2A 2AE
U.K.
n.l.biggs@lse.ac.uk

May 2008
CDAM Research Reports Series LSE-CDAM 2008-05

Abstract

Bracelet theory allows the chromatic polynomials of certain families of graphs to be written in a standard form. This form is particularly appropriate for studying the limit curves of the chromatic roots of these families. In this paper these techniques are applied to the quartic Möbius ladders. A simple explicit formula for the chromatic polynomials is obtained, and the limit curves are determined. There is remarkable agreement with the experimental evidence.

Chromatic Roots of the Quartic Möbius Ladders

1. Introduction

In April 2008 Gordon Royle asked me about a family of graphs, known as *quartic Möbius ladders*, that had turned up in his work with Mayhew and Whittle on matroids [**5**]. These are the graphs $R(2k+1)$ constructed by taking an odd cycle C_{2k+1} and adding diagonals joining each vertex to the two opposite vertices. Explicitly, if the vertices are the integers mod $2k+1$, each vertex x is joined to $x \pm 1$ and $x \pm k$. Royle was interested in the Tutte polynomial of $R(2k+1)$, and its specialisations, the chromatic and flow polynomials.

It occurred to me that it might be possible to represent $R(2k+1)$ as a *bracelet*; specifically, as n copies of K_3 with constant links between the copies. Clearly this could be done only when n is an odd number $2r+1$, so that $2k+1 = 3n = 6r+3$, and $k = 3r+1$. In fact, it works when $n = 5$: $R(15)$ contains five disjoint K_3's

$$\{0,7,8\} \qquad \{9,1,2\} \qquad \{3,10,11\} \qquad \{12,4,5\} \qquad \{6,13,14\},$$

and, if we denote these K_3's by $\{i_1, i_2, i_3\}$ $(i = 0,1,2,3,4)$, the additional edges are $i_1 - (i+1)_2$, $i_3 - (i+1)_1$, $i_3 - (i+1)_2$. In bracelet terms, this means that there is a constant linking set $L = \{12, 31, 32\}$.

Theorem 1 Let $B(n)$ be the bracelet formed by n copies of K_3 with linking set $L = \{12, 31, 32\}$. Then $B(2r+1)$ is isomorphic to $R(6r+3)$.

Proof As defined above, $R(6r+3)$ is the graph with vertices the integers mod $6r+3$ (denoted by $0, 1, \ldots, 6r+2$), and edges joining each vertex x to $x \pm 1$ and $x \pm (3r+1)$. Denote the vertices of $B(2r+1)$ by the $2r+1$ triples i_1, i_2, i_3, $(0 \le i \le 2r)$, each such triple forming a copy of K_3. Then it is easy to check that the following is a graph isomorphism.

$$i_1 \mapsto 3i(r+1), \quad i_2 \mapsto 3(ir+i+r)+1, \quad i_3 \mapsto 3(ir+i+r)+2, \quad (i = 0, 1, \ldots, 2r).$$

$\square$

2. The chromatic polynomials of the Royle family

It would be possible to study the chromatic polynomials of the quartic Möbius ladders by elementary recursive techniques, such as those that were used back in 1972. On the other hand, a more theoretical framework for studying bracelets in general is now available, and it is particularly useful for explaining the limiting behaviour of the chromatic roots. Since it turns out that the graphs $R(6r+3)$ provide a very good example of this theory, that approach will be followed in the present paper.

The theory of bracelets [**3**, **4**] was developed in order to study the limit curves formed by the chromatic roots of certain infinite families of graphs, say $\{G_n\}$. There are two stages: first, a standard formula for the chromatic polynomial of G_n is obtained, and second, the Beraha-Kahane-Weiss theorem [**1**] is applied to this formula to derive algebraic equations for the limit curves.

For bracelets based on complete graphs K_b and a constant linking set L, the standard formula can be be written in a very simple form. It is a sum of terms, one for each partition π of a non-negative integer $\ell = |\pi|$ with $\ell \leq b$:

$$P(G_n; z) \;=\; \sum_{\pi} m_\pi(z)\mathrm{tr}(T_{L,\pi}(z)^n).$$

Here $m_\pi(z)$ is a polynomial in z with rational coefficients (independent of L), and $T_{L,\pi}(z)$ is a matrix whose entries are polynomials in z with integer coefficients.

Fortunately, when $b = 3$ there is a simple recipe for calculating the relevant matrices $T_{L,\pi}(z)$ for any linking set L [**4**, section 6]. We briefly review this calculation for the case $L = \{12, 31, 32\}$ that produces the Royle family.

