NONCOHERENCE OF A CAUSAL WIENER ALGEBRA USED IN CONTROL THEORY

AMOL SASANE

Abstract. Let $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}$ and let $\mathcal{A}$ denote the ring

$$\mathcal{A} = \left\{ s(\in \mathbb{C}_+) \mapsto \widehat{f_a}(s) + \sum_{k=0}^{\infty} f_k e^{-st_k} \;\middle|\; \begin{array}{l} f_a \in L^1(0,\infty),\ (f_k)_{k\geq 0} \in \ell^1, \\ 0 = t_0 < t_1 < t_2 < \dots \end{array} \right\}$$

equipped with pointwise operations. (Here $\widehat{\cdot}$ denotes the Laplace transform.) It is shown that the ring $\mathcal{A}$ is not coherent, answering a question of Alban Quadrat [6, p. 30]. In fact, we present two principal ideals in the domain $\mathcal{A}$ whose intersection is not finitely generated.

CDAM Research Report LSE-CDAM-2008-2

1. Introduction

The aim of this paper is to show that the ring $\mathcal{A}$ (defined below) is not coherent.

We first recall the notion of a coherent ring.

Definition 1.1. Let R be a commutative ring with identity element 1, and let $R^n = R \times \dots \times R$ (n times). Let $f = (f_1, \dots, f_n) \in R^n$. An element $(g_1, \dots, g_n) \in R^n$ is called a *relation on* f if $g_1 f_1 + \dots + g_n f_n = 0$. The set of all relations on $f \in R^n$, denoted by $f^\perp$, is a R-submodule of the R-module R^n. The ring R is called *coherent* if for each $f \in R^n$, $f^\perp$ is finitely generated, that is, there exists a $d \in \mathbb{N}$ and there exist $g_j \in f^\perp$, $j \in \{1, \dots, d\}$, such that for all $g \in f^\perp$, there exist $r_j \in R$, $j \in \{1, \dots, d\}$ such that $g = r_1 g_1 + \dots + r_d g_d$.

An integral domain is coherent if and only if the intersection of any two finitely generated ideals of in the ring is again finitely generated; see [3, Theorem 2.3.2, p. 45].

The coherence of some rings of analytic functions has been investigated in earlier works. For example, W.S. McVoy and L.A. Rubel [4] showed that the Hardy algebra $H^\infty(\mathbb{D})$ is coherent, while the disc algebra $A(\mathbb{D})$ is not. Raymond Mortini and Michael von Renteln proved that the Wiener algebra W^+ (of all absolutely convergent Taylor series in the open unit disc) is not

1991 *Mathematics Subject Classification.* Primary 46J15; Secondary 13E15, 93D15.

Key words and phrases. coherent ring, Callier-Desoer algebra, control theory, Wiener algebra.

coherent [5]. In this article, we will show that the ring $\mathcal{A}$ (defined below, and which is useful in control theory) is coherent.

Throughout the article, we will use the following notation:

$$\mathbb{C}_+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}.$$

Definition 1.2. Let $\mathcal{A}$ denote the Banach algebra

$$\mathcal{A} = \left\{ f : \mathbb{C}_+ \to \mathbb{C} \;\middle|\; \begin{array}{l} f(s) = \widehat{f_a}(s) + \displaystyle\sum_{k=0}^{\infty} f_k e^{-st_k} \quad (s \in \mathbb{C}_+), \\ f_a \in L^1(0,\infty),\ (f_k)_{k\geq 0} \in \ell^1, 0 = t_0 < t_1 < t_2 < \dots \end{array} \right\}$$

equipped with pointwise operations and the norm:

$$\|f\|_{\mathcal{A}} := \|f_a\|_{L^1} + \|(f_k)_{k\geq 0}\|_{\ell^1}.$$

Here $\widehat{f_a}$ denotes the Laplace transform of f_a.

The above algebra arises as a natural class of transfer functions of stable distributed parameter systems in control theory; see [2], [9]. The relevance of the coherence property in control theory can be found in [7], [6].

