Balanced routing of random calls

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Abstract

We consider an online routing problem in continuous time, where calls have Poisson arrivals and exponential durations. The first-fit dynamic alternative routing algorithm sequentially selects up to d random two-link routes between the two endpoints of a call, via an intermediate node, and assigns the call to the first route with spare capacity on each link, if there is such a route. The balanced dynamic alternative routing algorithm simultaneously selects d random two-link routes; and the call is accepted on a route minimising the maximum of the loads on its two links, provided neither of these two links is saturated.

We determine the capacities needed for these algorithms to route calls successfully, and find that the balanced algorithm requires a much smaller capacity.

1 Introduction

We consider here an online routing problem in continuous time, where calls have Poisson arrivals and exponential durations. First, let us recall the following online routing problem in discrete time from [8], where calls do not

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end. There is a set $V = \{1, \dots, n\}$ of n nodes, each pair of which may wish to communicate. A *call* is an unordered pair $\{u, v\}$ of distinct nodes, that is an edge of the complete graph K_n on V. For each of the $N = \binom{n}{2}$ unordered pairs $\{u, v\}$ of distinct nodes, there is a *direct link*, also denoted by $\{u, v\}$, with capacity $D_1 = D_1(n)$. The direct link is used to route a call as long as it has available capacity. There are also two *indirect links*, denoted by uv and vu, each with capacity $D_2 = D_2(n)$. The indirect link uv may be used when for some w a call $\{u, w\}$ finds its direct link saturated, and we seek an alternative route via node v. Similarly vu may be used for alternative routes for calls $\{v, w\}$ via u.

We are given a sequence of M calls one at a time. For each call in turn, we must choose a route (either a direct link or an alternative two-link route via an intermediate node) if this is possible, before seeing later calls. These routes cannot be changed later, and calls do not end. The aim is to minimise the number of calls that fail to be routed successfully.

The calls are independent random variables $Z_1, Z_2, \dots, Z_M$, where each Z_j is uniformly distributed over the edges $e \in E(K_n)$. Let d be a (fixed) positive integer. A *general dynamic alternative routing algorithm* GDAR operates as follows. For each call $e = \{u, v\}$ in turn, the call is routed on the direct link if possible; and otherwise nodes $w_1, \dots, w_d$ are selected uniformly at random with replacement from $V \setminus \{u, v\}$ and the call is routed via one of these nodes if possible, along the two corresponding indirect links. The *first-fit dynamic alternative routing algorithm* FDAR is the version when we always choose the first possible alternative route, if there is one. The *balanced dynamic alternative routing algorithm* BDAR is the version when we choose an alternative route which minimises the larger of the current loads on its two indirect links, if possible.

Results for this model were first obtained in [4, 9], and later strengthened and extended in [8]. Consider the case where $M \sim cN$ for a constant $c > 0$. It is known that with the algorithm FDAR we need both link capacities D_1, D_2 of order $\sqrt{\frac{\ln n}{\ln \ln n}}$ to ensure that *asymptotically almost surely* (aas), that is 'with probability $\to 1$ as $n \to \infty$', all M calls are routed successfully. On the other hand, the balanced method BDAR succeeds with much smaller capacities. Specifically, there is a tight threshold value close to $\ln \ln n / \ln d$ for D_2 to guarantee that aas no call fails (and the precise value of D_1 is unimportant).

Here we consider a related continuous-time network model, with the desirable additional feature that calls end. Calls arrive in a Poisson process with rate $\lambda N = \lambda\binom{n}{2}$, where λ is a positive constant. The calls are iid random variables $Z_1, Z_2, \ldots$, where Z_j is the j-th call to arrive and is uniform over the edges of K_n for each j; also let T_j be the arrival time of call Z_j. For each edge $\{u, v\}$ there are two links, uv and vu, both with capacity $D = D(n) < \infty$. Since in [8] the use of direct links was found to have a minor effect on the total capacity requirements for efficient communication, here we do not use direct links but instead demand that each call be routed along a path consisting of a pair of indirect links. If a call is for $\{u, v\}$, then we pick d possible intermediate nodes uniformly at random with replacement, as in the GDAR algorithm. The FDAR algorithm always chooses the first possible alternative route, if there is one. The BDAR algorithm chooses an alternative route which minimises the larger of the current loads on its two links, if possible. Call durations are unit mean exponential random variables, independent of one another and of the arrivals and choices processes. Whenever a call terminates, both busy links are freed.

All random processes considered here are assumed to be right-continuous (as is standard). For each edge $e = \{u, v\} \in E$ and node $w \in V \setminus e$, let $X_t(e, w)$ denote the number of calls in progress at time t which are routed along the path consisting of links uw and vw, that is calls between the end nodes u and v of e routed via w. We call $X_t = (X_t(e, w) : e \in E, w \in V \setminus e)$ the *load vector* at time t, and let $\Omega = (\mathbb{Z}^+)^{N(n-2)}$ denote the set of all possible load vectors. The load vectors X_t for $t \geq 0$ form a continuous-time discrete-space Markov chain. By standard results, there exists a unique stationary distribution $\mathbf{\Pi}$; and, whatever the distribution of the starting state X_0, the distribution of the load vector X_t at time t converges to $\mathbf{\Pi}$ as $t \to \infty$.

Note that if there were no capacities then the calls in progress would form an immigration-death process with immigration rate $\lambda\binom{n}{2}$ and death rate 1. Thus in equilibrium the total number $\|X_t\|_1$ of calls in progress at time t is stochastically at most $Po(\lambda\binom{n}{2})$, and in particular

$$\mathbf{E}\,\|X_t\|_1 \leq \lambda\binom{n}{2} < \infty. \tag{1}$$

Our main interest is in the *blocking probability*, that is the probability that a new call fails to find an available route. As in the discrete version analysed in [8], or in the models analysed in [6] and [7], see also [2, 3, 11, 12], we observe the ‘power of two choices’ phenomenon; that is, with the BDAR algorithm for $d \geq 2$ the capacity required to ensure that most calls are routed successfully is much smaller than with the FDAR algorithm. Let us now state our two theorems.

Theorem 1.1 shows that, when the FDAR algorithm is used, capacity $D(n)$ of order $\frac{\ln n}{\ln \ln n}$ is needed in order to ensure that no call is lost in an interval of length polynomial in n. The set-up is that we have capacity $D = D(n) \sim \alpha \frac{\ln n}{\ln \ln n}$, and consider a time interval of length n^K. In the case $d = 2$, the result is roughly that, if $\alpha > K + 1$ then all calls are expected to be successful, and if $\alpha < K + 1$ then many calls are expected to fail. For general d we need to consider how K compares to $d - 2$, as there is a change of behaviour at $K = d - 2$.

We allow any initial configuration X_0 (for which $\mathbf{E}\,\|X_0\|_1$ is finite). If the system is in equilibrium at time 0 then our results apply to any time interval of length n^K: if the system is not in equilibrium at time 0 then we need to assume that the starting time t_0 of the interval is sufficiently large.

Theorem 1.1 *Let $\lambda > 0$ be fixed and let d be a fixed positive integer. Then there exists a constant $\kappa = \kappa(\lambda, d)$ such that the following holds. Let $K > 0$ and $\alpha > 0$ be constants. Let $D = D(n) \sim \alpha \frac{\ln n}{\ln \ln n}$. Let $t_1 = t_1(n) \geq \ln(\mathbf{E}\,\|X_0\|_1 + 1) + \kappa \ln n$, and consider the interval $[t_1, t_1 + n^K]$.*

(a) Suppose that $\alpha > (K + 2)/d$ if $K < d - 2$, and $\alpha > K + 3 - d$ if $K \geq d - 2$. Whatever version of GDAR we use, the mean number of calls lost during the interval is $o(1)$.

(b) Suppose that $\alpha < (K + 2)/d$ and $0 < \epsilon < K + 2 - d\alpha$ if $K < d - 2$; and that $\alpha < K + 3 - d$ and $0 < \epsilon < K + 3 - d - \alpha$ if $K \geq d - 2$. If we use the FDAR algorithm, then the mean number of calls lost during the interval is $\Omega(n^\epsilon)$.

If X_0 is in equilibrium then $\mathbf{E}\,\|X_0\|_1$ is finite by (1), so the conclusions above apply to an interval $[t_1, t_1 + n^K]$ for sufficiently large t_1; and hence they apply to all intervals $[t, t + n^K]$ for $t \geq 0$. Observe that for $K < d - 2$ the critical value of α is $(K + 2)/d$, and for $K \geq d - 2$ the critical value of α is $K + 3 - d$; and neither value depends on the arrival rate λ. As foreshadowed above, the next result shows that the BDAR algorithm requires significantly smaller capacities.

Colin: back as it was, we always keep λ fixed

Theorem 1.2 *Let $\lambda > 0$ be fixed and let $d \geq 2$ be a fixed integer. Let $K > 0$ be a constant. Then there exist constants $\kappa = \kappa(\lambda, d)$ and $c = c(\lambda, d, K) > 0$*

such that the following holds. Let $t_1 = t_1(n) \geq \ln(\mathbf{E} \|X_0\|_1 + 1) + \kappa \ln n$, *and consider the interval* $[t_1, t_1 + n^K]$.

(a) *If* $D(n) \geq \ln\ln n/\ln d + c$ *and we use the BDAR algorithm, then the expected number of failing calls during the interval is* $o(1)$.

(b) *If* $D(n) \leq \ln\ln n/\ln d - c$ *and we use any GDAR algorithm, then aas at least* $n^{K+2-d-o(1)}$ *calls are lost during the interval.*

We shall give further details, for example concerning the numbers of full links, when we prove Theorem 1.2.

We mention that a process similar to the one defined above, but also with direct links, was considered in [9], and then in [1]. The first of these works obtained, heuristically, some preliminary results. In [1], Anagnostopoulos et al. find an upper bound of $\ln\ln n/\ln d + o(\ln\ln n/\ln d)$ for the capacity required by the BDAR algorithm to ensure that, in equilibrium, an arriving call is accepted with probability tending to 1 as $n \to \infty$. Further, they identify a lower bound of $\Omega(\sqrt{\ln n/\ln\ln n})$ for the capacity needed by the FDAR algorithm to achieve the same effect. Here we prove sharper versions of these bounds, and supplement them with a lower bound on the performance of the BDAR algorithm and an upper bound on the performance of the FDAR algorithm. Further we extend them to prove upper and lower bounds on the performance of these algorithms over long time intervals. In comparison with [1], our bounds for the FDAR algorithm are of the order $\ln n/\ln\ln n$, not $\sqrt{\ln n/\ln\ln n}$; this is due to the fact that we do not allow direct routing between pairs of nodes.

Let us close this section by giving some further definitions and notation which we shall need shortly, and then a brief plan of the paper.

Given an edge $e = \{u, v\} \in E$, let $X_t(e) = \sum_{w \notin e} X_t(e, w)$ denote the number of calls between u and v in progress at time t. Also, given distinct nodes v and u, let $X_t(vu) = \sum_{w \neq u,v} X_t(\{v, w\}, u)$, which is the load of link vu at time t. Given a node v, let $X_t(v) = \sum_{u \neq v} X_t(vu)$, which is the number of calls with one end v at time t. Thus $\| X_t \|_1 = \frac{1}{2} \sum_{v \in V} X_t(v)$ is the total number of calls at time t. We say that a link is *saturated* (or full) if it has load equal to its capacity D. Given a node v, we let $\mathcal{S}_t^{\mathrm{D}}(\text{at } v)$ denote the set of saturated links vw for calls at v at time t; and let $S_t^{\mathrm{D}}(\text{at } v) = |\mathcal{S}_t^{\mathrm{D}}(\text{at } v)|$, which is the number of saturated links vw. Similarly, given a node w, we let $\mathcal{S}_t^{\mathrm{D}}(\text{via } w)$ denote the set of saturated links vw for calls at some node v at time t; and let $S_t^{\mathrm{D}}(\text{via } w) = |\mathcal{S}_t^{\mathrm{D}}(\text{via } w)|$. Also, for each time t we let ϕ_t

denote the σ-field generated by $(X_s : s \leq t)$, and let ϕ_{t-} denote the σ-field generated by $(X_s : s < t)$.

The rest of the paper is organised as follows. Section 2 contains some preliminary lemmas that will be needed in our proofs. In Section 3 we establish concentration of measure inequalities for the total number of calls as well as the number of saturated links. In Section 4 we prove Theorem 1.1, and in Section 5 we prove Theorem 1.2.

2 Preliminary results

In this section we give some elementary results which will be used in our proofs.

If X has the Poisson distribution with mean μ let us write $X \sim Po(\mu)$. For such a random variable X we shall use the bound

$$\mathbf{Pr}(X \geq k) \leq \mu^k/k! \leq (e\mu/k)^k$$

for each positive integer k. Given a positive integer D, let

$$p_D(\mu) = e^{-\mu} \sum_{k \geq D} \frac{\mu^k}{k!} \leq \mu^D/D!$$

be the probability that a $Po(\mu)$ random variable takes value at least D. The following are a pair of standard concentration inequalities for a binomial or Poisson random variable X with mean μ:

$$\mathbf{Pr}(X - \mu \geq \epsilon\mu) \leq \exp(-\frac{1}{3}\epsilon^2\mu) \tag{2}$$

and

$$\mathbf{Pr}(X - \mu \leq -\epsilon\mu) \leq \exp(-\frac{1}{2}\epsilon^2\mu) \tag{3}$$

for $0 \leq \epsilon \leq 1$ (see for example Theorem 2.3 (c) and inequality (2.8) in [10]).