The first step is to list the set $\mathcal{M}(L)$ of 'matchings' that form a subset of L: $\mathcal{M}(L) = \{\emptyset, \{12\}, \{31\}, \{32\}, \{12, 31\}\}$. Each $T_{L,\pi}$ matrix can be expressed in terms of U_M matrices, where $M \in \mathcal{M}(L)$,

$$T_{L,\pi} = U_\emptyset - (U_{12} + U_{31} + U_{32}) + U_{12,31}.$$

Matchings with $|M| \leq |\pi|$ make zero contribution and, since $\mathcal{M}(L)$ contains no matchings of size three, we need only consider the partitions π of $0, 1,$ and 2. According to the general theory we have $m_{[0]}(z) = 1$, and

$$m_{[1]}(z) = z - 1, \quad m_{[20]}(z) = \frac{1}{2}z(z-3), \quad m_{[11]}(z) = \frac{1}{2}(z-1)(z-2),$$

and it remains only to use the recipe given in [**4**] to calculate the corresponding matrices $T_{L,[0]}$, $T_{L,[1]}$, $T_{L,[20]}$, $T_{L,[11]}$.

For $\pi = [0]$ the U_M matrices are 1×1 matrices:

$U_\emptyset = [z(z-1)(z-2)], \quad U_{12} = U_{31} = U_{32} = [(z-1)(z-2)], \quad U_{12,31} = [z-2].$

Hence $T_{L,[0]} = [z^3 - 6z^2 + 12z - 8] = [(z-2)^3]$.
For $\pi = [1]$ the U_M matrices are 3×3 matrices: $U_\emptyset$ is the zero matrix and

$$U_{12} = \begin{pmatrix} 0 & (z-1)(z-2) & 0 \\ 0 & -(z-2) & 0 \\ 0 & -(z-2) & 0 \end{pmatrix}, \qquad U_{31} = \begin{pmatrix} -(z-2) & 0 & 0 \\ -(z-2) & 0 & 0 \\ (z-1)(z-2) & 0 & 0 \end{pmatrix},$$

$$U_{32} = \begin{pmatrix} 0 & -(z-2) & 0 \\ 0 & -(z-2) & 0 \\ 0 & (z-1)(z-2) & 0 \end{pmatrix}, \qquad U_{12,31} = \begin{pmatrix} 0 & z-2 & 0 \\ -1 & -1 & 0 \\ z-2 & 0 & 0 \end{pmatrix}.$$

Hence

$$T_{L,[1]} = -(U_{12} + U_{31} + U_{32}) + U_{12,31} = \begin{pmatrix} z-2 & -(z-1)(z-2) & 0 \\ z-3 & 2z-5 & 0 \\ -(z-2)^2 & z(z-2) & 0 \end{pmatrix}.$$

Note that the trace of $T^n_{L,[1]}$ is the same as the trace of the nth power of the 2×2 submatrix formed by the first two rows and columns.
Finally, for $\pi = [20]$ and $\pi = [11]$ there is only one non-zero matrix, $U_{12,31}$. In both cases it takes the form

$$U_{12,31} = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 0 & 0 \\ \gamma & 0 & 0 \end{pmatrix},$$

where $\alpha = -1$ when $\pi = [20]$ and $\alpha = 1$ when $\pi = [11]$. It follows that the trace of $T^n_{L,[20]}$ is $(-1)^n$ and the trace of $T^n_{L,[11]}$ is 1^n, so the corresponding terms in the standard formula are

$$\frac{1}{2}z(z-3)(-1)^n + \frac{1}{2}(z-1)(z-2)1^n = \begin{cases} z^2 - 3z + 1 & \text{if } n \text{ is even;} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

In the light of the isomorphism given in Theorem 1, we have the following result.

Theorem 2 Let $A(z)$ be the matrix

$$\begin{pmatrix} z-2 & -(z-2)(z-3) \\ z-3 & 2z-5 \end{pmatrix}.$$

The chromatic polynomial of the quartic mobius ladder $R(6r+3)$ is

$$P(R(6r+3);z) \;=\; (z-2)^{6r+3} + (z-1)\mathrm{tr}(A(z)^{2r+1}) + 1.$$

□

3. Equimodular curves and limit curves

For convenience, let $w = z - 2$. Then the chromatic polynomial of $R(6r+3)$ can be written as

$$(w^3)^{2r+1} + (w+1)(\lambda_1(w)^{2r+1} + \lambda_2(w)^{2r+1}) + 1,$$

where $\lambda_1(w)$ and $\lambda_2(w)$ are the eigenvalues of

$$\begin{pmatrix} w & -w(w-1) \\ w-1 & 2w-1 \end{pmatrix}.$$

The theorem of Beraha, Kahane and Weiss [**1**] tells us that as $r \to \infty$ the roots of the chromatic polynomial cluster around parts of the *equimodular curves*, comprising the points w such that any two of the values w^3, $\lambda_1(w)$, $\lambda_2(w)$, 1 are equal in modulus.