Our main result is the following:

Theorem 1.3. *The ring $\mathcal{A}$ is not coherent.*

The proof of the main result is inspired by the proof of the noncoherence of W^+ given by Mortini and von Renteln in [5].

In Section 3, we will give the proof of Theorem 1.3. But before doing that, in Section 2, we first prove a few technical results needed in the sequel.

2. Preliminaries

Notation 2.1. Let $\mathfrak{m}_0$ denote the kernel of the complex homomorphism $f \mapsto f(0) : \mathcal{A} \to \mathbb{C}$, that is,

$$\mathfrak{m}_0 := \{f \in \mathcal{A} \mid f(0) = 0\}.$$

Then $\mathfrak{m}_0$ is a maximal ideal of $\mathcal{A}$, and this maximal ideal plays an important role in the proof of our main result in the next section. We will prove a few technical results about $\mathfrak{m}_0$ in this section, which will be used in the sequel. The following result is analogous to [5, Lemma 1]:

Lemma 2.2. *Let $L \neq (0)$ be an ideal in $\mathcal{A}$ contained in the maximal ideal $\mathfrak{m}_0$. If $L = L\mathfrak{m}_0$, that is, if every function $f \in L$ can be factorized in a product $f = hg$ of two functions $h \in L$ and $g \in \mathfrak{m}_0$, then L cannot be finitely generated.*

Proof. Suppose that

$$L = (f_1, \dots, f_N) \neq (0)$$

is a finitely generated ideal in $\mathcal{A}$ contained in the maximal ideal $\mathfrak{m}_0$. By our assumption there are functions $h_n \in L$, $g_n \in \mathfrak{m}_0$ with

$$f_n = h_n g_n \quad (n = 1, \dots, N).$$

Since $h_n \in L$, there exist functions $q_k^{(n)} \in \mathcal{A}$ with

$$h_n = \sum_{k=1}^{N} q_k^{(n)} f_k \quad (n = 1, \dots N;\ \ k = 1, \dots, N).$$

From this it follows that

$$\sum_{n=1}^{N} |h_n| \leq NC \sum_{n=1}^{N} |f_n| = NC \sum_{n=1}^{N} |h_n g_n| \quad \text{in } \mathbb{C}_+,$$

where C is a constant chosen so that

$$\big\| q_k^{(n)} \big\|_\infty \leq C \quad \text{for all } k \text{ and } n.$$

(Here $\|\cdot\|_\infty$ denotes the sup-norm over $\mathbb{C}_+$.) This implies together with the Cauchy-Schwarz inequality that

$$\sum_{n=1}^{N} |h_n|^2 \leq \left(\sum_{n=1}^{N} |h_n|\right)^2 \leq N^2C^2\left(\sum_{n=1}^{N} |h_n g_n|\right)^2 \leq N^2C^2\left(\sum_{n=1}^{N} |h_n|^2\right)\left(\sum_{n=1}^{N} |g_n|^2\right).$$

This inequality holds for all $s \in \mathbb{C}_+$. With $\delta := 1/(N^2C^2)$, we obtain the inequality

$$\delta \leq \sum_{n=1}^{N} |g_n(s)|^2 \tag{1}$$

for all points $s \in E$, where

$$E := \left\{ s \in \mathbb{C}_+ \;\middle|\; \sum_{n=1}^{N} |h_n(s)|^2 > 0 \right\}.$$

Since $L \neq (0)$, E is a dense subset of $\mathbb{C}_+$ (for otherwise, if $s_0 \in \mathbb{C}_+$ is such that it has a neighbourhood V in $\mathbb{C}_+$ where there is no point of E, then each h_n is identically zero in V, and by the identity theorem for holomorphic functions, each h_n is zero; consequently each f_n is zero, and so $L = (0)$, a contradiction). So by continuity, this inequality (1) holds in $\mathbb{C}_+$. But this contradicts the fact that each g_n vanishes at 0. $\square$

Since every maximal ideal is closed, $\mathfrak{m}_0$ is a commutative Banach subalgebra of $\mathcal{A}$, but obviously without identity element. But there is a substitute, namely the notion of the approximate identity, which turns out to be useful.