We shall use the following version of Talagrand's inequality, see for example Theorem 4.3 in [10]. (In the notation in [10], the function h below is a (c^2r)-configuration function.)

Lemma 2.1 *Let $\mathbf{X} = (X_1, X_2, \ldots)$ be a finite family of independent random variables, where the random variable X_j takes values in a set Ω_j. Let $\Omega = \prod_j \Omega_j$.*

Let c and r be positive constants, and suppose that the non-negative real-valued function h on Ω satisfies the following two conditions for each $\mathbf{x} \in \Omega$.

- *Changing the value of a co-ordinate x_j can change the value of $h(\mathbf{x})$ by at most c.*
- *If $h(\mathbf{x}) = s$, then there is a set of at most rs co-ordinates such that $h(\mathbf{x}') \geq s$ for any $\mathbf{x}' \in \Omega$ which agrees with $\mathbf{x}$ on these co-ordinates.*

Let m be a median of the random variable $Z = h(\mathbf{X})$. For each $x \geq 0$

$$P(Z \geq m + x) \leq 2 \exp\left(-\frac{x^2}{4c^2 r(m+x)}\right), \tag{4}$$

and

$$P(Z \leq m - x) \leq 2 \exp\left(-\frac{x^2}{4c^2 rm}\right). \tag{5}$$

The next lemma concerns hitting times of a generalised random walk with 'drift'. It is the 'reverse' of Lemma 7.2 in [6], and can be deduced from that result by replacing the Y_i with $-Y_i$; we omit the details.

Lemma 2.2 *Let $\phi_0 \subseteq \phi_1 \subseteq \ldots$ be a filtration, and let $Y_1, Y_2, \ldots$ be random variables taking values in $\{-1, 0, 1\}$ such that each Y_i is ϕ_i-measurable. Let $E_0, E_1, \ldots,$ be events where $E_i \in \phi_i$ for each i, and let $E = \cap_i E_i$. For each $t \in \mathbb{N}$, let $R_t = R_0 + \sum_{i=1}^{t} Y_i$. Let $0 \leq p \leq 1/3$, let r_0 and r_1 be integers such that $r_1 < r_0$, and let m be an integer such that $pm \geq 2(r_0 - r_1)$. Assume that for each $i = 1, \ldots, m$,*

$$\mathbf{Pr}(Y_i = 1 | \phi_{i-1}) \leq p \quad \text{on } E_{i-1} \cap (R_{i-1} > r_1),$$

and

$$\mathbf{Pr}(Y_i = -1 | \phi_{i-1}) \geq 2p \quad \text{on } E_{i-1} \cap (R_{i-1} > r_1).$$

Then

$$\mathbf{Pr}\left(E \cap \{R_t > r_1 \ \ \forall t \in \{1, \ldots, m\}\} | R_0 = r_0\right) \leq \exp\left(-\frac{pm}{28}\right).$$

We can use the last lemma to upper bound hitting times for a type of discrete-time 'immigration-death' process.

Lemma 2.3 *Let $\phi_0 \subseteq \phi_1 \subseteq \ldots,$ be a filtration, and let $Y_1, Y_2, \ldots$ be random variables taking values in $\{-1, 0, 1\}$ such that each Y_i is ϕ_i-measurable. Let $E_0, E_1, \ldots$ be events where $E_i \in \phi_i$ for $i = 0, 1, \ldots$. Let $a, b > 0$ be constants. Let $\tilde{r}$ and r be integers with $2a/b \leq r \leq \tilde{r} - 1$.*

Let $R_0 = r_0$, and let $R_t = R_0 + \sum_{i=1}^{t} Y_i$. Assume that for each $i = 1, 2, \ldots$ and each $y \geq r + 1$

$$\mathbf{Pr}(Y_i = 1 | \phi_{i-1}) \leq a \quad \text{on} \quad E_{i-1} \cap (R_{i-1} > r);$$

and

$$\mathbf{Pr}(Y_i = -1 | \phi_{i-1}) \geq by \quad \text{on} \quad E_{i-1} \cap \ (R_{i-1} = y)$$

for each $y = r + 1, \ldots, \tilde{r}$, and

$$\mathbf{Pr}(Y_i = -1 | \phi_{i-1}) \geq b\tilde{r} \quad \text{on} \quad E_{i-1} \cap \ (R_{i-1} > \tilde{r}).$$

Let $m' = \lceil \frac{4}{b} \rceil \ \lceil \log_2 \frac{\tilde{r}}{r} \rceil$, and let E be the event $\cap_{i=1}^{m'} E_i$. Then

$$\mathbf{Pr}(E \cap \{R_t > r \ \forall t \in \{1, \ldots, m'\}\}) \leq 2 \exp\left(-\frac{r}{14}\right). \tag{6}$$

Proof. Let $k = \lceil \log_2 \frac{\tilde{r}}{r} \rceil - 1$, so that $2^k r < \tilde{r} \leq 2^{k+1} r$. Let $T_0, T_1, \ldots, T_k$ be the hitting times to cross the k intervals from r_0 down to $2^k r$, from $2^k r$ down to $2^{k-1} r$, and so on, ending with the interval from $2r$ down to r. Thus

$$T_0 = \min\{t \geq 0 : R_t = 2^k r\},$$

and for $j = 1, \ldots, k$,

$$T_j = \min\{t > T_{j-1} : R_t = 2^{k-j} r\}.$$

Consider $j \in \{0, \ldots, k\}$. We may use the last lemma with p as $p_j = b\, 2^{k-j-1} r$, $r_0 = 2^{k-j+1} r$ (except that for $j = 0$ we let $r_0 = \tilde{r}$), $r_1 = 2^{k-j} r$ and m as $m_j = \lceil \frac{4}{b} \rceil$. Note that $p_j m_j \geq 2^{k-j+1} r$, which is at least twice the length of the interval. (It may look at first sight that we are 'giving away' rather a lot on the 'upward' probability but this makes only a constant factor difference.) Hence, with $T_{-1} \equiv 0$,

$$\mathbf{Pr}(E \cap \{T_j - T_{j-1} > m_j\}) \leq \exp\left(-\frac{p_j m_j}{28}\right) \leq \exp\left(-\frac{2^{k-j} r}{14}\right).$$

But now

$$\begin{aligned}
\mathbf{Pr}(E \cap \{R_t > r \ \forall t \in \{1,\ldots,m'\}\}) &\leq \sum_{j=0}^{k} \mathbf{Pr}(E \cap (T_j - T_{j-1} > m_j)) \\
&\leq \sum_{j=0}^{k} \exp\left(-\frac{2^{k-j} r}{14}\right) \\
&\leq e^{-\frac{r}{14}}/(1 - e^{-\frac{r}{14}}).
\end{aligned}$$

Hence

$$\mathbf{Pr}(E \cap \{R_t > r \ \forall t \in \{1,\ldots,m\}\}) \leq 2e^{-\frac{r}{14}},$$

by the above if $e^{-\frac{r}{14}} \leq \frac{1}{2}$ and trivially otherwise. ∎

The next lemma is Lemma 7.3 in [6], and shows that if we try to cross an interval against the drift we rarely succeed.

Lemma 2.4 *Let a be a positive integer. Let p and q be reals with $q > p \geq 0$ and $p + q \leq 1$. Let $\phi_0 \subseteq \phi_1 \subseteq \phi_2 \subseteq \ldots$ be a filtration, and let $Y_1, Y_2, \ldots$ be random variables taking values in $\{-1, 0, 1\}$ such that each Y_i is ϕ_i-measurable. Let $E_0, E_1, \ldots$ be events where each $E_i \in \phi_i$, and let $E = \cap_i E_i$. Let $R_0 = 0$ and let $R_k = \sum_{i=1}^{k} Y_i$ for $k = 1, 2, \ldots$. Assume that for each $i = 1, \ldots, m$,*

$$\mathbf{Pr}(Y_i = 1 | \phi_{i-1}) \leq p \ \ \text{on } E_{i-1} \cap (0 \leq R_{i-1} \leq a-1),$$

and

$$\mathbf{Pr}(Y_i = -1 | \phi_{i-1}) \geq q \ \ \text{on } E_{i-1} \cap (0 \leq R_{i-1} \leq a-1).$$

Let

$$T = \inf\{k \geq 1 : R_k \in \{-1, a\}\}.$$

Then

$$\mathbf{Pr}\left(E \cap \{R_T = a\}\right) \leq (p/q)^a.$$

We shall require another lemma, similar to Lemma 2.1 in [6]. Consider the n-node case, with set $\Omega = (\mathbb{Z}^+)^{N(n-2)}$ of all load vectors. Let us say that a real-valued function f on Ω has *bounded increase at* a node v if whenever s and t are times with $s < t$, then $f(x_t)$ is at most $f(x_s)$ plus the total number of arrivals in the interval $(s, t]$ for v; f has *bounded increase via* a node v if whenever s and t are times with $s < t$, then $f(x_t)$ is at most $f(x_s)$ plus twice the total number of arrivals in the interval $(s,t]$ routed via v as the intermediate node; and f has *strongly bounded increase at* a node v if $f(x_t)$ is at most $f(x_s)$ plus the maximum number of arrivals for v in the interval $(s,t]$ which use any given link incident on v. Thus for example, given $v \in V$, $f(x) = x(v)$ has bounded increase at v, $f(x) = |\{w \in V \setminus \{v\} : x(wv) \geq D\}|$ has bounded increase via v, and $f(x) = \max_{u \in V \setminus \{v\}} x(vu)$ has strongly bounded increase at v.

Lemma 2.5 *Let v be a node in V. Let $\sigma, \tau > 0$ and let a, b be non-negative integers. Let $f : \Omega \to \mathbb{R}$ and $g, h : \mathbb{R} \to \mathbb{R}$ be functions.Let $E \in \phi_{t_1}$. Suppose that, for all $a \in \mathbb{R}$ and all times $t_1 \leq t \leq t_1 + \tau$, $\mathbf{Pr}\Big(E \cap \{f(X_t) \leq a\}\Big) \leq g(a)$ and $\mathbf{Pr}\Big(E \cap \{f(X_t) \geq a\}\Big) \leq h(a)$. Assume further that*

(a) f has bounded increase at v and $\theta = \mathbf{Pr}(Po(\lambda(n-1)\sigma) \geq b+1)$, or

(b) f has strongly bounded increase at v and $\theta = (n-1)\mathbf{Pr}(Po(\lambda d\sigma) \geq b+1)$.

Then

$$\mathbf{Pr}\Big[E \cap \{f(X_t) \leq a \text{ for some } t \in [t_1, t_1+\tau]\}\Big] \leq (\frac{\tau}{\sigma}+1)\,(g(a+b)+\theta)\,, \quad (7)$$

and

$$\mathbf{Pr}\Big[E \cap \{f(X_t) \geq a+b \text{ for some } t \in [t_1, t_1+\tau]\}\Big] \leq (\frac{\tau}{\sigma}+1)\,(h(a)+\theta)\,. \quad (8)$$

(c) Suppose that f has bounded increase via v and $\theta = \mathbf{Pr}(Po(\lambda d(n-1)\sigma/2) \geq b+1)$. Then

$$\mathbf{Pr}\Big[E \cap \{f(X_t) \leq a \text{ for some } t \in [t_1, t_1+\tau]\}\Big] \leq (\frac{\tau}{\sigma}+1)\,(g(a+2b)+\theta)\,, \quad (9)$$

and

$$\mathbf{Pr}\Big[E \cap \{f(X_t) \geq a+2b \text{ for some } t \in [t_1, t_1+\tau]\}\Big] \leq (\frac{\tau}{\sigma}+1)\,(h(a)+\theta)\,. \quad (10)$$

Proof. Consider first the case (a), when f has bounded increase at v. Note that the $j = \lfloor \frac{\tau}{\sigma} \rfloor + 1$ disjoint intervals $[t_1 + (r-1)\sigma, t_1 + r\sigma)$ for $r = 1, \ldots, j$ cover $[t_1, t_1+\tau]$. Let A_r denote the event that there are at least $b+1$

arrivals for node v in the interval $[t_1 + (r-1)\sigma, t_1 + r\sigma)$, so that $\mathbf{Pr}(A_r) = \mathbf{Pr}[Po(\lambda(n-1)\sigma) \geq b+1] = \theta$. But

$$\begin{aligned} & E \cap \{f(X_t) \leq a \text{ for some } t \in [t_1, t_1+\tau]\} \\ \subseteq E \cap \Big\{ & \big(\cup_{r=1}^{j}\{f(X_{t_1+(r-1)\sigma}) \leq a+b\}\big) \ \cup \ \big(\cup_{r=1}^{j} A_r\big)\}, \end{aligned}$$

and (7) follows. Similarly

$$\begin{aligned} & E \cap \{f(X_t) \geq a+b \text{ for some } t \in [t_1, t_1+\tau)\} \\ \subseteq E \cap \Big\{ & \big(\cup_{r=0}^{j-1}\{f(X_{t_1+(r-1)\sigma}) \geq a\}\big) \ \cup \ \big(\cup_{r=1}^{j} A_r\big) \Big\}, \end{aligned}$$

and (8) follows.