The three values w^3, $\lambda_1(w)$, $\lambda_2(w)$ are the eigenvalues of the 3×3 matrix

$$C = \begin{pmatrix} w^3 & 0 & 0 \\ 0 & w & -w(w-1) \\ 0 & w-1 & 2w-1 \end{pmatrix}.$$

The theory developed in [**2**] provides implicit equations for the equimodular curves associated with C.

Theorem 3 Two of the values $w^3, \lambda_1(w), \lambda_2(w)$ are equal in modulus if and only if there is a real number $t \in [0,4]$ such that

$$tw^3 - 9w^2 + 6w - 1 = 0 \quad \text{or} \quad w^6 - (3t-6)w^4 + (t^2-3t)w^3 + 9w^2 - 3tw + t = 0.$$

Proof The characteristic equation of C is

$$\lambda^3 - (w^3 + 3w - 1)\lambda^2 + 3w^4\lambda - w^6.$$

According to [**2**], substituting the coefficients

$$a_1(w) = -(w^3 + 3w - 1), \qquad a_2(w) = 3w^4, \qquad a_3(w) = -w^6$$

in the 'generic' expression

$$r_3(t, a_1, a_2, a_3) = (t-1)a_3^2 - (t-1)(t+2)a_1a_2a_3 + ta_2^2 + ta_1^3a_3 - a_1^2a_2^2,$$

produces an expression

$$v(t,w) = w^6(tw^3-9w^2+6w-1)(w^6-(3t-6)w^4+(t^2-3t)w^3+9w^2-3tw+t)$$

which has the required property. □

The expression $v(t,w)$ factorises because C is reducible [**2**, section 6]. The factor

$$f(t,w) = w^6 - (3t-6)w^4 + (t^2-3t)w^3 + 9w^2 - 3tw + t$$

defines the curve $\mathcal{C}_f$ where $|w^3|$ is equal to one of $|\lambda_1(w)|$, $|\lambda_2(w)|$. The factor

$$g(t,w) = tw^3 - 9w^2 + 6w - 1$$

defines the curve $\mathcal{C}_g$ where $|\lambda_1(w)| = |\lambda_2(w)|$.

The curves can be plotted directly from the information given above, but it is instructive to pursue the theoretical analysis a bit further. Since f is a polynomial of degree 6 in w, the curve $\mathcal{C}_f$ can be thought of as six images of the interval $[0,4]$. These arcs will be smooth unless the Jacobian J_f of the transformation $w \mapsto f(t,w)$ vanishes. In fact, J_f is also the discriminant of f as a polynomial in w, which turns out to be

$$J_f = 729t^3(t-4)^7(t^2+t+7)^2.$$

This shows that there are no non-smooth points corresponding to values of t strictly between 0 and 4. It also tells us that there will be coincidences of the arcs at the points corresponding to $t=0$ and $t=4$. In fact,

$$f(0,w) = w^2(w^2+3)^2, \qquad f(4,w) = (w-1)^4(w+2)^2.$$

The first formula shows that two arcs join up at each of the $t=0$ points $0, i\sqrt{3}, -i\sqrt{3}$. The second formula shows that four arcs join up at 1, and two arcs at -2. These facts are evident from a plot of the curve (Figure 1). If we choose the notation for the 'quadratic' roots so that, at each point w, $|\lambda_1(w)| \geq |\lambda_2(w)|$, then the outer loop of the curve corresponds to $|w^3| = |\lambda_1(w)|$, and the inner loop corresponds to $|w^3| = |\lambda_2(w)|$.

The Cartesian equation of $\mathcal{C}_f$ can be obtained by substituting $w = x+iy$ in $f(t,w)$, equating the real and imaginary parts to zero (remembering that t must be real), and eliminating t from these two equations. The eliminant has three factors, the one which defines $\mathcal{C}_f$ being

$$x^4 - 3x^2 + 2x + 2x^2y^2 - 3y^2 + y^4 = 0.$$

A curve of this type is known as a *limaçon*.