Definition 2.3. Let R be a commutative Banach algebra (without identity element). We say that R has a *(strong) approximate identity* if there exists a bounded (sequence) net $(e_\alpha)_\alpha$ of elements e_α in R such that for any $f \in R$,

$$\lim_\alpha \|e_\alpha f - f\| = 0.$$

We will now prove the following result, which shows that the maximal ideal $\mathfrak{m}_0$ in $\mathcal{A}$ has a strong approximate identity.

Theorem 2.4. *Let*

$$e_n := \frac{s}{s+\frac{1}{n}}, \quad n \in \mathbb{N}.$$

Then $(e_n)_{n\in\mathbb{N}}$ *is an approximate identity for* $\mathfrak{m}_0$*.*

The existence of an approximate identity for the maximal ideal $\mathfrak{m}_0$ in $\mathcal{A}$ is not obvious (since $\mathcal{A}$ and therefore $\mathfrak{m}_0$ is not a function algebra). In order to prove Theorem 2.4, we will need the following lemma.

Lemma 2.5. *Suppose* $\widehat{f} \in \mathfrak{m}_0$*. Then for each* $\epsilon > 0$*, there exists a* $\widehat{p} \in \mathfrak{m}_0$ *such that* p *has compact support in* $[0, \infty)$*, and* $\|\widehat{f} - \widehat{p}\|_{\mathcal{A}} < \epsilon$*.*

Proof. Let $\epsilon > 0$ be given. Let

$$f = f_a + \sum_{k=0}^{\infty} f_k \delta(\cdot - t_k),$$

where $f_a \in L^1(0,\infty)$, $(f_k)_{k\geq 0} \in \ell^1$, and $0 = t_0 < t_1 < t_2 < \cdots$. Choose a compactly supported $p_a \in L^1(0,\infty)$ such that

$$\|p_a - f_a\|_{L^1} < \frac{\epsilon}{4}.$$

Furthermore, select $N \in \mathbb{N}$ such that

$$\sum_{k>N} |f_k| < \frac{\epsilon}{4}.$$

Now let $T \in (0,\infty)$ be any number satisfying $t_N < T < t_{N+1}$, and define

$$f_T := -\left(\int_0^\infty p_a(t)dt + \sum_{k\leq N} f_k \right).$$

Set

$$p := p_a + \sum_{k\leq N} f_k \delta(\cdot - t_k) + f_T \delta(\cdot - T).$$

Then $\widehat{p} \in \mathcal{A}$ and

$$\widehat{p}(0) = \int_0^\infty p(t)dt = \int_0^\infty p_a(t)dt + \sum_{k\leq N} f_k + f_T = 0.$$

So $\widehat{p} \in \mathfrak{m}_0$. Clearly p has compact support contained in $[0, \infty)$. We have

$$\begin{aligned}
|f_T| &= \left| \int_0^\infty p_a(t)dt + \sum_{k \le N} f_k \right| \\
&= \left| \int_0^\infty f_a(t)dt + \sum_{k=0}^{\infty} f_k + \int_0^\infty \big(p_a(t) - f_a(t)\big)dt - \sum_{k>N} f_k \right| \\
&\le \left| \int_0^\infty f(t)dt \right| + \|p_a - f_a\|_{L^1} + \sum_{k>N} |f_k| \\
&= |\widehat{f}(0)| + \|p_a - f_a\|_{L^1} + \sum_{k>N} |f_k| \\
&< 0 + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.
\end{aligned}$$

Thus

$$\|\widehat{f} - \widehat{p}\|_{\mathcal{A}} = \|f_a - p_a\|_{L^1} + \sum_{k>N} |f_k| + |f_T| < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.$$

This completes the proof. □

We are now ready to prove the existence of an approximate identity for the maximal ideal $\mathfrak{m}_0$ in $\mathcal{A}$.