To handle the case (b) when f has strongly bounded increase at v, note that the arrival process onto any given link vu is stochastically dominated by a Poisson process with rate

$$(n-2)\lambda \frac{(n-2)^d - (n-3)^d}{(n-2)^d} \leq \lambda d.$$

Thus if B_r denotes the event that there are at least $b+1$ arrivals in the interval $[t_1 + (r-1)\sigma, t_1 + r\sigma)$ that are routed on some link vu, $u \neq v$, then

$$\mathbf{Pr}(B_r) \leq (n-1)\mathbf{Pr}[Po(\lambda d \sigma) \geq b+1];$$

and we can complete the proof as above.

Finally, in the case (c) the arrival process onto links with v as the intermediate node is stochastically dominated by a superposition of $\binom{n-1}{2}$ independent Poisson processes, each with rate

$$\lambda \frac{(n-2)^d - (n-3)^d}{(n-2)^d} \leq \frac{\lambda d}{n-2}.$$

If C_r denotes the event that there are at least $b+1$ arrivals in the interval $[t_1 + (r-1)\sigma, t_1 + r\sigma)$ that are routed via v, then $\mathbf{Pr}(C_r) \leq \mathbf{Pr}[Po(\lambda d(n-1)\sigma/2) \geq b+1]$. The rest of the proof is as above. ∎

Consider a continuous-time Markov process (X_t) with countable state space S and with q-matrix $q = (q(x,y) : x, y \in S)$. Under certain conditions we can compare features of its behaviour with that of independent immigration-death processes. We shall need the following lemma to handle the lower bound part of Theorem 1.1.

Let N be a positive integer and let the index j run over $\{1, \ldots, N\}$. For each j let $\mathbf{e}_j$ denote the jth unit N-vector and let f_j be a function from S to the non-negative integers; and write $\mathbf{f}(x)$ for $(f_1(x), \ldots, f_N(x))$. Assume that the following two conditions hold:

(i) for all distinct x and y in S such that $q(x, y) > 0$ we have $\mathbf{f}(y) = \mathbf{f}(x) \pm \mathbf{e}_j$ for some j ; and

(ii) for each $x \in S$ and each j

$$\sum_{y \in S: f_j(y) = f_j(x) - 1} q(x, y) = f_j(x).$$

Now define $\lambda_j(x)$ for each $x \in S$ and each j by setting

$$\lambda_j(x) = \sum_{y \in S: f_j(y) = f_j(x) + 1} q(x, y).$$

Lemma 2.6 *For each j let $\lambda_j > 0$ be a constant. Let $0 \leq t_1 < t_2$. Let $\mathbf{Y}_t = ((Y_j)_t : j = 1, \ldots, N)$ be a vector of independent immigration-death processes where $(Y_j)_t$ has immigration rate λ_j and death rate 1, and has population 0 at time t_1. Let $F \subseteq S$ be such that for each $x \in F$ and each j we have $\lambda_j(x) \geq \lambda_j$, and let A be the event that $X_t \in F$ for each $t \in [t_1, t_2]$. Then for each downset B in $\{0, 1, \ldots\}^N$,*

$$\mathbf{Pr}(\{\mathbf{f}(X_{t_2}) \in B\} \cap A) \leq \mathbf{Pr}(\mathbf{Y}_{t_2} \in B).$$

Now let n_j be a given positive integer for each j. Let $\tilde{\mathbf{Y}}_t = ((\tilde{Y_j})_t : j = 1, \ldots, N)$ be like $\mathbf{Y}_t$ except that $(\tilde{Y_j})_t$ has upper population limit n_j. Let $\tilde{F} \subseteq S$ be such that, for each $x \in \tilde{F}$ and each $j = 1, \ldots, N$, if $f_j(x) < n_j$ then $\lambda_j(x) \geq \lambda_j$. Let $\tilde{A}$ be the event that $X_t \in \tilde{F}$ for each $t \in [t_1, t_2]$. Then for each downset B in $\{0, 1, \ldots\}^N$,

$$\mathbf{Pr}(\{\mathbf{f}(X_{t_2}) \in B\} \cap \tilde{A}) \leq \mathbf{Pr}(\mathbf{Y}_{t_2} \in B).$$

Proof. Let $x_0 \in F$, and condition on $X_{t_1} = x_0$. Then we may assume that $\lambda_j(x) \geq \lambda_j$ for each $x \in S$, since the values $\lambda_j(x)$ for $x \notin F$ are irrelevant;

and then we may ignore the event A. But now an easy coupling argument shows that

$$\mathbf{Pr}(\mathbf{f}(X_{t_2}) \in B | X_{t_1} = x_0) \leq \mathbf{Pr}(\mathbf{Y}_{t_2} \in B),$$

and since this is true for each $x_0 \in F$ the result follows. The second part, with population limits, may be proved similarly. ∎

3 Failure probability, total load and saturated links

In this section we give upper and lower bounds on the failure probability of a call, upper and lower bounds on the total number of calls for a node v, and upper bounds on the number of saturated links incident on v over long periods of time. All the results are valid for any GDAR algorithm.

First we consider the failure probability of a call. For each time t and node v, let $d_t(v) = S_t^{\mathrm{D}}(\text{at } v)$, and let $\Delta_t = \max_v d_t(v)$. Recall that T is the random departure time of the last one of the $\|X_0\|_1$ initial calls, and for $j = 1, 2, \ldots$ the call Z_j arrives at time T_j.

Lemma 3.1 *On $T_j = t$*

$$\mathbf{Pr}(Z_j \textit{ fails } | \phi_{t-}) \leq \left(\frac{2\Delta_{t-}}{n-2} \right)^d, \tag{11}$$

and

$$\mathbf{Pr}(Z_j \textit{ fails } | \phi_{t-}) \leq 2^{d+1} n^{-1} \sum_{v \in V} \left(\frac{d_{t-}(v)}{n-2} \right)^d; \tag{12}$$

also,

$$\mathbf{Pr}(Z_j \textit{ fails } | \phi_{t-}) \geq n^{-1} \sum_{v \in V} \left(\frac{d_{t-}(v)}{n-2} \right)^d. \tag{13}$$

and so

$$\mathbf{Pr}(Z_j \textit{ fails } | \phi_{t-}) \geq \left(\frac{\min_v d_{t-}(v)}{n-2} \right)^d. \tag{14}$$

Proof. Recall that $N = \binom{n}{2}$. On $T_j = t$ we have

$$\begin{aligned}\mathbf{Pr}(Z_t \text{ fails } |\phi_{t-}) &\leq \frac{1}{N}\sum_{u\neq v}\left(\frac{d_{t-}(u)+d_{t-}(v)}{n-2}\right)^d \\ &\leq \frac{2^{d-1}}{N(n-2)^d}\sum_{u\neq v}(d_{t-}(u)^d + d_{t-}(v)^d),\end{aligned}$$

and both upper bounds follow. (For the second inequality we used the fact that $(x+y)^d \leq 2^{d-1}(x^d+y^d)$ for $x, y > 0$.)

On the other hand, on $T_j = t$

$$\mathbf{Pr}(Z_t \text{ fails } |\phi_{t-}) \geq \frac{1}{N}(n-2)^{-d}\sum_{v\in V}\sum_{u\neq v}(d_{t-}(v) - \mathbb{I}_{X_{t-}(vu)=D})^d.$$

But for each $v \in V$,

$$\sum_{u\neq v}(d_{t-}(v)-\mathbb{I}_{X_{t-}(vu)=D})^d = (n-1-d_{t-}(v))d_{t-}(v)^d+d_{t-}(v)(d_{t-}(v)-1)^d \geq \frac{1}{2}(n-1)d_{t-}(v)^d$$

for $n \geq 3$. Hence on $T_j = t$

$$\begin{aligned}\mathbf{Pr}(Z_t \text{ fails } |\phi_{t-}) &\geq \frac{1}{N}(n-2)^{-d}\sum_{v\in V}\frac{1}{2}(n-1)d_{t-}(v)^d \\ &= n^{-1}\sum_{v\in V}\left(\frac{d_{t-}(v)}{n-2}\right)^d,\end{aligned}$$

and so both lower bounds (13) and (14) follow. ∎

To obtain our estimates for the total number of calls for a node v, and upper bounds on the number of saturated links incident on v, we compare the process (X_t) to a 'superprocess' $(\tilde{X}_t)$ which satisfies $\tilde{X}_0 = X_0$ and evolves as follows. The unordered pairs of distinct nodes u and v receive independent rate λ Poisson arrival streams of calls; each link uv has infinite capacity; and each call throughout its duration occupies d two-link routes chosen uniformly at random with replacement. (If a route is chosen more than once by a given call, the call will still be counted only once on the corresponding two links.) All call durations are unit mean exponentials independent of one another and of the arrivals and choices processes. For each pair of distinct nodes u and v, $\tilde{X}_t(\{u,v\}) = \sum_{w\neq u,v} \tilde{X}_t(\{u,v\},w)$ denotes the number of calls in progress between u and v at time t. Note that the process $(\tilde{X}_t(\{u,v\}) : u,v \in V, u \neq v)_{t\geq 0}$ is itself Markov, since the capacities are infinite. It has a unique equilibrium distribution, and in equilibrium the $\tilde{X}_t(\{u,v\})$ are all independent $Po(\lambda)$ random variables. Thus, in equilibrium the total number $\|\tilde{X}_t\|_1$ of ongoing calls at time t is $Po(\lambda\binom{n}{2})$; and, for each v, the total number $\tilde{X}_t(v)$ of ongoing calls with one end v is $Po(\lambda(n-1))$.

We shall use T_v to denote the time that the last of the $X_0(v)$ initial calls with one end v departs. Also, we let $T = \max_{v\in V} T_v$. For various events A we shall give an upper bound on $\mathbf{Pr}(A \cap \{T \leq t\})$. We may later obtain an upper bound on $\mathbf{Pr}(A)$ using

$$\mathbf{Pr}(A) \leq \mathbf{Pr}(A \cap \{T \leq t\}) + \mathbf{Pr}(T > t), \tag{15}$$

and noting that

$$\mathbf{Pr}(T > t) \leq \mathbf{E}\,\|X_0\|_1\, e^{-t}. \tag{16}$$

To see why (16) holds, temporarily let S_t be the number of initial calls surviving to time t, and observe that

$$\mathbf{Pr}(T \geq t) = \mathbf{Pr}(S_t > 0) \leq \mathbf{E}S_t = \mathbf{E}\,\|X_0\|_1\, e^{-t}$$

The next lemma shows that for any node $v \in V$, $X_t(v)$ is unlikely to deviate far above $\lambda(n-1)$ once the initial calls have gone.

Lemma 3.2 *Let $0 < \delta < 1$, let n be a positive integer, and let A_t be the event that $X_t(v) \geq (1+\delta)\lambda(n-1)$ for some vertex v. Then for all times $t \geq 0$*

$$\mathbf{Pr}(A_t \cap \{T \leq t\}) \leq ne^{-\frac{1}{3}\delta^2\lambda(n-1)}. \tag{17}$$

(The value of D is not relevant here.)

Proof. Consider links vu incident on a given node v. The total number of calls on those links is stochastically dominated by the number of calls for v corresponding to the process $(\tilde{X}_t)$. Let $(\tilde{Y}_t) = (\tilde{Y}_t(\{u,v\}) : u,v \in V)$ for $t \geq 0$ be a Markov process with the same q-matrix as $(\tilde{X}_t)$ but in equilibrium. We couple (X_t), $(\tilde{X}_t)$ and $(\tilde{Y}_t)$ as follows. We assume that $X_0 = \tilde{X}_0$. All subsequent arrival and potential departure times of new calls are the same for the three processes, except that the departures of calls that were not accepted in (X_t) are ignored in that process. Additionally, every one of the

$\|X_0\|_1$ initial calls in (X_t) is coupled with a corresponding initial call in $(\tilde{X}_t)$, and the paired calls have the same departure times.

Under the coupling, on the event $T_v \leq t$, for all times t and all $u \neq v$,

$$X_t(vu) \leq \tilde{X}_t(vu) \leq \tilde{Y}_t(vu),$$

and so also

$$X_t(v) \leq \tilde{X}_t(v) \leq \tilde{Y}_t(v).$$

But $\tilde{Y}_t(v)$ is a Poisson random variable with mean $\lambda(n-1)$, and so by the concentration inequality (2), we have

$$\mathbf{Pr}\left(\{X_t(v) \geq (1+\delta)\lambda(n-1)\} \cap \{T_v \leq t\}\right) \leq e^{-\frac{1}{3}\delta^2\lambda(n-1)}.$$

Now (17) follows. ∎

We shall need to upper bound the number of saturated links around any given node, as in the following lemma.

Lemma 3.3 *Let n and D be positive integers, and let $k \geq 4p_D(d\lambda)(n-1)$. Then for each $t \geq 0$*

$$\mathbf{Pr}(\{S_t^D(at\ v) \geq k\} \wedge \{T \leq t\}) \leq 2\exp\left(-\frac{k}{2d^2D}\right) \tag{18}$$

and

$$\mathbf{Pr}(\{S_t^D(via\ v) \geq k\} \wedge \{T \leq t\}) \leq 2\exp\left(-\frac{k}{8D}\right). \tag{19}$$

Observe that if $\delta > 0$ and $D = D(n) \to \infty$ then for n sufficiently large we may for example take k as δn.

Proof. We use the coupling of the three processes (X_t), $(\tilde{X}_t)$ and $(\tilde{Y}_t)$ described in the proof of Lemma 3.2. Let $v \in V$ be a node. Note that, for each $u \neq v$, the load $\tilde{Y}_t(vu)$ of link vu is a Poisson random variable with mean

$$\lambda(n-2)\frac{(n-2)^d - (n-3)^d}{(n-2)^d} \leq d\lambda.$$

We write $\tilde{\mathcal{S}}_t^{\mathrm{D}}(\text{at } v)$ to denote the set of links vw for calls at v that have load at least D at time t in the stationary superprocess $(\tilde{Y}_t)$; and we write $\tilde{S}_t^{\mathrm{D}}(\text{at } v) = |\tilde{\mathcal{S}}_t^{\mathrm{D}}(\text{at } v)|$. Also, for $w \in W$, $\tilde{\mathcal{S}}_t^{\mathrm{D}}(\text{via } w)$ denotes the set of links uw for calls at some node u, and routed via w, that have load at least D at time t in

$(\tilde{Y}_t)$; and $\tilde{S}_t^{\mathrm{D}}(\text{via } w) = |\tilde{\mathcal{S}}_t^{\mathrm{D}}(\text{via } w)|$. Then $\mathbf{E}[\tilde{S}_t^{\mathrm{D}}(\text{at } v)] \leq (n-1)p_D(d\lambda)$ and $\mathbf{E}[\tilde{S}_t^{\mathrm{D}}(\text{via } w)] \leq (n-1)p_D(d\lambda)$ for all times $t \geq 0$.

For a given $v \in V$, the loads $\tilde{Y}_t(vu)$ of links vu for $u \neq v$ are determined by a set of $(n-1)(n-2)^d$ independent Poisson random variables each with mean $\lambda/(n-2)^d$ (corresponding to $n-1$ choices of the other end node w and $(n-2)^d$ choices of d routes for a call with end nodes v and w), and so there is strong concentration of measure. Note that the median $m(v)$ of $\tilde{S}_t^{\mathrm{D}}(\text{at } v)$ is at most $2(n-1)p_D(d\lambda)$. We can use Talagrand's inequality Lemma 2.1, with $c = d$ and $r = D$. This gives, for all $t \geq 0$,

$$\mathbf{Pr}(\tilde{S}_t^{\mathrm{D}}(\text{at } v) \geq m(v) + z) \leq 2\exp\left(-\frac{z^2}{4d^2D(m(v)+z)}\right).$$

Now take $z \geq 2(n-1)p_D(d\lambda) \geq m(v)$, so that

$$\mathbf{Pr}(\tilde{S}_t^{\mathrm{D}}(\text{at } v) \geq 2z) \leq 2\exp\left(-\frac{z}{8d^2D}\right).$$

Similarly, given $w \in V$, the loads $\tilde{Y}_t(uw)$ of links uw for $u \neq w$ are determined by a set of $\binom{n-1}{2}[(n-2)^d - (n-3)^d]$ independent random variables $Po(\lambda/(n-2)^d)$ (corresponding to calls for all possible pairs of distinct nodes $v, u \in V \setminus \{w\}$ choosing a route via node w). Applying Talagrand's inequality with $c = 2$ and $r = D$, we have, for $t \geq 0$ and $z \geq 2(n-1)p_D(d\lambda)$,

$$\mathbf{Pr}(\tilde{S}_t^{\mathrm{D}}(\text{via } w) \geq 2z) \leq 2\exp\left(-\frac{z}{32D}\right).$$

On the event $\{T \leq t\}$

$$X_t(vu) \leq \tilde{X}_t(vu) \leq \tilde{Y}_t(vu)$$

for each link vu, and we deduce that inequalities (18) and (19) hold. ∎

The next lemma uses the last one to show that for any node $v \in V$, for all sufficiently large times t, $X_t(v)$ is unlikely to deviate much below $\lambda(n-1)$.

Lemma 3.4 *Let $0 < \delta < 1$ be a constant. Let the capacity $D = D(n)$ be such that $p_D(d\lambda) \leq \delta/32$ and $D = o(n)$. Let B_t be the event that $X_t(v) \leq (1-\delta)\lambda(n-1)$ for some vertex v. Then there exists a constant $\eta = \eta(\delta) > 0$ such that, for each positive integer n we have* changes here and to the proof

$$\mathbf{Pr}(B_{t_2} \cap \{T \leq t_1\}) \leq 2e^{-\eta n/D} \tag{20}$$

for all times $t_1 \geq 0$ and $t_2 \geq t_1 + \ln(4/\delta)$.

Proof. Observe that the left hand side of (20) is non-decreasing in t_1, and so we need only consider the case when $t_2 = t_1 + \ln(4/\delta)$. For each $t_1 \geq 0$, let $t_2 = t_1 + \ln(4/\delta)$; and each $v \in V$ let $A_{t_1}(v)$ be the event that

I did not always remember to put in the factor $1/D$ - now done

$$S_t^{\mathrm{D}}(\text{at } v) \leq (n-2)\delta/4 \text{ for all } t \in [t_1, t_2]$$

and let $A_{t_1} = \cap_v A_{t_1}(v)$. By (18) in Lemma 3.3, we have

$$\mathbf{Pr}(\{S_t^{\mathrm{D}}(\text{at } v) \geq (n-2)\delta/8\} \cap \{T \leq t_1\}) \leq 2\exp\left(-\frac{(n-2)\delta}{16d^2D}\right). \qquad (21)$$

We now apply Lemma 2.5, part (a), with $a = b = (n-2)\delta/8$, $\sigma = \delta/d\lambda$, and E as the event that $T \leq t_1$. Thus

$$\mathbf{Pr}(\{T \leq t_1\} \cap \overline{A_{t_1}(v)}) \leq (\frac{\ln(4/\delta)}{\sigma} + 1)(h(a) + \theta),$$

where $h(a)$ is the right hand side of (21), and

$$\theta = \mathbf{Pr}(Po(\lambda(n-1)\sigma \geq b+1) \leq e^{-\delta(n-1)/3d},$$

by inequality (2). Now summing over all $v \in V$ we obtain

$$\mathbf{Pr}(\{T \leq t_1\} \cap \overline{A_{t_1}}) \leq n\,(d\ln(4/\delta)\lambda/\delta + 1)\left(2\exp\left(-\frac{(n-2)\delta}{16d^2D}\right) + e^{-\delta(n-1)/3d}\right).$$

Thus there exists a constant $\eta = \eta(\delta)$ such that for all $n \in \mathbb{N}$ and all $t_1 \geq 0$ we have

do not care about large t here

$$\mathbf{Pr}(\{T \leq t_1\} \cap \overline{A_{t_1}}) \leq 2e^{-\eta n/D}.$$

what do you mean?? now changed

Now, by (11) in Lemma 3.1, on the event that $S_{t-}^{\mathrm{D}}(\text{at } u) \leq (n-2)\delta/4$ for each vertex u, the probability, conditional on ϕ_{t-} and on the end points of the call, that a new call arriving at time t would not be accepted is at most $\delta/2$, and so the rate at which calls for a given node v are accepted is at least $(n-1)\lambda(1-\delta/2)$. We may now apply Lemma 2.6 in the special case with $N = 1$, $f_1(x)$ as the number of calls in progress with one end v, and $\lambda_1 = (n-1)\lambda(1-\delta/2)$. Also $1 - e^{-(t_2-t_1)} = 1 - e^{-\ln(4/\delta)} = 1 - \delta/4$. Hence, on A_{t_1}, $X_{t_2}(v)$ stochastically dominates a Poisson random variable $Po((n-1)\lambda(1-3\delta/4))$. Then, using the concentration inequality (3), for all v,

$$\mathbf{Pr}(A_{t_1} \cap \{X_{t_2}(v) \leq (1-\delta)\lambda(n-1)\}) \leq e^{-\delta^2\lambda(n-1)/128}.$$

Combining the above estimates we have that $\mathbf{Pr}(\{T \le t_1\} \cap \{X_{t_2}(v) \le (1-\delta)\lambda(n-1)\})$ is at most

$$\mathbf{Pr}\left(\{T \le t_1\} \cap \overline{A_{t_1}}\right) + \mathbf{Pr}(A_{t_1} \cap \{X_{t_2}(v) \le (1-\delta)\lambda(n-1)\}) \le 2e^{-\eta n/D}$$

for all n with a suitable new value of η: thus

$$\mathbf{Pr}(\{T \le t_1\} \cap \{X_{t_2}(v) \le (1-\delta)\lambda(n-1)\}) \le 2e^{-\eta n/D}, \qquad (22)$$

and the lemma follows easily. ∎

To end this section we put together two of the results above to show that we are unlikely to observe large deviations of $X_t(v)$ from $\lambda(n-1)$ for any node v even during very long time intervals.

Lemma 3.5 *Given $0 < \delta < 1$, there exists a constant $\beta = \beta(\delta) > 0$ such that the following holds. Let the capacity $D = D(n)$ be such that $p_D(d\lambda) \le \delta/32$, and say $D = O(n^{\frac{1}{2}})$. Let $\theta > 0$ and let $t_0 = \ln(\mathbf{E}\,\|X_0\|_1 + 1) + (\theta+1)\ln n$. Let C_t denote the event that $|X_t(v) - \lambda(n-1)| > \delta\lambda(n-1)$ for some vertex v. Then for each positive integer n and each time $t_1 \ge t_0 + \ln(4/\delta)$*

$$\mathbf{Pr}(C_t \text{ holds for some } t \in [t_1, t_1 + e^{\eta n/D}]) = o(n^{-\theta}).$$

Proof. Observe that $C_t = A_t \cup B_t$, where A_t and B_t are defined in Lemmas 3.2 and 3.4 respectively. Hence by these lemmas there exists a constant $\gamma > 0$ such that for any time $t \ge t_0 + \ln(4/\delta)$ we have

$$\mathbf{Pr}\Big(C_t \cap \{T \le t_0\}\Big) \le 2e^{-\gamma n/D}.$$

Let $\beta = \gamma/3$. We may now apply case (a) of Lemma 2.5, with $\tau = e^{\beta n}$ and $\sigma = \delta/4$, and $b = \delta\lambda(n-1)/2$. We use inequality (7) with $a = (1-\delta)\lambda(n-1)$ for deviations below the mean, and inequality (8) with $a = (1-\delta/2)\lambda(n-1)$ for deviations above the mean. Thus for all positive integers n and all times $t_1 \ge t_0 + \ln(4/\delta)$ we have

$$\mathbf{Pr}\left(\{C_t \text{ for some } t \in [t_1, t_1 + e^{\beta n/D}]\} \cap \{T \le t_0\}\right) \le 2e^{-\beta n/D}.$$

We may now use (15) and (16) to complete the proof. ∎

4 Proof of Theorem 1.1

Recall that T denotes the departure time of the last one among those calls that were present in the system at time 0. Let $\tau_1 = \tau_1(n)$ be any time such that

$$\tau_1 \geq \ln(\mathbf{E}\,\|X_0\|_1 + 1) + (K+3)\ln n. \tag{23}$$

4.1 Theorem 1.1: upper bound

Since $D \sim \alpha \ln n / \ln\ln n$, we have $p_D(d\lambda) = n^{-\alpha+o(1)}$. Then

$$\mathbf{Pr}(T \geq \tau_1) \leq \mathbf{E}\,\|X_0\|_1\, e^{-\tau_1} \leq n^{-K-3}.$$

Let N_A be the number of calls that arrive in the interval $[\tau_1, \tau_1 + n^K]$. Thus $N_A \sim Po(\lambda\binom{n}{2}n^K$. Let N_F be the number of calls that fail in the interval $[\tau_1, \tau_1 + n^K]$. We must show that $\mathbf{E}N_F = o(1)$.

Suppose first that $K < d-2$ and $\alpha > (K+2)/d$. We are going to upper bound Δ_t in order to use (11), and for that we argue as in the proof of Lemma 3.3. By Talagrand's inequality (Lemma 2.1), for each $v \in V$,

$$\begin{aligned}
& \mathbf{Pr}\left(\{d_t(v) > 4(n-1)p_D(d\lambda) + \ln^3 n\} \cap \{T < t\}\right) \\
\leq\ & \mathbf{Pr}\left(\tilde{S}_t^{\mathrm{D}}(\text{at } v) > 4(n-1)p_D(d\lambda) + \ln^3 n\right) \\
=\ & \exp(-\Omega(\ln^3 n/D)) = \exp(-\Omega(\ln^2 n)).
\end{aligned}$$

For $t' > t > 0$, let $A_{t,t'}$ be the event that $\Delta_s \leq 4(n-1)p_D(d\lambda) + \ln^3 n$ for all $s \in [t, t')$. By the above inequality and Lemma 2.5,

$$\mathbf{Pr}(\overline{A_{t,t+n^K}} \cap \{T < t\}) = \exp(-\Omega(\ln^2 n)),$$

and it follows using (15) and (16) that $\mathbf{Pr}(\overline{A_{\tau_1,\tau_1+n^K}}) = o(n^{-K-3})$. was $\overline{A_{\tau_1,n^K}}$

By Lemma 3.1 inequality (11), on $A_{t,t'} \cap \{T_j = t'\}$ was $\phi_{t+t'-}$

$$\mathbf{Pr}(Z_j \text{ fails } |\phi_{t'-}) \leq \left(\frac{8(n-1)p_D(d\lambda) + 2\ln^3 n}{n-2}\right)^d = p_0 = o(n^{-K-2}).$$

Let $N_0 = \lceil 2\mathbf{E}N_A \rceil = \lceil 2\lambda\binom{n}{2}n^K \rceil$. Then

$$\begin{aligned}
\mathbf{E}N_F &\leq \mathbf{E}\left[N_A \mathbb{I}_{N_A > N_0}\right] + N_0 p_0 + \mathbf{E}\left[N_F \mathbb{I}_{\overline{A}_{\tau_1,\tau_1+n^K}} \mathbb{I}_{N_A \leq N_0}\right] \\
&\leq o(1) + \mathbf{E}\left[N_F \mathbb{I}_{\overline{A}_{\tau_1,\tau_1+n^K}} \mathbb{I}_{N_A \leq N_0}\right] \\
&\leq o(1) + N_0 \mathbf{Pr}(\overline{A_{\tau_1,\tau_1+n^K}}) \\
&= o(1).
\end{aligned}$$

This completes the proof of the case $K < d-2$ and $\alpha > (K+2)/d$. In the case where $K \geq d-2$ and $\alpha > K+3-d$, the proof is somewhat longer. Note first that $\alpha > 1$. By Lemma 3.1 inequality (12), for each time $t > 0$, on $T_j = t$

$$\mathbf{Pr}(Z_j \text{ fails} \cap \{T < t\} | \phi_{t-}) \quad \leq \quad \frac{2^{d+1}}{n(n-2)^d} \sum_{v \in V} d_{t-}(v)^d \mathbb{I}_{T<t}.$$

Recall from Section 3 that, for all $v \in V$, on $T < t$, the number of full links ending in v is stochastically dominated by the number of links vu, $u \neq v$ such that $\tilde{Y}_t(vu) \geq D$, where $\tilde{Y}_t$ is a stationary copy of the superprocess. Let us consider time $t = 0$ say, and call this quantity $\tilde{d}(v)$. Therefore for each time $t \geq \tau_1$, on $T_j = t$ we have

$$\begin{aligned}
\mathbf{Pr}(Z_j \text{ fails}) \quad & \leq \quad \mathbf{Pr}(T \geq t) + \mathbf{Pr}(Z_j \text{ fails} \cap \{T < t\}) \\
& \leq \quad \mathbf{Pr}(T \geq \tau_1) + \frac{2^{d+1}}{n(n-2)^d} \sum_{v \in V} \mathbf{E}[d_{t-}(v)^d \mathbb{I}_{T<t}] \\
& \leq \quad O(n^{-K-3}) + \frac{2^{d+1}}{n(n-2)^d} \sum_{v \in V} \mathbf{E}[\tilde{d}(v)^d]. \qquad (24)
\end{aligned}$$

Consider a fixed node $v \in V$. We show next that

$$\mathbf{E}[\tilde{d}(v)^d] = \mathbf{E}[\tilde{d}(v)](1 + o(1)) = n^{1-\alpha+o(1)}. \qquad (25)$$

Consider the superprocess at time 0. Let $u_1, \ldots, u_d$ be distinct nodes in $V \setminus \{v\}$. Let $N(u_i)$ be the number of live calls with one end v that have selected the link vu_i but none of the links vu_j for $j \neq i$. Let $\tilde{N}$ be the number of live calls that have selected at least two of the links vu_i. Then the $N(u_i)$ are iid, each is Poisson with mean at most λd, and $\tilde{N}$ is Poisson with mean $O(1/n)$.

Let $x = d + \alpha$, and let A be the event that $\tilde{N} \leq x$. Note that $\mathbf{Pr}(\bar{A}) = O(n^{-x})$; and

$$\mathbf{E}\left[\prod_{i=1}^{k} \mathbb{I}_{\tilde{Y}_0(vu_i) \geq D} \mathbb{I}_A\right] \leq \mathbf{E}\left[\prod_{i=1}^{k} \mathbb{I}_{N(u_i) \geq D-x}\right] = \mathbf{Pr}(N(u_1) \geq D-x)^k.$$

Now let a_k be the number of partitions of $1, \ldots, d$ into exactly k non-empty blocks. In the sums below the w_j run over $V \setminus \{v\}$. We find

$$\mathbf{E}[\tilde{d}(v)^d \mathbb{I}_A] \quad = \quad \mathbf{E}\left[\prod_{j=1}^{d} \left(\sum_{w_j} \mathbb{I}_{\tilde{Y}_0(vw_j) \geq D} \mathbb{I}_A\right)\right]$$

$$
\begin{aligned}
&= \sum_{w_1,\ldots,w_d} \mathbf{E}[\prod_{j=1}^{d} \mathbb{I}_{\tilde{Y}_0(vw_j)\geq D}\mathbb{I}_A] \\
&= \sum_{k=1}^{d} a_k (n-1)_k \mathbf{E}[\prod_{i=1}^{k} \mathbb{I}_{\tilde{Y}_0(vu_i)\geq D}\mathbb{I}_A] \\
&\leq \mathbf{E}[\tilde{d}(v)\mathbb{I}_A] + \sum_{k=2}^{d} a_k n^k \mathbf{Pr}(N(u_1) \geq D - x)^k \\
&\leq \mathbf{E}[\tilde{d}(v)] + O(\sum_{k=2}^{d} (n^{1-\alpha+o(1)})^k) \\
&= \mathbf{E}[\tilde{d}(v)] + n^{-2(\alpha-1)+o(1)} \\
&= n^{1-\alpha+o(1)}.
\end{aligned}
$$

Also

$$
\mathbf{E}[\tilde{d}(v)^d \mathbb{I}_{\bar{A}}] \leq n^d \mathbf{Pr}(\bar{A}) = O(n^{d-x}) = O(n^{-\alpha}),
$$

and so (25) holds, as desired. Thus on $T_j \geq \tau_1$

$$
\begin{aligned}
\mathbf{Pr}(Z_j \text{ fails }) &\leq O(n^{-K-3}) + \frac{2^{d+1}}{n(n-2)^d}\; n\; n^{1-\alpha+o(1)} \\
&= O(n^{-K-3}) + n^{1-d-\alpha+o(1)} \\
&= o(n^{-K-2})
\end{aligned}
$$

since $\alpha > K + 3 - d$. It follows that

$$
\mathbf{E}[N_F] \;\leq\; \mathbf{E}\left[N_A \mathbb{I}_{N_A > N_0}\right] + o(N_0\; n^{-K-2}) = o(1).
$$

as required.

Note that a much easier proof works in the case $d = 1$. Let $\alpha > K + 2$. On $T_j \geq \tau_1$, by symmetry

$$
\mathbf{Pr}(Z_j \text{ fails }) \leq 2\mathbf{Pr}(\tilde{Y}_0(vu) \geq D) + \mathbf{Pr}(T \geq \tau_1)
$$

for any pair of distinct nodes $u, v \in V$; and so

$$
\mathbf{Pr}(Z_j \text{ fails }) \leq n^{-\alpha+o(1)} + o(n^{-K-2}) = o(n^{-K-2}).
$$

It now follows as above that $\mathbf{E}[N_F] = o(1)$.

4.2 Theorem 1.1: lower bound

Suppose first that $K < d-2$ and $0 < \alpha < (K+2)/d$; or that $K \geq d-2$ and $0 < \alpha < 1$. Let $D \sim \alpha \ln n / \ln\ln n$. Let $0 < \delta < (K+2)/d - \alpha$. We shall use Lemma 3.1 inequality (13) to obtain a lower bound on the probability that a call Z_t is lost; this will entail lower bounding the quantity $\sum_v d_{t-}(v)^d$. deleted 'We use notation much as earlier'

For $0 \leq t_1 \leq t_2$ let $A^1_{t_1,t_2}$ be the event that $S^{\mathrm{D}}_t(\text{at } v) \leq (n-2)\delta/2$ for all vertices v and all times $t \in [t_1, t_2]$. Then for any $v \in V$, any link vj and any time $t \in [t_1, t_2]$, on $A^1_{t_1,t_2}$ the probability that a call for a node v arriving at time t which selects link vj as its first choice is blocked by the 'partner' link uj (where u is the random other end of the call) is at most $\delta/2$.

Fix a node v. We apply Lemma 2.6 with $N = n-1$. For each load vector x and each node $j \neq v$, we let $f_j(x)$ be the number of calls in progress on the link vj. Also, for each j we let $\lambda_j = \lambda(1-\delta/2)$ and $n_j = D$. It follows that on $A^1_{t_1,t_2}$ the random variable $S^{\mathrm{D}}_{t_2}(\text{at } v)$ stochastically dominates $\tilde{S}^D_{t_2-t_1}(\text{at } v)$, where

$$\tilde{S}^D_t(\text{at } v) = \sum_{j \neq v} \mathbb{I}_{\tilde{Y}^{(vj)}_t = D},$$

and the $\tilde{Y}^{(vj)}$ are independent immigration-death processes each with arrival rate $\lambda(1-\delta/2)$, death rate 1, population 0 at time t_1 and population limit D. It is well known that in equilibrium the $n-1$ immigration-death processes $\tilde{Y}^{(vj)}_t$ are iid random variables with a Poisson distribution $Po(\lambda(1-\delta/2))$ truncated at D, that is the Erlang distribution with parameters λ and D. Since, by standard theory, each $\tilde{Y}^{(vj)}_t$ converges to equilibrium exponentially fast, there exists a constant $\tilde{c} > 0$ such that, for all $t \geq \tilde{c} \ln n$ and all $j \neq v$ we have $\mathbf{Pr}(\tilde{Y}^{(vj)}_t = D) \geq n^{-\alpha+o(1)}$, and so $\mathbf{E}[\tilde{S}^D_t(\text{at } v)] \geq n^{1-\alpha+o(1)}$.

We have assumed that $0 < \alpha < 1$, and so $\mathbf{E}[\tilde{S}^D_t(\text{at } v)]$ tends to infinity as $n \to \infty$. Let $0 < \epsilon < K+2-d\alpha$ and refine the condition on δ so that it now must satisfy $\delta < \frac{K+2}{d} - \alpha - \frac{\epsilon}{d}$. Using inequality (3), on $A^1_{t_1,t_2}$

$$\mathbf{Pr}(\tilde{S}^D_t(\text{at } v) \leq n^{1-\alpha-\delta}) \leq \exp(-n^{1-\alpha+o(1)})$$

for all t such that $t_1 + \tilde{c} \ln n \leq t \leq t_2$. For $0 \leq t_1 \leq t_2$ let A_{t_1,t_2} denote the event that $d_t(v) \geq n^{1-\alpha-\delta}$ for all $v \in V$ and all $t \in [t_1, t_2]$.

Recall that τ_1 was introduced in (23). Let $\tau_2 = \tau_1 + \tilde{c} \ln n$. Let I denote the interval $[\tau_2, \tau_2 + n^K]$, and let A denote the event A_{τ_2,τ_2+n^K}. By the above and Lemma 2.5,

$$\mathbf{Pr}(\overline{A} \cap A^1_{\tau_1,\tau_2+n^K}) = o(n^{-K}).$$

Also, we know by Lemmas 3.3 and 2.5 that

$$\mathbf{Pr}\left(\overline{A^1_{\tau_1,\tau_2+n^K}} \cap \{T \le \tau_1\}\right) = o(n^{-K}).$$

Further by (16)

$$\mathbf{Pr}(T > \tau_1) \le \mathbf{E}\,\|X_0\|_1 \;\; e^{-\tau_1} = o(n^{-K}).$$

It thus follows that

$$\mathbf{Pr}(\overline{A}) = o(n^{-K}).$$

By Lemma 3.1 equation (14), for each $t \in I$, on $A \cap \{T_j = t\}$ we have

$$\mathbf{Pr}(Z_j \text{ fails} \mid \phi_{t-}) \ge n^{-d(\alpha+\delta)} = p_0.$$

Let F be the event that fewer than n^ϵ calls arriving during the interval I fail; and let N_F be the number of calls that fail during this interval. Let $N_0 = \lfloor \frac{\lambda}{2}\binom{n}{2} n^K \rfloor$. Then

$$\mathbf{Pr}(F) \le \mathbf{Pr}(\overline{A}) + \mathbf{Pr}(Po(2N_0) < N_0) + \mathbf{Pr}(Bin(N_0, p_0) < n^\epsilon) = o(n^{-K})$$

and hence $\mathbf{E}N_F = \Omega(n^\epsilon)$.

Now consider the last case remaining, when $K \ge d-2$ and $1 \le \alpha < K+3-d$. As in the case $0 < \alpha < 1$ considered above, for each $t \ge \tau_2$ and $v \in V$, $\mathbf{E}[d_t(v)] \ge n^{1-\alpha+o(1)}$. By Lemma 3.1 inequality (13), for each $t \ge 0$, on $T_j = \tau_2 + t$

$$\begin{aligned}
\mathbf{Pr}(Z_j \text{ fails }) &\ge n^{-1}(n-2)^{-d} \sum_v \mathbf{E}[d_{(\tau_2+t)-}(v)^d] \\
&\ge n^{-1-d} \sum_v \mathbf{E}[d_{(\tau_2+t)-}(v)] \\
&\ge n^{-1-d} \cdot n \cdot n^{1-\alpha+o(1)} \\
&= n^{1-d-\alpha+o(1)},
\end{aligned}$$

and hence $\mathbf{E}N_F = \Omega(n^{K+3-d-\alpha+o(1)}) = \Omega(n^\epsilon)$, as required.

5 Proof of Theorem 1.2

Fix an integer $d \ge 2$ and a constant $K > 0$. Fix a constant $0 < \delta < 1$. Let θ be a constant with $\theta > (100+K)/\ln 2$. We now define times τ_0, τ_1, τ_2

depending on n (not quite as when we introduced τ_1 in (23) in the last section): we let

$$\tau_0 \geq \ln(\mathbf{E}\,\|X_0\|_1 + 1) + \theta \ln n,\ \ \tau_1 = \tau_0 + \theta \ln n \text{ and } \tau_2 = \tau_1 + n^K. \qquad (26)$$

For each $t \in [\tau_0, \tau_2]$, let A_t^0 be the event

$$\{(1-\delta)\lambda(n-1) \leq X_s(v) \leq (1+\delta)\lambda(n-1) \quad \forall s \in [\tau_0, t],\ \forall v\}\,;$$

by Lemma 3.5, $\mathbf{Pr}(\overline{A_{\tau_2}^0}) = o(n^{-K-1})$. Also, let A_t^1 be the event

$$\left\{S_s^{\mathrm{D}}(\text{via } v) \leq (n-2)\delta/4 \quad \forall s \in [\tau_0, t],\ \forall v\right\}\,;$$

then $\mathbf{Pr}(\overline{A_{\tau_2}^1}) = o(n^{-K-1})$, by Lemmas 3.3 and Lemma 2.5.

Recall that for each link vw, $X_t(vw)$ is the load of link vw at time t, that is the number of channels in use at time t. For $h = 0, 1, \ldots$, let $L_t(v,h)$ be the number of links vw at v with $X_t(vw) \geq h$ (so, in particular, $L_t(v,0) = n-1$ for all t). For an integer j, if $T_j \leq t$ and the call Z_j is for node v routed via node w and is still in progress at time t, then the *height* $\mathcal{H}_t(j,v)$ at v at time t of the call is one plus the number of calls in progress on vw at time t that arrived before it; and the call has height 0 at v at time t if the conditions do not all hold. Thus we have $\mathcal{H}_t(j,v) \leq X_t(vw)$. For $h = 1, 2, \ldots$, we define $H_t(v,h)$ to be the total number of calls on the links vw for $w \in V \setminus \{v\}$ with height at least h at time t. Clearly, $L_t(v,h) \leq H_t(v,h)$ for each node v and each positive integer h.

Let $c = \max\{c_1, c_2\}$, where c_1 and c_2 are constants respectively defined in Sections 5.1 and 5.2 below.

5.1 Upper bound

Let the constant $c_1 = c_1(\lambda, d, K)$ be as in (29) below, and let $D = D(n) \geq \frac{\ln\ln n}{\ln d} + c_1$. We shall show that aas no calls arriving during the interval $[\tau_1, \tau_2]$ of length n^K fail.

Given a positive integer h_0, for $h = h_0, h_1 + 1, \ldots$ let $B_t(h,\alpha)$ denote the event that $H_t(v,h) \leq \alpha$ for each v. Also, given numbers α_h for $h = h_0, h_0+1, \ldots$ and times t_h for $h = h_0, h_0+1, \ldots$ satisfying $\tau_0 \leq t_h \leq \tau_1$, let

$$B(h_0) = \{L_t(v,h) \leq 2\alpha_{h_0}\ \forall t \in [t_{h_0}, \tau_2],\ \forall v\},$$

and for $h = h_0 + 1, h_0 + 2, \ldots$ let

$$B(h) = \{H_t(v,h) \leq 2\alpha_h \ \forall t \in [t_h, \tau_2], \ \forall v\}.$$

The idea of the proof is to choose a sequence of about $\ln\ln n/\ln d$ numbers α_h decreasing quickly from a constant multiple of n to zero, and an increasing sequence of times t_h for $h = 0, 1, 2, \ldots$ satisfying $\tau_0 \leq t_h \leq \tau_1$ for all h. Then the aim is to show that $B(h_0)$ holds aas, and if $B(h)$ holds aas then so does $B(h+1)$; and to deduce that $B(h)$ holds aas for some $h \leq D$ with $\alpha_h = 0$. Thus aas no link is ever saturated during $[t_h, \tau_2]$, and so no call can fail during that interval.

Let $h_0 = \lceil \max\{8\lambda, 768\lambda^2\}\rceil$. We choose a decreasing sequence of numbers $\alpha_h \geq 0$ as follows. First, let

$$\alpha_{h_0} = \min\{\frac{n-1}{8}, \frac{n-1}{768\lambda}\}.$$

Note that $\alpha_{h_0} \geq \lambda(n-1)/h_0$. Hence, on $A^0_{\tau_2}$, for each $t \in [\tau_0, \tau_2]$, since $X_t(v) \leq 2\lambda(n-1)$ we have $L_t(v, h_0) \leq 2\lambda(n-1)/h_0 \leq 2\alpha_{h_0}$; and so $A^0_{\tau_2} \subseteq B(h_0)$. Next let

$$\frac{\alpha_h}{n-1} = 6\lambda \left(\frac{8\alpha_{h-1}}{n-1}\right)^d, \tag{27}$$

for $h = h_0 + 1, h_0 + 2, \ldots$, until $\alpha_h < 14(K+2)\ln n$. [We shall see shortly that there is such an h.] When this first occurs, we let $h^* = h^*(n)$ be the current value of h and increase α_{h^*} to $14(K+2)\ln n$. Finally we set $\alpha_{h^*+1} = 2K + 5$. Observe that on $B(h^*+1)$, for each $t \in [t_{h^*+1}, \tau_2]$ we have $\max_v X_t(v) \leq h^* + 2K + 10$. Note that the recurrence (27) can be rewritten as

$$\tilde{\alpha}_h = 48\lambda \cdot \tilde{\alpha}^d_{h-1}, \tag{28}$$

where $\tilde{\alpha}_h = 8\alpha_h/(n-1)$. It follows that for $h_0 + 1 \leq h \leq h^* - 1$

$$\tilde{\alpha}_h = (48\lambda)^{1+d+\cdots+d^{h-h_0-1}} \tilde{\alpha}_{h_0}^{d^{h-h_0}} \leq (48\lambda \cdot \tilde{\alpha}_{h_0})^{1+d+\cdots+d^{h-h_0-1}}$$

since $\tilde{\alpha}_{h_0} \leq 1$. But now, since $48\lambda \cdot \tilde{\alpha}_{h_0} \leq \frac{1}{2}$, for $h_0 \leq h \leq h^* - 1$ we have

$$\frac{8\alpha_h}{n-1} = \tilde{\alpha}_h \leq (0.5)^{\frac{d^{h-h_0-1}}{d-1}},$$

and so $h^*(n) = \ln\ln n/\ln d + O(1)$. We now set

$$c_1 = \sup_k \{h^*(k) + 4K + 11 - \frac{\ln\ln k}{\ln d}\}, \tag{29}$$

so that

$$D \geq \frac{\ln\ln n}{\ln d} + c_1 \geq h^*(n) + 4K + 11.$$

Now define an increasing sequence t_h of times as follows. Let $\gamma_h = 48\lceil \log_2 (2\alpha_h/\alpha_{h+1})\rceil$ for $h = h_0, \ldots, h^*-2$, let $\gamma_{h^*-1} = 48\log_2 n$, and let $\gamma_{h^*} = (K+3)\ln n$. Note that $\gamma_{h^*-1} \geq 48\lceil \log_2(2\alpha_{h^*-1}/\alpha_{h^*})\rceil$ for n sufficiently large. Let $t_{h_0} = \tau_0$, and let $t_h = t_{h-1} + \gamma_{h-1}$ for $h = h_0+1, h_0+2, \ldots, h^*+1$. Thus $t_{h^*+1} = \tau_0 + \sum_{h=h_0}^{h^*} \gamma_h$. We shall show that with high probability $B(h^*+1)$ holds and so throughout the interval $[t_{h^*+1}, \tau_2]$ there are no full links.

Note that

$$\begin{aligned} t_{h^*-1} - \tau_0 &= \sum_{h=h_0}^{h^*-2} \gamma_h = 48 \sum_{h=h_0}^{h^*-2} \lceil \log_2 (2\alpha_h/\alpha_{h+1})\rceil \\ &\leq 96(h^*-h_0) + 48 \sum_{h=h_0}^{h^*-2} (\log_2 \alpha_h - \log_2 \alpha_{h+1}) \\ &\leq 96h^* + 48\log_2 \alpha_{h_0} \leq 49 \log_2 n \end{aligned}$$

for n sufficiently large. It follows that

$$t_{h^*+1} - \tau_0 = t_{h^*-1} - \tau_0 + \gamma_{h^*-1} + \gamma_{h^*} \leq \theta \ln n$$

for n large enough, as $\theta \geq (100+K)/\ln 2$.

Recall that $A^0_{\tau_2} \subseteq B(h_0)$, as we noted earlier. We shall show that $\mathbf{Pr}(\overline{B(h)} \cap B(h-1))$ is small for each $h = h_0+1, \ldots, h^*+1$, which will yield that $\mathbf{Pr}(B(h^*+1)$ is close to 1. Hence, as we discussed earlier, aas throughout $[t_{h^*+1}, \tau_2]$ there are no full links. Since $t_{h^*+1} \leq \tau_1$, this shows that

$$\mathbf{Pr}[\exists j : \{T_j \in [\tau_1, \tau_2]\} \cap \{Z_j \text{ is blocked}\}] \leq \mathbf{Pr}(\overline{B(h^*+1)}) = o(n^{-k}). \quad (30)$$

This yields the desired upper bound of Theorem 1.2.

The main step is to prove that $\mathbf{Pr}(\overline{B(h)} \cap B(h-1))$ is small for each h. With this aim in mind, we first show that if $B(h-1)$ holds then aas for each v there exists a time $t_h(v) \in (t_{h-1}, t_h]$ such that $H_{t_h(v)}(v,h) \leq \alpha_h$. We then show that aas $H_t(v,h) \leq 2\alpha_h$ for all $t \in (t_h(v), \tau_2]$ and all $v \in V$. For each node $v \in V$ and for each integer $h = h_0+1, \ldots, h^*+1$, let

$$C(v,h) = \{\exists t_h(v) \in (t_{h-1}, t_h] : H_{t_h(v)}(v,h) \leq \alpha_h\}.$$

Let also $C(h) = \cap_v C(v,h)$, so that

$$\overline{C(h)} = \{\exists w : \ H_t(w,h) > \alpha_h \quad \forall t \in (t_{h-1}, t_h]\}$$

is the event that there is a node u such that the number of calls with height at least h at u is greater than α_h throughout $(t_{h-1}, t_h]$.

Lemma 5.1 *Uniformly over all $h = h_0 + 1, \dots, h^* + 1$,*

$$\mathbf{Pr}(\overline{C(h)} \cap B(h-1)) = o(n^{-K-1}).$$

Proof. Fix a node v and a height h with $h_0 + 1 \leq h \leq h^*$. Let $J_0(v) = t_{h-1}$, and enumerate the jump times of the process of arrivals (possibly failing) and terminations of calls with one end v after time $J_0(v)$ as $J_1(v), J_2(v), \dots$. For $k = 0, 1, \dots$ let $R_k = H_{J_k(v)}(v,h)$ and for $k = 1, 2, \dots$ let $Y_k = R_k - R_{k-1}$, so that

$$R_k = R_0 + \sum_{j=1}^{k} Y_j.$$

Note that each $Y_k \in \{-1, 0, 1\}$ and is $\phi_{J_k(v)}$-measurable, and that the sum $\sum_{k: t_{h-1} < J_k(v) \leq t_h} Y_k$ is the net change in $H_t(v,h)$ during the interval $(t_{h-1}, t_h]$. For $h = h_0, \dots, h^* - 1$, let $m_h = \lceil 12\lambda n \rceil \lceil \log_2(2\alpha_h/\alpha_{h+1}) \rceil \leq \frac{1}{2}\gamma_h \lambda(n-1)$ for $n \geq 2$. Note that for each $h = h_0 + 1, \dots, h^* + 1$ we have $J_{m_{h-1}}(v) \leq t_h$ aas, since by inequality (3)

$$\Pr(J_{m_{h-1}}(v) > t_h) \leq \mathbf{Pr}(Po(\lambda(n-1)\gamma_{h-1}) < m_{h-1}) \leq e^{-\gamma_{h-1}\lambda(n-1)/8},$$

which is $e^{-\Omega(n)}$.

For $k = 0, 1, \dots$ let

$$E_k = A^0_{J_{k+1}(v)-} \cap B_{J_{k+1}(v)-}(h-1) = A^0_{J_k(v)} \cap B_{J_k(v)}(h-1);$$

and let $E = \cap_{k=0}^{m_{h-1}-1} E_k$. We saw earlier that $\mathbf{Pr}(\overline{A^0_{t_h}}) = o(n^{-K-1})$. Thus

$$\mathbf{Pr}(\overline{E} \cap B(h-1)) \leq \mathbf{Pr}(J_{m_{h-1}}(v) > t_h) + \mathbf{Pr}(\overline{A^0_{t_h}}) = o(n^{-K-1}).$$

Now we obtain an upper bound q_h^+ for the probability of positive steps and a lower bound q_h^- for the probability of negative steps. On $E_{k-1} \cap (R_{k-1} >$

α_h), upper bounding $\mathbf{Pr}(J_k(v)$ is an arrival time) by 1, we obtain

$$\begin{aligned}\mathbf{Pr}(Y_k = 1|\phi_{J_k(v)-}) &\leq \left(\frac{2\max_w L_{J_k(v)-}(w,h-1)}{n-2}\right)^d \\ &\leq \left(\frac{2\max_w H_{J_k(v)-}(w,h-1)}{n-2}\right)^d \\ &\leq \left(\frac{8\alpha_{h-1}}{n-1}\right)^d = q_h^+\end{aligned}$$

(for $n \geq 3$). On $A^0_{J_k(v)-}$, the probability that $J_k(v)$ is a departure time of a call with one end v is at least $\frac{1}{\lambda(2+\delta)(n-1)} \geq \frac{1}{3\lambda(n-1)}$ for $n \geq 3$. It follows that, for $n \geq 3$, on $A^0_{J_k(v)-}$

$$\mathbf{Pr}(Y_k = -1|\phi_{J_k(v)-}) \geq \frac{H_{J_k(v)-}(v,h)}{3\lambda(n-1)} \geq \frac{R_{k-1}}{3\lambda(n-1)};$$

and so, for each $y \geq \alpha_h$, on $E_{k-1} \cap \{R_{k-1} = y\}$,

$$\mathbf{Pr}(Y_k = -1|\phi_{J_k(v)-}) \geq \frac{y}{3\lambda(n-1)} \geq \frac{\alpha_h}{3\lambda(n-1)} = q_h^-.$$

We note that for $h \leq h^*$

$$q_h^+ = \left(\frac{8\alpha_{h-1}}{n-1}\right)^d = \frac{\alpha_h}{6\lambda(n-1)} = q_h^-/2.$$

Let $a = \frac{\alpha_h}{6\lambda(n-1)}$ and let $b = \frac{1}{3\lambda(n-1)}$. By Lemma 2.3, with $r = \alpha_h$ and any value of $\alpha_h + 1 \leq \tilde{r} \leq 2\alpha_{h-1}$,

$$\begin{aligned}\mathbf{Pr}(E \cap \{H_{J_k(v)}(v,h) > \alpha_h \ \forall k \leq m_{h-1}\}|\ H_{J_0(v)}(v,h) = r_0) &\leq 2e^{-\alpha_h/14} \\ &= o(n^{-K-1}).\end{aligned}$$

It follows that, uniformly over $h_0 + 1 \leq h \leq h^*$,

$$\mathbf{Pr}(\overline{C}(h) \cap B(h-1)) = o(n^{-K-1}).$$

Now we consider $h = h^* + 1$. Let $J'_0(v) = t_{h^*}$, and enumerate arrival times of calls with one end v after time t_{h^*} as $J'_1(v), J'_2(v), \ldots$. Define $m_{h^*} = 2\lambda(n-1)\gamma_{h^*}$, which is $O(n \ln n)$, recalling that $\gamma_{h^*} = (K+3)\ln n$.

Let $N_t(v)$ denote the number of new calls for v arriving during $[t_{h^*}, t]$ with height at v at least $h^* + 1$ on arrival. For $k = 0, 1, \ldots$ let

$$E'_k = A^0_{J'_{k+1}(v)-} \cap B_{J'_{k+1}(v)-}(h^*).$$

Further let $E' = \cap_{k=0}^{m_{h^*}-1} E'_k$.

Consider the call that arrives at time $J'_k(v)$. On E'_{k-1} it has probability at most $p_1 = \left(56(K+2)n^{-1}\ln n\right)^d$ of choosing a link vw with at least h^* calls for n large enough. Further we note that, for each positive integer r,

$$\mathbf{Pr}(Bin(m_{h^*}, p_1) \geq r) \leq (m_{h^*}p_1)^r = O((n^{-d+1}(\ln n)^{d+1})^r).$$

Then, for each integer $r \geq K + 2$,

$$\begin{aligned} & \mathbf{Pr}(\{N_{t_{h^*+1}}(v) \geq r\} \cap B(h^*)) \\ \leq \; & \mathbf{Pr}(Bin(m_{h^*}, p_1) \geq r) + \mathbf{Pr}(Po(\lambda(n-1)\gamma_{h^*}) > m_{h^*}) + \mathbf{Pr}(\overline{E'} \cap B(h^*)) \\ = \; & O((n^{-d+1}(\ln n)^d)^{K+2}) + e^{-\Omega(n)} = o(n^{-K-1}). \end{aligned}$$

Also, on $B(h^*)$ the probability that some call with height at least $h^* + 1$ present at time t_{h^*} survives to time t_{h^*+1} is at most $28(K+2)\ln n \; e^{-\gamma_{h^*}} = O(n^{-K-2})$. It follows that

$$\mathbf{Pr}(\overline{C(h^*+1)} \cap B(h^*)) = o(n^{-K-1})$$

as required. ∎

We now show that for each $h = h_0 + 1, \ldots, h^* + 1$, aas there will be no 'excursions' that cross upwards from α_h to at least $2\alpha_h$, that is $H_t(v, h)$ cannot exceed $2\alpha_h$ during $(t_v(h), \tau_2]$ for any $v \in V$ and any $h = h_0 + 1, \ldots, h^* + 1$.

Lemma 5.2 *For $h_0 + 1 \leq h \leq h^* + 1$, let $B'(h) = B(h-1) \cap C(h)$. Then uniformly over $h = h_0 + 1, \ldots, h^* + 1$*

$$\mathbf{Pr}(\overline{B(h)} \cap B'(h)) = o(n^{-K-1}).$$

Proof. The only possible start times for a crossing are arrival times during $[t_{h-1}, \tau_2]$. Let $N_0 = 2\lambda(n-1)\tau_2$. The probability that more than N_0 arrivals for any given node v occur during $(t_h(v), \tau_2]$ is $o(n^{-K-1})$. We apply Lemma 2.4 with $p = q_h^+$, $q = q_h^-$, $a = \lfloor \alpha_h \rfloor - 1$, and

$$E_k = A^0_{J_{k+1}(v)-} \cap B_{J_{k+1}(v)-}(h-1, 2\alpha_{h-1}).$$

By Lemma 2.4, the probability that any given excursion leads to a 'crossing' is at most $(q_h^+/q_h^-)^{\lfloor \alpha_h \rfloor -1}$, and so for $h = h_0+1, \ldots, h^*+1$

$$\mathbf{Pr}(\overline{B(h)} \cap B'(h)) \leq Nn(q_h^+/q_h^-)^{\lfloor \alpha_h \rfloor -1} + \mathbf{Pr}(\overline{E}) + o(n^{-K-1}).$$

For $h = h_0+1, \ldots, h^*$

$$(q_h^+/q_h^-)^{\lfloor \alpha_h \rfloor -1} \leq 2^{-\alpha_h+2} = 2^{-14(K+2)\ln n+2} = o(n^{-2K-4});$$

for $h = h^*+1$

$$\begin{aligned}(q_h^+/q_h^-)^{\alpha_h-2} &\leq (3\lambda/2K+5)^{2K+3}(112(K+2)\ln n)^{d(2K+3)}(n-1)^{-(d-1)(2K+3)} \\ &= O\left(n^{-2K-3}(\ln n)^{4K+6}\right),\end{aligned}$$

and the lemma follows. ∎

We may now complete the proof. As $A^0_{\tau_2} \subseteq B_{h_0}$ and $\mathbf{Pr}(\overline{A^0_{\tau_2}}) = o(n^{-K-1})$, we have

$$\begin{aligned}&\mathbf{Pr}(\overline{B(h^*+1)}) \\ \leq\ &\mathbf{Pr}(\overline{A^0_{\tau_2}}) + \mathbf{Pr}(\overline{B(h_0)} \cap A^0_{\tau_2}) + \sum_{h=h_0+1}^{h^*+1} \mathbf{Pr}(\overline{B(h)} \cap B(h-1)) \\ =\ &\mathbf{Pr}(\overline{A^0_{\tau_2}}) + \sum_{h=h_0+1}^{h^*+1} \mathbf{Pr}(\overline{C(h)} \cap B(h-1)) + \sum_{h=h_0+1}^{h^*+1} \mathbf{Pr}(\overline{B(h)} \cap C(h) \cap B(h-1)) \\ =\ &o(n^{-K}).\end{aligned}$$

This completes the proof of (30) and thus of the upper bound of Theorem 1.2.

5.2 Lower bound

Let $0 < \epsilon < \min\{1, (K+1)/d\}$. Recall that θ is a constant satisfying $\theta > (100+K)/\ln 2$. Let the constant $c_2 = c_2(\lambda, d, K)$ be as defined below, and let $D = D(n) \leq \frac{\ln \ln n}{\ln d} - c_2$. Recall that τ_0, τ_1 and τ_2 are defined in (26) at the start of this section. We consider the interval $[\tau_1, \tau_2]$ of length n^K. We shall show that aas for each v at least $(n-1)^{1-\epsilon}$ links vw incident on v are saturated (and so unavailable) throughout the interval, and hence aas at least $n^{K+2-d-o(1)}$ calls arriving during the interval fail.

Given a non-negative integer h and a real value $\alpha > 0$, let $B_t(h, \alpha)$ denote the event that $L_t(v, h) \geq \alpha$ for each v. Given also a sequence of numbers

α_h, $h = 0, 1, 2, \ldots$ and a sequence of times t_h, $h = 0, 1, \ldots$, let $B(h) = \cap_{t\in[t_h,\tau_2]} B_t(h, \alpha_h)$; thus $B(h)$ is the event that for every $v \in V$, throughout the interval $[t_h, \tau_2]$, the number $L_t(v, h)$ of links with one end v that carry at least h calls remains at least α_h. We shall choose positive numbers $\alpha_0, \alpha_1, \ldots$, starting with $\alpha_0 = n - 1$ and decreasing rapidly. These will satisfy $2\alpha_{h+1} \leq \alpha_h(1 - e^{-1})$, so that $(\alpha_h - 2\alpha_{h+1})^d \geq (\alpha_h/e)^d$. We shall further choose an increasing sequence of times t_h, $h = 0, 1, \ldots$, such that $\tau_0 \leq t_h \leq \tau_1$ for each h. We want to show that $B(D)$ occurs aas, with a value $\alpha_D = \Omega(n)$, so that there are always many saturated links. Analogously to the upper bound proof in Section 5.1, the main task is to show that $\mathbf{Pr}(\overline{B(h)} \cap B(h-1))$ is small for each $h \leq D$.

Again similarly to the upper bound proof, we first show that on $B_{t_{h-1},t_h}(h-1, \alpha_{h-1})$ aas there exists a time $t_h(v) \in (t_{h-1}, t_h]$ such that $L_{t_h(v)}(v, h) \geq 2\alpha_h$. We then show that on $B(h-1)$ aas $L_t(v, h)$ never falls below α_h during $(t_h(v), \tau_2]$. The numbers α_h are given as follows. We let $\alpha_0 = n - 1$, and for $h = 1, 2, \ldots$

$$\frac{\alpha_h}{n-1} = \frac{\min\{1, \lambda\}}{24e^d h} \left(\frac{\alpha_{h-1}}{n-1}\right)^d. \tag{31}$$

Thus $2\alpha_h \leq \alpha_{h-1}(1 - e^{-1})$ as required, since $\frac{1}{12} \leq e - 1$. We need to choose the constant c_2 in the upper bound on $D(n)$ above such that for n sufficiently large

$$\alpha_D \geq (n-1)^{1-\epsilon}.$$

It is easy to check that such a choice is possible. To see this, let $\nu = \frac{\min\{1,\lambda\}}{24e^d}$ and let $\beta_h = \frac{\alpha_h}{n-1}$. Then $0 < \nu < 1$, $\beta_0 = 1$ and for $h = 1, 2, \ldots$

$$\beta_h = \frac{\nu}{h}\beta_{h-1}^d. \tag{32}$$

It follows that for each positive integer h

$$\beta_h = \frac{\nu^{1+d+\ldots+d^{h-1}}}{\prod_{i=1}^h i^{d^{h-i}}}.$$

To upper bound the denominator, note that

$$\ln(h(h-1)^d(h-2)^{d^2}\ldots 2^{d^{h-2}}) = d^h \sum_{i=2}^h d^{-i} \ln i \leq c_3 d^h$$

for some constant $c_3 > 0$, and so the denominator is at most $e^{c_3 d^h}$. It follows that for each $h \in \mathbb{N}$
$$\beta_h \geq e^{-d^h(\ln(\frac{1}{\nu})+c_3)}.$$
Let c_4 be such that $d^{-c_4}(\ln(\frac{1}{\nu}) + c_3) \leq \epsilon$. Hence if $h \leq \ln\ln(n-1)/\ln d - c_4$ then

$$\begin{aligned}\beta_h &\geq \exp(-(\ln(n-1))d^{-c_4}(\ln(1/\nu) + c_3)) \\ &\geq \exp(-\epsilon \ln(n-1)) = (n-1)^{-\epsilon}.\end{aligned}$$

For $h = 0, 1, \ldots$ we let $\gamma_h = \frac{4}{\max\{1,\lambda\}(h+1)}$. Now define an increasing sequence of times t_h as follows. Let $t_0 = \tau_0$, and for $h = 1, \ldots,$ let $t_h = t_{h-1} + \gamma_{h-1}$. Then

$$t_D - \tau_0 = \sum_{h=0}^{D-1} \gamma_h \leq 4 \sum_{h=1}^{D} \frac{1}{h} \leq 4 \ln(D+1) = O(\ln\ln\ln n).$$

It follows that $t_D \leq \tau_1$ for n sufficiently large.

We now recall that

$$B(h) = \{L_t(v,h) \geq \alpha_h \quad \forall t \in [t_h, \tau_2], \forall v\}, \qquad h = 0, 1, \ldots.$$

Thus $\mathbf{Pr}(B(0)) = 1$; and we prove by induction that $\mathbf{Pr}(\overline{B}(h)) = o(n^{-K-1})$ for $h = 1, \ldots, D$, so that aas throughout $[\tau_1, \tau_2]$ for each v there are at least $(n-1)^{1-\epsilon}$ saturated links vw incident on v.

Fix a node v and an integer $h \geq 1$. Let $J_0(v) = t_{h-1}$ and enumerate the jump times of the process after time $J_0(v)$ that concern node v as $J_1(v), J_2(v), \ldots$. For $k = 0, 1, \ldots$ let $R_k = L_{J_k(v)}(v,h)$ and for $k = 1, 2, \ldots$ let $Y_k = R_k - R_{k-1}$, so that

$$R_k = R_0 + \sum_{j=1}^{k} Y_j.$$

Then each $Y_k \in \{-1, 0, 1\}$, is $\phi_{J_k(v)}$-measurable, and $\sum_{k: t_{h-1} < J_k(v) \leq t_h} Y_k$ is the net change in $L_t(v,h)$ during $(t_{h-1}, t_h]$. Let $m_h = 2\min\{1,\lambda\}(n-1)/(h+1) = \frac{1}{2}\lambda(n-1)\gamma_h$. Note that $J_{m_{h-1}}(v) \leq t_h$ with high probability for $h \leq D$, since

$$\mathbf{Pr}(J_{m_{h-1}}(v) > t_h) \leq \mathbf{Pr}(Po(\lambda(n-1)\gamma_{h-1} < m_{h-1}) \leq e^{-\gamma_{h-1}\lambda(n-1)/8} = o(n^{-K-1}).$$

For $k = 0, 1, \ldots$ let

$$E_k = A^0_{J_{k+1}(v)-} \cap A^1_{J_{k+1}(v)-} \cap B_{J_{k+1}(v)-}(h-1).$$

Let $E = \cap_{k=0}^{m_{h-1}-1} E_k$. Recall that $\mathbf{Pr}(\overline{A^0_{t_h} \cup A^1_{t_h}}) = o(n^{-K-1})$. Thus

$$\mathbf{Pr}(\overline{E} \cap B(h-1)) \leq \mathbf{Pr}(J_{m_{h-1}}(v) > t_h) + \mathbf{Pr}(\overline{A^0_{t_h} \cup A^1_{t_h}}) = o(n^{-K-1}).$$

Now we want to give a lower bound on the probability that $Y_k = 1$. First note that on $A^0_{J_k(v)-}$ the probability that $J_k(v)$ is an arrival time (for a call for v) is at least $1/(2+\delta) \geq \frac{1}{3}$. Now consider picking the random other end u of the call and the random intermediate nodes $w_1, \ldots, w_d$ in the order $w_1, u, w_2, \ldots, w_d$. On $A^1_{J_k(v)-}$ we have $S^{\mathrm{D}}_{J_k(v)-}(\text{via } w) \leq (n-2)/2$ for all nodes w; and so, whatever w_1 is picked, the probability that uw_1 is saturated is at most $\frac{1}{2}$. Hence on $A^0_{J_k(v)-} \cap A^1_{J_k(v)-}$ we have

$$\begin{aligned} & \mathbf{Pr}(Y_k = 1 | \phi_{J_k(v)-}) \\ \geq \;& \frac{1}{3} \cdot \frac{L_{J_k(v)-}(v,h-1) - L_{J_k(v)-}(v,h)}{n-1} \cdot \frac{1}{2} \cdot \left(\frac{L_{J_k(v)-}(v,h-1) - 1 - L_{J_k(v)-}(v,h)}{n-2} \right)^{d-1} \\ \geq \;& \frac{1}{6} \left(\frac{L_{J_k(v)-}(v,h-1) - 1 - L_{J_k(v)-}(v,h)}{n-1} \right)^{d}. \end{aligned}$$

Similarly on $A^0_{J_k(v)-}$ the probability that $J_k(v)$ is a departure time of a given call with one end v is at most $\frac{1}{\lambda(2-\delta)(n-1)} \leq \frac{1}{\lambda(n-1)}$, and so

$$\mathbf{Pr}(Y_k = -1 | \phi_{J_k(v)-}) \leq \frac{h(L_{J_k(v)-}(v,h) - L_{J_k(v)-}(v,h+1))}{\lambda(n-1)}.$$

It follows that on $E_{k-1} \cap (R_{k-1} < 2\alpha_h)$

$$\mathbf{Pr}(Y_k = 1 | \phi_{J_k(v)-}) \geq \frac{1}{6} \left(\frac{\alpha_{h-1} - 2\alpha_h}{n-1} \right)^d \geq \frac{e^{-d}}{6} \left(\frac{\alpha_{h-1}}{n-1} \right)^d = q_h^+.$$

Also for each $y < 2\alpha_h$, on $E_{k-1} \cap (R_{k-1} = y)$

$$\begin{aligned} \mathbf{Pr}(Y_k = -1 | \phi_{J_k(v)-}) \;&\leq\; \frac{hy}{\lambda(n-1)} \leq \frac{2h\alpha_h}{\lambda(n-1)} \\ &\leq\; \frac{2h\alpha_h}{\min\{1,\lambda\}(n-1)} = q_h^-. \end{aligned}$$

We note that $q_h^- = \frac{1}{2} q_h^+$ for each positive integer h.

For each node $v \in V$ and each positive integer h let

$$C(v,h) = \{\exists t_h(v) \in (t_{h-1}, t_h] : L_{t_h(v)}(v,h) \geq 2\alpha_h\},$$

and let $C(h) = \cap_v C(v,h)$. We now show that, uniformly over $h = 1, \ldots, D$,

$$\mathbf{Pr}(\overline{C(h)} \cap B(h-1)) = o(n^{-K-1}).$$

Let $p = q_h^+$, let $r_1 = 2\alpha_h$, and let r_0 be any positive integer less than $2\alpha_h$. Note that $q_h^+ m_{h-1} \geq 4\alpha_h \geq 2(r_1 - r_0)$. By a natural 'reversed' version of Lemma 2.2

$$\begin{aligned} &\mathbf{Pr}(E \cap (L_{J_k(v)}(v,h) < 2\alpha_h \; \forall k \in \{1,, \ldots, m_{h-1})| \; L_{J_0(v)}(v,h-1) = r_0) \\ \leq \quad & e^{-\alpha_h/7} \leq e^{-\alpha_D/7} \leq e^{-\Omega(n^{1-\epsilon})}. \end{aligned}$$

It follows that

$$\mathbf{Pr}(\overline{C(h)} \cap B(h-1)) = o(n^{-K-1}).$$

We now need to prove that for each $h = 1, 2, \ldots, D$, aas there will be no excursions that cross downwards from $2\alpha_h$ to less than α_h, that is none of the numbers $L_t(v,h)$ can drop below α_h during $(t_v(h), \tau_2]$.

For each positive integer h let $B'(h) = B(h-1) \cap C(h)$. We shall show that uniformly over $1 \leq h \leq D$

$$\mathbf{Pr}(\overline{B(h)} \cap B'(h)) = o(n^{-K-1}).$$

The only possible start times for a crossing are departure times affecting links at v during $[t_{h-1}, \tau_2]$. Let $N_0 = 4\lambda\tau_2(n-1)$. Let F denote the event that there are more than N_0 such departures in $[t_{h-1}, \tau_2]$. Then

$$\mathbf{Pr}(F \cap A^0_{\tau_2}) = o(n^{-K-1}).$$

We apply a reversed version of Lemma 2.4 with $p = q_h^-$, $q = q_h^+$, $a = \lfloor \alpha_h \rfloor - 1$, and

$$E_k = A^0_{J_{k+1}(v)-} \cap A^1_{J_{k+1}(v)-} \cap B_{J_{k+1}(v)-}(h-1).$$

The probability that any given excursion leads to a 'crossing' is at most $(q_h^-/q_h^+)^{\lfloor \alpha_h \rfloor - 1} \leq (1/2)^{\alpha_h - 2}$. It follows that

$$\begin{aligned} \mathbf{Pr}(\overline{B(h)} \cap B'(h)) &\leq nN_0(0.5)^{\alpha_h - 2} + n\mathbf{Pr}(F \cap A^0_{\tau_2}) + \mathbf{Pr}(\overline{E}) \\ &\leq 4\lambda\tau_2(n-1)n(0.5)^{\alpha_h-2} + o(n^{-K-1}) \\ &\leq 4\lambda n^{K+2}(0.5)^{\alpha_D - 2} + o(n^{-K-1}) = o(n^{-K-1}). \end{aligned}$$

The proof may now be completed in the same way as the proof of the upper bound. We have

$$\begin{aligned}\mathbf{Pr}(\overline{B(D)}) &\leq \mathbf{Pr}(\overline{B(0)}) + \sum_{h=1}^{D} \mathbf{Pr}(\overline{B(h)} \cap B(h-1)) \\ &= \sum_{h=1}^{D} \mathbf{Pr}(\overline{C(h)} \cap B(h-1)) + \sum_{h=1}^{D} \mathbf{Pr}(\overline{B(h)} \cap C(h) \cap B(h-1)) \\ &= o(n^{-K}).\end{aligned}$$

For $t \in [\tau_1, \tau_2]$ let F_t be the event that for each v at least $(n-1)^{1-\epsilon}$ links vw with one end v stay saturated throughout the interval $[\tau_1, t]$. By the above, provided that $D \leq \frac{\ln \ln n}{\ln d} - c_2$ the event F_{τ_2} holds aas. Now let $s \in [\tau_1, \tau_2]$ and consider a new call arriving at time s. On F_{s-} the probability that this call is blocked is at least

$$p_1 = \left(\frac{(n-1)^{1-\epsilon} - 1}{n-2} \right)^d \geq \frac{1}{2} n^{-\epsilon d}$$

for n sufficiently large. Let $N_1 = \lceil \frac{1}{2} \lambda \binom{n}{2} n^K \rceil$. Let B^* be the event that fewer than $b^* = \frac{1}{16} \lambda n^{K+2-d\epsilon}$ calls are lost. Then

$$\mathbf{Pr}(B^*) \leq \mathbf{Pr}(\overline{F_{\tau_2}}) + \mathbf{Pr}(Po(\lambda \binom{n}{2} n^K) < N_1) + \mathbf{Pr}(Bin(N_1, p_1) < b^*) = o(1).$$

This completes the proof of the lower bound of Theorem 1.2.

6 Concluding remarks

We have considered the performance of two algorithms for a continuous-time network routing problem, strengthening and extending the earlier results in [9] and [1].

The analysis in [9] (see also [4]) suggests that the performance of the model can be upper and lower bounded by differential equations. While this analysis is non-rigorous, it is hoped that a suitable differential equation approximation, and concentration of measure bounds, can indeed be obtained. The main challenge is to disentangle the complex dependencies within subsets of links to obtain a tractable asymptotic approximation for the generator of the underlying Markov process. The details will appear in [5].

For simplicity we have assumed throughout that the underlying network is a complete graph, but our results will carry over in a straightforward way to a suitably 'dense' subnetwork. Consider for example the upper bound in Theorem 1.2 part (a). Let $\delta > 0$, and suppose that, in the network with n vertices, for each pair of distinct vertices u and v the number of possible intermediate nodes is at least δn. Then minor alterations to the proof of Theorem 1.2 part (a) show that we obtain the same conclusion: if $D(n) \geq \ln \ln n / \ln d + c$ and we use the BDAR algorithm, then the expected number of failing calls during the interval of length n^K is $o(1)$. The only difference is that now the constant c depends also on δ. Note that the leading term $\ln \ln n / \ln d$ depends only on the problem size n and the number d of choices, and not on δ (or on λ or K).

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