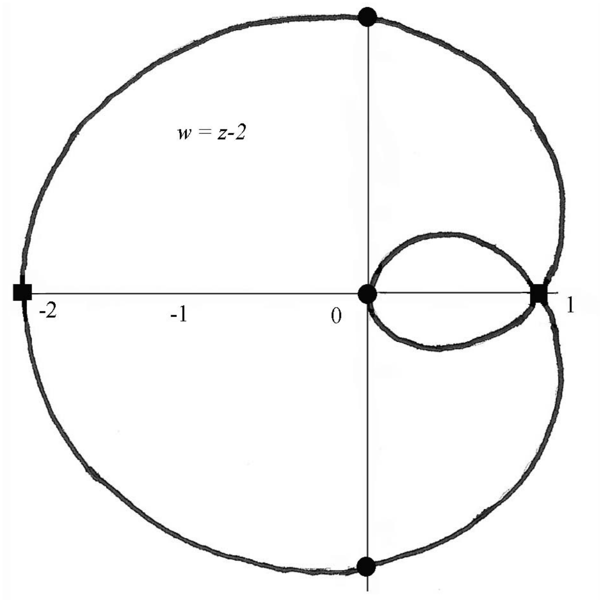

Fig.1: the curve $\mathcal{C}_f$, showing the six arcs

A similar analysis can be applied to $\mathcal{C}_g$. Here there are three arcs and

$$J_g = -27t(t-4), \qquad g(0,w) = -(3w-1)^2, \qquad g(4,w) = (4w-1)(w-1)^2.$$

The $t = 0$ points are $w = 1/3$ (twice) and ∞, since the coefficient of w^3 in $g(t,w)$ is t. The $t = 4$ points are $w = 1/4$ and $w = 1$ (twice). The three arcs combine to form the interval $[1/4, \infty)$ on the real axis, as indicated in Figure 2. This corresponds to the fact that the eigenvalues $\lambda_1(w)$, $\lambda_2(w)$ are complex conjugates when w belongs to this interval.

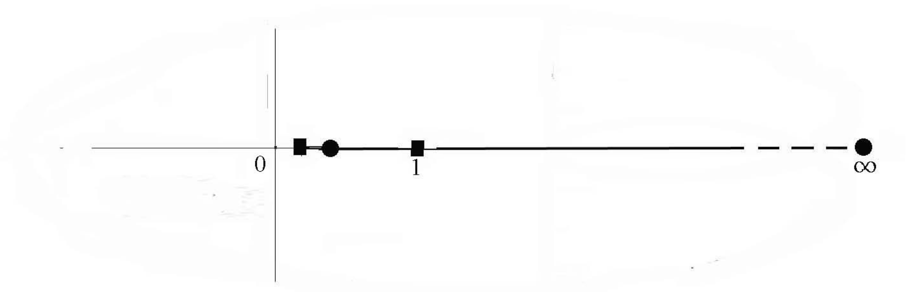

Fig.2: the curve $\mathcal{C}_g$, showing the three arcs

Thus far we have determined the equimodular curves given by equality between the moduli of two of the first three of the values w^3, $\lambda_1(w)$, $\lambda_2(w)$, 1. The curves involving the value 1 are easily dealt with. The curve $|w^3| = 1$ is just the circle $|w| = 1$, and the curves $|\lambda_1(w)| = 1$ and $|\lambda_2(w)| = 1$ can be identified as the result of a happy accident. Since $\lambda_1(w)\lambda_2(w) = w^3$, it follows that $|\lambda_1(w)| = 1$ if and only if $|\lambda_2(w)| = |w^3|$, and $|\lambda_2(w)| = 1$ if and only if $|\lambda_1(w)| = |w^3|$. Hence these curves are, respectively, the inner and outer loops of the curve $\mathcal{C}_f$ that we have already determined. The totality of the equimodular curves is thus as illustrated in Figure 3.