Proof of Theorem 2.4. Given $\widehat{f} \in \mathcal{A}$, and $\epsilon > 0$ arbitrarily small, in view of Lemma 2.5, we can find a $\widehat{p} \in \mathfrak{m}_0$ such that p has compact support and $\|\widehat{f} - \widehat{p}\|_{\mathcal{A}} < \epsilon$. Then

$$\|e_n \widehat{f} - \widehat{f}\|_{\mathcal{A}} \le \|e_n \widehat{p} - \widehat{p}\|_{\mathcal{A}} + \|e_n\|_{\mathcal{A}} \|\widehat{f} - \widehat{p}\|_{\mathcal{A}} + \|\widehat{f} - \widehat{p}\|_{\mathcal{A}}.$$

So it is enough to prove that

$$\lim_{n \to \infty} \|e_n \widehat{p} - \widehat{p}\|_{\mathcal{A}} = 0$$

for all $\widehat{p} \in \mathfrak{m}_0$ such that p has compact support in $[0, \infty)$. We do this below.

We have

$$e_n \widehat{p} - \widehat{p} = \frac{s + \frac{1}{n} - \frac{1}{n}}{s + \frac{1}{n}} \widehat{p} - \widehat{p} = -\frac{1}{n} \frac{1}{s + \frac{1}{n}} \widehat{p} = -\frac{1}{n} (\widehat{e^{-t/n} * p}).$$

Let C denote the convolution $e^{-t/n} * p$:

$$C(t) := \int_0^t e^{-\frac{t-\tau}{n}} p(\tau) d\tau.$$

We note that $C \in L^1(0, \infty)$, since $L^1(0, \infty)$ is an ideal in $\mathcal{A}$. Let $T > 0$ be such that

$$\operatorname{supp}(p) \subset [0, T].$$

We have

$$\|e_n\widehat{p}-\widehat{p}\|_{\mathcal{A}}=\frac{1}{n}\|C\|_{L^1}=\frac{1}{n}\int_0^\infty |C(t)|dt=\underbrace{\frac{1}{n}\int_0^T |C(t)|dt}_{(I)}+\underbrace{\frac{1}{n}\int_T^\infty |C(t)|dt}_{(II)}.$$

We estimate (I) as follows:

$$\begin{aligned}(I)=\frac{1}{n}\int_0^T |C(t)|dt &= \frac{1}{n}\int_0^T \left|\int_0^t e^{-\frac{t-\tau}{n}}p(\tau)d\tau\right|dt\\ &\leq \frac{1}{n}\int_0^T\int_0^t e^{-\frac{t-\tau}{n}}|p(\tau)|d\tau dt\\ &\leq \frac{1}{n}\underbrace{\left(\int_0^T\int_0^t 1\cdot|p(\tau)|d\tau dt\right)}_{(III)}.\end{aligned}$$

Since the integral (III) does not depend on n, we obtain that

$$\lim_{n\to\infty}\frac{1}{n}\int_0^T |C(t)|dt=0.$$

Furthermore,

$$\begin{aligned}(II) &= \frac{1}{n}\int_T^\infty |C(t)|dt=\frac{1}{n}\int_T^\infty e^{-\frac{t}{n}}\left|\int_0^t e^{\frac{\tau}{n}}p(\tau)d\tau\right|dt\\ &= \frac{1}{n}\int_T^\infty e^{-\frac{t}{n}}\left|\int_0^\infty e^{\frac{\tau}{n}}p(\tau)d\tau\right|dt \;\;\text{(since } \operatorname{supp}(p)\subset[0,T])\\ &= \frac{1}{n}\int_T^\infty e^{-\frac{t}{n}}\left|\widehat{p}\Big(-\frac{1}{n}\Big)\right|dt\end{aligned}$$

Since p has compact support in $[0,T]$, $\widehat{p}$ is an entire function by the Payley-Wiener theorem; see for instance [8, Theorem 7.2.3, p. 122]. Consequently,

$$(II)=\frac{1}{n}\int_T^\infty e^{-\frac{t}{n}}\left|\widehat{p}\Big(-\frac{1}{n}\Big)\right|dt=e^{-\frac{T}{n}}\left|\widehat{p}\Big(-\frac{1}{n}\Big)\right|\stackrel{n\to\infty}{\longrightarrow}1\cdot|\widehat{p}(0)|=1\cdot 0=0.$$

This completes the proof. □

We will also need the following, which is based on a key step from Browder's proof of Cohen's factorization theorem; see [1, Theorem 1.6.5, p. 74]. We will need this version since in our application in the proof of Theorem 1.3, we cannot use Cohen's factorization theorem directly.

Lemma 2.6. *Let $f_1,f_2\in\mathfrak{m}_0$ and $\delta>0$. Let $G(\mathcal{A})$ denote the set of all invertible elements in $\mathcal{A}$. Then there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that*

1. *for all $n\in\mathbb{N}$, $g_n\in G(\mathcal{A})$.*
2. *$(g_n)_{n\in\mathbb{N}}$ is convergent in $\mathcal{A}$ to a limit $g\in\mathfrak{m}_0$.*
3. *for all $n\in\mathbb{N}$, $\|g_n^{-1}f_i-g_{n+1}^{-1}f_i\|_{\mathcal{A}}\leq\delta/2^n$, $i=1,2$.*

Proof. Let $(e_n)_{n\in\mathbb{N}}$ denote the strong approximate identity for $\mathfrak{m}_0$ from Theorem 2.4. Let $K>1$ be such that $\|e_n\|_{\mathcal{A}}\le K$ for all $n\in\mathbb{N}$. Choose c such that

$$0<c<\frac{1}{4K}<\frac{1}{4}.$$

(A): If $e\in\mathfrak{m}_0$ and $\|e\|_{\mathcal{A}}\le 2$, then $1-c+ce\in G(\mathcal{A})$: indeed,

$$\left\|\frac{c}{c-1}e\right\|_{\mathcal{A}}<\frac{1/(4K)}{3/4}\cdot K=\frac{1}{3}<1,$$

and so

$$(1-c+ce)^{-1}=\frac{1}{1-c}\sum_{k=0}^{\infty}\left(\frac{c}{c-1}\right)^k e^k.$$

(B): Furthermore, we now show that if $\|eF-F\|_{\mathcal{A}}$ is small for some F, then so is $\|EF-F\|_{\mathcal{A}}$, where $E:=(1-c+ce)^{-1}$. Since

$$1=\frac{1}{1-c}\sum_{k=0}^{\infty}\left(\frac{c}{c-1}\right)^k,$$

we have

$$\begin{aligned}\|EF-F\|_{\mathcal{A}} &= \left\|\frac{1}{1-c}\sum_{k=0}^{\infty}\left(\frac{c}{c-1}\right)^k(e^kF-F)\right\|_{\mathcal{A}}\\ &\le \frac{1}{1-c}\sum_{k=0}^{\infty}\left(\frac{c}{c-1}\right)^k\|e^kF-F\|_{\mathcal{A}}.\end{aligned}$$

But

$$\begin{aligned}\|e^kF-F\|_{\mathcal{A}} &= \left\|\sum_{j=0}^{k-1}e^{j+1}F-F\right\|_{\mathcal{A}}\le\sum_{j=0}^{k-1}\|e^j\|_{\mathcal{A}}\|eF-F\|_{\mathcal{A}}\\ &\le \|eF-F\|_{\mathcal{A}}\sum_{j=0}^{k-1}\|e\|_{\mathcal{A}}^j<\|eF-F\|_{\mathcal{A}}\frac{K^k}{K-1}.\end{aligned}$$

Hence

$$\|EF-F\|_{\mathcal{A}}<\|eF-F\|_{\mathcal{A}}\frac{1}{1-c}\sum_{k=0}^{\infty}\frac{1}{K-1}\left(\frac{1}{4(1-c)}\right)^k<\frac{2}{K-1}\|eF-F\|_{\mathcal{A}}.$$

This estimate will be used in constructing the sequence of g_n's.