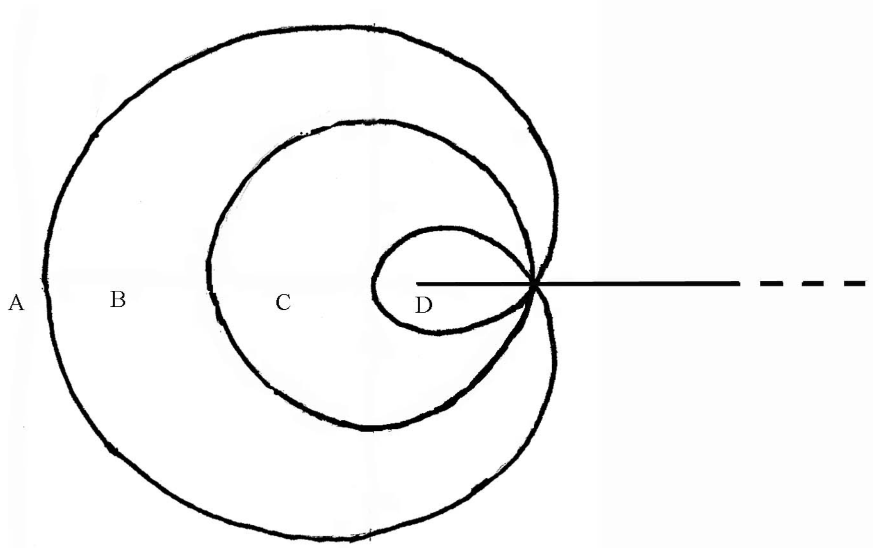

Fig.3: the equimodular curves

According to the Beraha-Kahane-Weiss theorem, the limit curves comprise only those parts of the equimodular curves that are *dominant*: that is, where the two values defining the curve are larger in modulus than the others. Define four regions A, B, C, D of the complex plane as in Figure 3.

A: outside the outer loop of $\mathcal{C}_f$;
B: between the outer loop of $\mathcal{C}_f$ and the circle $|w| = 1$;
C: between the circle $|w| = 1$ and the inner loop of $\mathcal{C}_f$;
D: inside the inner loop of $\mathcal{C}_f$.

Then it follows from the analysis given above that the ranking of the moduli in these regions is:

A : $\quad |w^3| > |\lambda_1(w)| > |\lambda_2(w)| > 1.$

$B:\quad |\lambda_1(w)| > |w^3| \geq 1 > |\lambda_2(w)|.$
$C:\quad |\lambda_1(w)| > 1 \geq |w^3| > |\lambda_2(w)|.$
$D:\quad 1 > |\lambda_1(w)| \geq |\lambda_2(w)| > |w^3|.$

Thus $|w^3|$ dominates in A, $|\lambda_1(w)|$ dominates in $B \cup C$, and 1 dominates in D. The equimodular curves $|\lambda_1(w)| = |\lambda_2(w)|$ (that is, $\mathcal{C}_g = [1/4, \infty)$) and $|w^3| = 1$ are not dominant, because they lie in regions where another value dominates. The final conclusion is that the limit curve is precisely $\mathcal{C}_f$.

This conclusion is confirmed by a plot of the roots of the chromatic polynomials of all quartic Möbius ladders with up to 59 vertices, including the graphs $R(6r+3)$ for $1 \leq r \leq 9$ (Figure 4).

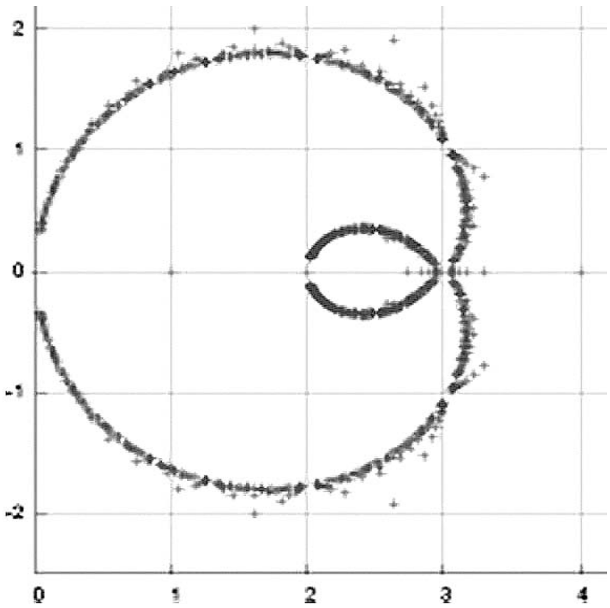

Fig.4: Royle's plot of the roots of quartic Möbius ladders ($z = w + 2$)

Acknowledgement I am grateful to Gordon Royle for bringing this problem to my attention and for supplying Figure 4.

References

1 S. Beraha, J. Kahane, N.J. Weiss. Limits of zeros of recursively defined families of polynomials, *Studies in foundations and combinatorics, Adv. Math. Suppl. Studies* 1 (1978) 213-232.

2 N.L. Biggs. Equimodular curves, *Discrete Mathematics* 259 (2002) 37-57.

3 N.L. Biggs. Specht modules and chromatic polynomials, *J. Combinatorial Theory B* 92 (2004) 359-377.

4 N.L. Biggs, M.H. Klin, P. Reinfeld. Algebraic methods for chromatic polynomials, *European J. Combinatorics* 25 (2004) 147-160.

5 D. Mayhew, G. Royle, G. Whittle. The internally 4-connected binary matroids with no $M(K_{3,3})$ minor, submitted.