We shall inductively define a sequence $(e_{m_k})_{k\in\mathbb{N}}$ with terms from the approximate identity for $\mathfrak{m}_0$ such that if

$$g_n:=c\sum_{k=1}^{n}(1-c)^{k-1}e_{m_k}+(1-c)^n, \tag{2}$$

then we have $\|f-g_1^{-1}f\|_{\mathcal{A}}<\delta/2$ and

(P1) for all $n\in\mathbb{N}$, $g_n\in G(\mathcal{A})$

(P2) for all $n \in \mathbb{N}$, $\|g_n^{-1} f_i - g_{n+1}^{-1} f_i\|_{\mathcal{A}} < \delta/2^n$, $i = 1, 2$.

Choose e_{m_1} such that

$$\|e_{m_1} f_i - f_i\|_{\mathcal{A}} < \frac{\delta}{4}(K-1), \quad i = 1, 2.$$

Define $g_1 = c e_{m_1} + 1 - c$. So by (A), $g_1 \in G(\mathcal{A})$ and using the calculation in (B), we see that

$$\|f - g_1^{-1} f\|_{\mathcal{A}} < \frac{\delta}{2}.$$

Suppose that $e_{m_1}, \dots, e_{m_n}$ have been constructed, so that g_n defined by (2) satisfies (P1) and (P2). We assert that if we choose $e_{m_{n+1}}$ such that

$$\|e_{m_{n+1}} f_i - f_i\|_{\mathcal{A}} \;\; (i = 1, 2) \quad \text{ and } \quad \|e_{m_{n+1}} e_{m_k} - e_{m_k}\|_{\mathcal{A}} \;\; (1 \le k \le n)$$

are sufficiently small, then g_{n+1} defined by (2) satisfies (P1) and (P2), completing the induction step.

Indeed, if $E := (1 - c - c e_{m_{n+1}})^{-1}$, we have

$$\begin{aligned} g_n &= E^{-1} c \sum_{k=1}^{n} (1-c)^{k-1} E e_{m_k} + (1-c)^n \text{ and} \\ g_{n+1} &= E^{-1} \left(c \sum_{k=1}^{n} (1-c)^{k-1} E e_{m_k} + (1-c)^n \right). \end{aligned}$$

Let $G_n := c \sum_{k=1}^{n} (1-c)^{k-1} E e_{m_k} + (1-c)^n$. Then

$$\begin{aligned} \|G_n - g_n\|_{\mathcal{A}} &< \|E e_{m_k} - e_{m_k}\|_{\mathcal{A}} c \sum_{k=1}^{n} (1-c)^{k-1} < \max_{1 \le k \le n} \|E e_{m_k} - e_{m_k}\|_{\mathcal{A}} \\ &< \frac{2}{K-1} \max_{1 \le k \le n} \|e_{m_{n+1}} e_{m_k} - e_{m_k}\|_{\mathcal{A}}. \end{aligned}$$

Hence $G_n \in G(A)$ and moreover $\|G_n^{-1} - g_n^{-1}\|_{\mathcal{A}}$ is small, provided only that $\|e_{m_{n+1}} e_{m_k} - e_{m_k}\|_{\mathcal{A}}$ is small for $k = 1, \dots, n$.

Since $g_{n+1} = E^{-1} G_n$, we have then $g_{n+1} \in G(\mathcal{A})$, $g_{n+1}^{-1} = G_n^{-1} E$, and so for $i = 1, 2$,

$$\begin{aligned} \|g_{n+1}^{-1} f_i - g_n^{-1} f_i\|_{\mathcal{A}} &= \|G_n^{-1} E f_i - g_n^{-1} f_i\|_{\mathcal{A}} \\ &\le \|G_n^{-1} E f_i - g_n^{-1} E f_i\|_{\mathcal{A}} + \|g_n^{-1} E f_i - g_n^{-1} f_i\|_{\mathcal{A}} \\ &\le \|G_n^{-1} - g_n^{-1}\|_{\mathcal{A}} \|E f_i\|_{\mathcal{A}} + \|g_n^{-1}\|_{\mathcal{A}} \|E f_i - f_i\|_{\mathcal{A}}. \end{aligned}$$

Thus if $\|e_{m_{n+1}} f_i - f_i\|_{\mathcal{A}}$ $(i = 1, 2)$ and $\|e_{m_{n+1}} e_{m_k} - e_{m_k}\|_{\mathcal{A}}$ $(1 \le k \le n)$ are sufficiently small, we will have $\|g_{n+1}^{-1} f_i - g_n^{-1} f_i\|_{\mathcal{A}}$ as small as we please. This completes the induction step.

Since $\|e_{m_k}\|_{\mathcal{A}} \le K$ and $0 < 1 - c < 1$,

$$g_n \longrightarrow c \sum_{k=1}^{\infty} (1-c)^{k-1} e_{m_k} =: g \in \mathfrak{m}_0,$$

and the proof is completed. $\square$

3. Noncoherence of $\mathcal{A}$

Proof of Theorem 1.3. We will use the characterization that an integral domain is coherent if and only if the intersection of any two finitely generated ideals of in the ring is again finitely generated; see [3, Theorem 2.3.2, p. 45]. In fact, we present two finitely generated ideals I and J such that $I \cap J$ is not finitely generated.

Let

$$p = (1 - e^{-s})^3 \quad \text{and} \quad S = e^{-\frac{1+e^{-s}}{1-e^{-s}}}.$$

Clearly $p \in \mathfrak{m}_0$.

It is known, see for example [5, Remark after Theorem 1, p. 224], that

$$(1-z)^3 e^{-\frac{1+z}{1-z}} \in W^+ := \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \; (z \in \overline{\mathbb{D}}) \;\middle|\; \sum_{n=0}^{\infty} |a_n| < \infty \right\}.$$

Here $\overline{\mathbb{D}} := \{z \in \mathbb{C} \mid |z| \leq 1\}$. So if a_n's are defined via

$$(1-z)^3 e^{-\frac{1+z}{1-z}} = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots, \quad z \in \mathbb{D}, \tag{3}$$

then

$$\sum_{k=0}^{\infty} |a_k| < \infty.$$

If Re$(s)> 0$, then $e^{-s} \in \mathbb{D}$, and so from (3), we have

$$pS = a_0 + a_1 e^{-s} + a_2 e^{-2s} + a_3 e^{-3s} + \dots, \quad \text{Re}(s) > 0. \tag{4}$$

Since $\sum_{k=0}^{\infty} |a_k| < \infty$, the right hand side in (4) belongs to $\mathcal{A}$. So $pS \in \mathcal{A}$.

We define the ideals $I = (p)$ and $J = (pS)$.

Let

$$K := \{pSf \mid f \in \mathcal{A} \text{ and } Sf \in \mathcal{A}\}.$$

We claim that $K = I \cap J$. Trivially $K \subset I \cap J$. To prove the reverse inclusion, let $g \in I \cap J$. Then there exist two functions f and h in $\mathcal{A}$ such that $g = ph = pSf$. Hence $Sf = h \in \mathcal{A}$. So $g \in K$.

Let L denote the ideal

$$L := \{f \in \mathcal{A} \mid Sf \in \mathcal{A}\}.$$

Then $K := pSL$. Since S has a singularity at $s = 0$, it follows that $L \subset \mathfrak{m}_0$. We will show that $L = L\mathfrak{m}_0$. Let $f \in L$. We would like to factor $f = hg$ with $h \in L$ and $g \in \mathfrak{m}_0$. Applying Lemma 2.6 with $f_1 := f \in \mathfrak{m}_0$ and $f_2 := Sf \in \mathfrak{m}_0$, for any $\delta > 0$, there exists a sequence $(g_n)_{n \in \mathbb{N}}$ in $\mathcal{A}$ such that

(1) for all $n \in \mathbb{N}$, $g_n \in G(\mathcal{A})$.
(2) $(g_n)_{n \in \mathbb{N}}$ is convergent in $\mathcal{A}$ to a limit $g \in \mathfrak{m}_0$.
(3) for all $n \in \mathbb{N}$, $\|g_n^{-1} f - g_{n+1}^{-1} f\|_{\mathcal{A}} \leq \frac{\delta}{2^n}$ and $\|g_n^{-1} Sf - g_{n+1}^{-1} Sf\|_{\mathcal{A}} \leq \frac{\delta}{2^n}$.

Put

$$h_n := g_n^{-1} f \quad \text{and} \quad H_n := g_n^{-1} S f.$$

Then $h_n \in \mathfrak{m}_0$. Also $H_n \in \mathfrak{m}_0$, since $|S|$ is bounded by 1 on $\text{Re}(s) > 0$ and $f(0) = 0$. The estimates above imply that $(h_n)_{n\in\mathbb{N}}$ and $(H_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathcal{A}$. Since $\mathfrak{m}_0$ is closed, they converge to elements h and H, respectively, in $\mathfrak{m}_0$, that is, $h_n = g_n^{-1} f \to h$ and $H_n = g_n^{-1} S f = S h_n \to H$. Let H^∞ denote the Hardy space of all bounded analytic functions in the open right half plane equipped with the norm $\|\varphi\|_\infty := \sup_{\text{Re}(s)>0} |\varphi(s)|$, $\varphi \in H^\infty$. Since convergence in $\mathcal{A}$ implies convergence in H^∞, it follows that

$$\begin{aligned}
h_n &\xrightarrow{H^\infty} h && (\text{since } h_n \xrightarrow{\mathcal{A}} h)\\
S h_n &\xrightarrow{H^\infty} S h && (\text{since } h_n \xrightarrow{\mathcal{A}} h \text{ and } S \in H^\infty)\\
S h_n &\xrightarrow{H^\infty} H && (\text{since } H_n \xrightarrow{\mathcal{A}} H)
\end{aligned}$$

and so $Sh = H$. Also, in $\mathcal{A}$ norm we have

$$f = \lim_{n\to\infty} h_n g_n = hg.$$

Since h and $Sh = H$ belong to $\mathfrak{m}_0 \subset \mathcal{A}$, we see that $h \in L$. Moreover, as $g \in \mathfrak{m}_0$, we have got the desired factorization and $L = L\mathfrak{m}_0$.

But $L \neq (0)$, since $p \in L$. By Lemma 2.2, it follows that L cannot be finitely generated. Therefore, $pSL = I \cap J$ is not finitely generated. $\square$

References

[1] A. Browder. *Introduction to Function Algebras.* W.A. Benjamin, Inc., New York, 1969.

[2] F.M. Callier and C.A. Desoer. An algebra of transfer functions for distributed linear time-invariant systems. *IEEE Transactions on Circuits and Systems*, no. 9, CAS-25:651-662, 1978.

[3] S. Glaz. *Commutative Coherent Rings.* Lecture Notes in Mathematics, 1371, Berlin, 1989.

[4] W.S. McVoy and L.A. Rubel. Coherence of some rings of functions. *Journal of Functional Analysis*, no. 1, 21:76-87, 1976.

[5] R. Mortini and M. von Renteln. Ideals in the Wiener algebra W^+. *Journal of the Australian Mathematical Society (Series A)*, 46: 220-228, 1990.

[6] A. Quadrat. An introduction to internal stabilization of linear infinite dimensional systems. Course Notes, *École Internationale d'Automatique de Lille (02-06/09/02): Contrôle de systèmes à paramètres répartis: Théorie et Applications*, 2002. Available at: `http://www-sop.inria.fr/cafe/Alban.Quadrat/Pubs/Germany2.pdf`

[7] A. Quadrat. The fractional representation approach to synthesis problems: an algebraic analysis viewpoint. Part I: (Weakly) doubly coprime factorizations. *SIAM Journal on Control and Optimization*, 42:266-299, 2004.

[8] R.S. Strichartz. *A Guide to Distribution Theory and Fourier Transforms.* World Scientific, Singapore, 2003.

[9] M. Vidyasagar. *Control System Synthesis: A Factorization Approach.* MIT Press Series in Signal Processing, Optimization, and Control, 7, MIT Press, Cambridge, MA, 1985.

E-mail address: `A.J.Sasane@lse.ac.uk`

Mathematics Department, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom.