Equilibria of Two-Sided Matching Games
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Abstract

Problems of matching have long been studied in the operations research literature (assignment problem, secretary problem, stable marriage problem). All of these consider a centralized mechanism whereby a single decision maker chooses a complete matching which optimizes some criterion. This paper analyzes a more realistic scenario in which members of the two groups (buyers-sellers, employers-workers, males-females) randomly meet each other in pairs (interviews, dates) over time and form couples if there is mutual agreement to do so. We assume members of each group have common preferences over members of the other group. Generalizing an earlier model of Alpern and Reyniers (2005), we assume that one group (called males) is r times larger than the other, $r \geq 1$. Thus all females, but only $1/r$ of the males, end up matched. Unmatched males have negative utility $-c$. We analyze equilbria of this matching game, depending on the parameters r and c. In a region of (r, c) space with multiple equilibria, we compare these, and analyze their 'efficiency' in several respects. This analysis should prove useful for designers of matching mechanisms when information about individuals is not available centrally but only statistical properties of the groups are known.

1 Introduction

The problem of pairwise matching of individuals from distinct sets (or sexes) X and Y occurs in many guises: buyers and sellers, employers and employees, medical schools and interns, males and females. We shall use the terminology of the last case, calling the larger group X the males. We assume that individuals of each group have common preferences over whom they would like to be matched with in the other group.

The so-called 'stable marriage' problem proposed by Gale and Shapley (1962) seeks a matching among equal sized finite sets X and Y such that for any two matched pairs (x_1, y_1) and (x_2, y_2), in neither unmatched couple (x_1, y_2) or (x_2, y_1) would each member prefer (with an arbitrary preference relation) their new partner to the one in the original matching. To analyze such questions one must look at complete matchings without considering how they might arise in practice. This 'centralized' problem has received much study (see Roth and Sotomayor (1990).

More recently, the processes by which complete matchings may arise over time have been analyzed as dynamic games played by the individuals in the two groups. The utilities of these players are often modeled (and will be so here) as 'common preferences' by all members of one sex over individuals of the other. For this reason we can give each individual a 'type' (called x for males, y for females) such that when a couple (x, y) is formed, the male x gets utility y, and the female y gets utility x. We can normalize these types to the unit interval $[0, 1]$ by identifying an individual's utility with relative rank of their partner within his or her group. A male who is unmated at the end of the n'th (final) period gets a utility $-c$, where c is a known parameter representing the cost of failure to mate. In the 'mutual choice', or 'two-sided', models we shall extend in this paper, individuals are randomly paired in each period (that is, the smaller group of females is randomly paired with an equally large randomly chosen set of males - the remaining males are not paired in that period). Then if each member of a matched pair chooses to accept the other rather than go into the next period unmated, they form a couple and are permanently mated. In the final period, players always accept. We call this game $\Gamma_n(r, c)$, where $r \geq 1$ (the 'sex ratio') is the initial number of males divided by the initial number of females. This game has been analyzed by Alpern and Reyniers (2005) in the symmetric case $r = 1$. Johnstone (1997) considered a similar dynamic game model and Kalick and Hamilton (1986) simulated a social psychology version.

A strategy for a player in $\Gamma_n(r,c)$ is a rule specifying which potential matches to accept in each period, by determining the least acceptable mate. A strategy profile is called an *equilibrium* if prospective mates are accepted if and only if their type (utility) exceeds the expected utility of the chooser of going into the next period unmated - this is essentially a subgame perfect Nash equilibrium. In the symmetric case ($r = 1$) studied by Alpern and Reyniers (2005), only a single equilibrium was found. In this generalization to $r \geq 1$, we find a region of (r, c) space having multiple equilibria. For example, when $n = 2$ we find three equilibria: a *choosy* equilibrium, where both groups have high acceptance standards; an *easy* equilibrium, where both groups have low but positive acceptance standards; and a *one-sided* (female choice) equilibrium, where males accept anyone. Much of the paper is devoted to analyzing and comparing these in terms of dynamical stability and marital stability (a randomized version of Gale's stability condition based on that of Eriksson and Strimling (2004) and Eriksson and Häggström (2007)). For $n = 2$ (and numerically, for higher n) we find that choosiness at equilibrium goes in the same direction for males and females; equilibria with choosy males have choosy females). We find that the choosy and one-sided equilibria are dynamically stable (attracting fixed points of a dynamical system); but the easy equilibrium is dynamically unstable. The equilibrium where both sexes are choosy has the highest marital stability; the equilibrium where only females choose has the lowest. We note that the *existence* of an equilibrium follows from a simple application of Brouwer's Fixed Point Theorem in the same way as established for $r = 1$ by Alpern and Reyniers (2005). As shown there, equilibria are fully determined by a pair of nonincreasing $n - 1$ tuples of threshold values $(u_1, u_2, \ldots, u_{n-1})$ and $(v_1, v_2, \ldots, v_{n-1})$, where u_i is the lowest type female that a top male ($x = 1$) will accept in period i (similarly for v_i for female choice). At equilibrium, a pairing (x, y) in period i will mutually accept and form a couple if and only if $x \geq v_i$ and $y \geq u_i$. The v_i will always be positive. If all the u_i are 0, we call it a 'one-sided' (or female choice) equilibrium; otherwise we call it a 'two-sided' (or mutual choice) equilibrium.

From the point of view of a single player, a sort of 'secretary problem' (see Ferguson (1989)) is being played out over time, in that he is being presented with a random succession of secretaries. As in the original secretary problem, he may not go back and accept someone he has rejected. However there are many differences: The distribution in each period depends on previous choices of other players; a secretary may reject him; the objective is expected rank. The closest version of the secretary problem is that of Eriksson, Sjöstrand, J. and Strimling (2007).

In contrast to two-sided search models such as the well known one of McNamara and Collins (1990), our model is not steady-state. Each period is different: the sex ratio increases and the distribution of types changes according to the strategies employed. The cohorts are initially uniformly distributed but not in any future period. At all equilibria, individuals become less choosy over time, as suggested in the Pennebaker et al (1979) social science analysis of the country and western song "Don't the girls get prettier at closing time". A good analysis of the effects of changing and uncertain distributions of male quality on female choice has been given in by Collins, McNamara and Ramsey (2006).

Two-sided matching models have been used in various aspects of economic theory, principally by Burdett and Coles (1997,1999), Bloch and Ryder (2000), Eeckhout (2000) and Eriksson and Häggström (2007). In biology and psychology, they have been used to describe and analyze mating behaviour in animals (Alpern and Reyniers (1999), Alpern, Katrantzi and Reyniers (2005), Bergstrom and Real (2000)), and in humans (Kalick and Hamilton (1986)). Connections with two-sided spatial matching ('rendezvous search') will be discussed in the Conclusions section.

Some notes on terminology. As our model involves two matching processes, the random pairing of unmated individuals at the start of each period and the permanent coupling of pairs who accept each other, we distinguish these by calling the former process *matching* and the latter *mating*. Some results are obtained numerically, and these will be denoted as *Propositions,* covering the region $1 \leq r \leq 2.5$, $0 \leq c \leq 2.5$.

The paper is organized as follows. Section 2 gives a complete treatment of the two period problem. We find formulae for the three equilibria: e^1 (one-sided), e^2 (easy), e^3 (choosy). We determine the regions of (r, c) space where they exist (Theorem 1). We show that male and female choosiness vary in the same way at equilibria (Monotonicity Lemma 4). We show that only e^1 and e^3 are dynamically stable (Proposition 5); We show that e^1 is the most maritally stable whereas e^1 is the least (Proposition 6). In Section 3 we use both analytical and numerical methods to establish that these properties of equilibria for $n = 2$ periods tend to hold for models with $n > 2$ periods.

We wish to thank an anonymous referee of Alpern and Reyniers (2005) for suggesting that an extension of that paper with a nontrivial sex ratio might yield new phenomenae – which it has.

2 The Two Period Game $\Gamma_2(r,c)$

We begin with populations of females and males, with types (quality) uniformly distributed on $[0,1]$. The females have unit density (and unit population), while the males have density (and population) r (the sex ratio) which is at least 1. Let u and v be the male and female first period cutoff strategies; females accept a male x iff $x \geq > v$ while males accept female y iff $y \geq u$. A matched male-female pair which types (x,y) will be mated by mutual acceptance if both $x \geq u$ and $y \geq v$ and with random matching the number of such couples will be

$$k = (1-u)(1-v). \tag{1}$$

as shown in the unshaded regions of both the female and male populations of Figure 1. In the left square, females are located according to their type (horizontal y axis) and the type of the male they are matched with (vertical x axis). Those in the left rectangle are rejected by their partner and those in the bottom right rectangle reject their partner. The rectangle on the right similarly plots all males, with the additional lower rectangle of unmatched males.

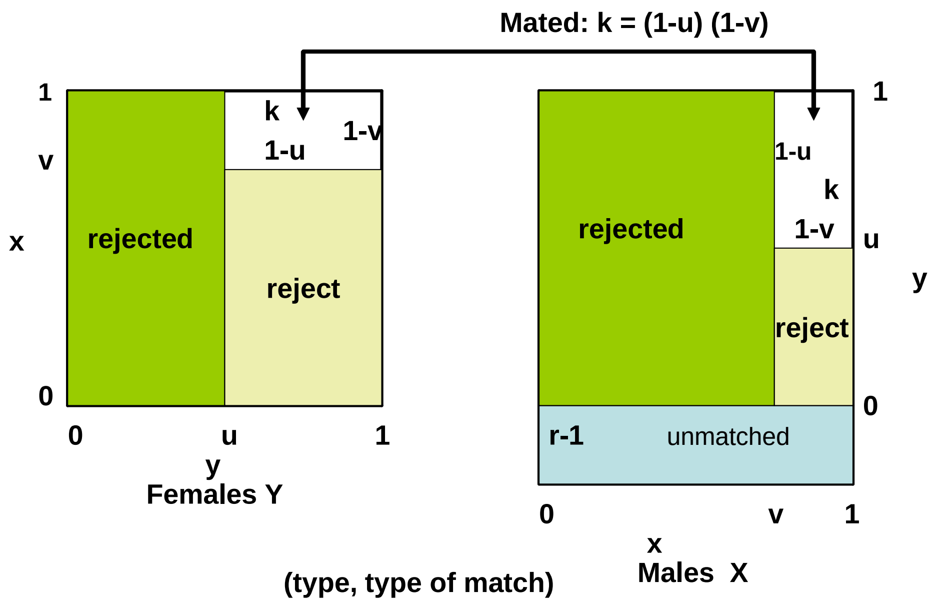

Figure 1. Couple formation

The mean value μ_x of the $r-k$ males x that enter the final period unmated (those not in upper right unshaded rectangle) is calculated by dividing them into those with $x < v$ (of average type $v/2$) and those with $x \geq v$ (of average type $(1+v)/2$). The first group of males have population (area) rv, while the second have population $(1-v)(r-1+u)$. Hence

$$\mu_x = \phi(u,v) = \frac{[rv](v/2) + [(1-v)(r-1+u)](1+v)/2}{r-k}. \tag{2}$$

The value (expected payoff) of any female who enters the final period unmated is simply the mean type of the second period male population, that is, μ_x. So in period 1 she should accept a male x iff $x \geq \mu_x$. Hence the female equilibrium condition (f.e.c.) is simply

$$v = \phi(u,v). \tag{3}$$

For v in the range of ϕ, we can solve uniquely for u, giving the f.e.c. (3) as

$$u = f(v) = \frac{-r-2v+2rv+v^2+1}{-2v+v^2+1}. \tag{4}$$

To calculate the corresponding male equilibrium equation (m.e.c.) we first need to obtain the mean female type μ_y in the final period. By symmetry when $r=1$, we obtain this by interchanging u and v in (2):

$$\mu_y = \frac{u(u/2) + (1-u)(v)(1+u)/2}{1-(1-u)(1-v)}. \tag{5}$$

Unlike the simpler case for females, a male entering the final period will not obtain an expected payoff of μ_y - he will only get this if he is lucky enough to be matched. Otherwise he will have the cost c (utility $-c$). The probability p of a male being matched in the second period is the inverse sex ratio

$$p = \frac{1-k}{r-k}. \tag{6}$$

Hence a male entering the final period has an expected payoff, which we will call $\tilde{\mu}_y$, given by

$$\tilde{\mu}_y = p\,\mu_y + (1-p)(-c), \text{ or} \tag{7}$$

$$\tilde{\mu}_y = \psi(u,v) \equiv \frac{-2c-2u+2cr+2ru+u^2}{-2u+u^2+1} \tag{8}$$

The justification for the notation $\tilde{\mu}_y$ is that we may add to the final period female population a number $r-1$ of imaginary females with type $-c$, and in this case $\tilde{\mu}_y$ would indeed be the mean of such a population. Hence the male equilibrium equation (m.e.c.) is

$$u = (\psi(u,v))^+, \text{ where } (a)^+ = \max(a,0), \tag{9}$$

since a cutoff value of 0 is equivalent to a negative one. Solving the m.e.c. (9) for v as a function of u, for u in the range of ψ^+, gives

$$v = g(u) \equiv \frac{u^2 + (2r-2)u + 2c(r-1)}{(u-1)^2} \tag{10}$$

Definition: A pair (u,v), $0 \le u, v \le 1$, satisfying (3) and (9) is called an *equilibrium* of the game $\Gamma_2(c,r)$. If $u=0$, the equilibrium (u,v) is called a *female-choice equilibrium (f.e.c.)* (or, one-sided choice equilibrium) and if $u>0$ it is called a *mutual-choice equilibrium* $(m.e.c.)$ (or two-sided equilibrium). Let E_1 denote the set of all one-sided equilibria, E_2 the two-sided ones, and $E = E_1 \cup E_2$ the set of all equilibria.

Note that at *any* equilibrium we have $v>0$, since the mean of the final period males is always positive.

2.1 Equilibrium Theorem

In the symmetric case $r=1$ studied in Alpern and Reyniers (2005) (where c is irrelevant, as all males end up mated), the unique equilibrium was shown to be the mutual-choice equilibrium $u = v = \dfrac{3-\sqrt{5}}{2} \approx 0.381\,97$. For general r and c the situation is more complicated, though indeed for r sufficiently close to 1 (depending on c) there is still a unique equilibrium which is of mutual-choice type. More generally, we show in Theorem 1 that equilibrium behavior partitions c,r space by two curves: $r = r_F(c)$, called the 'F' curve; and $r = r_M(c)$, called the 'M' curve. Female choice equilibria exist only on or above the F curve, while mutual choice equilibria exist only on or below the (higher) M curve. These curves are defined as

$$r = r_F(c) = \frac{(c+1/2)^2}{c(c+1)} \text{ and} \tag{11}$$

$$r = r_M(c) = \begin{cases} \dfrac{27(1+c) - \sqrt{27(-5-10c+27c^2)}}{32}, & c \geq 1 \\ r_F(c), & c < 1 \end{cases} \tag{12}$$

For $c \geq 1$, we have $r_M(c) \geq r_F(c)$ with equality only at $c = 1$ (where they are both 9/8, and tangent to each other). Figure 2 shows how the two curves F $(r = r_F(c))$ and M $(r = r_M(c))$ divide c, r space into three open regions defined by

$$I = \{(r,c) : r > r_M(c) \text{ and } c \leq 1, \text{ or } r > r_F(c) \text{ and } c > 1\}, \tag{13}$$

$$II = \{(r,c) : r < r_F(c)\ \}, \text{ and} \tag{14}$$

$$III = \{\{(r,c) : r_F(c) < r < r_M(c) \text{ and } c < 1\ \}\}. \tag{15}$$

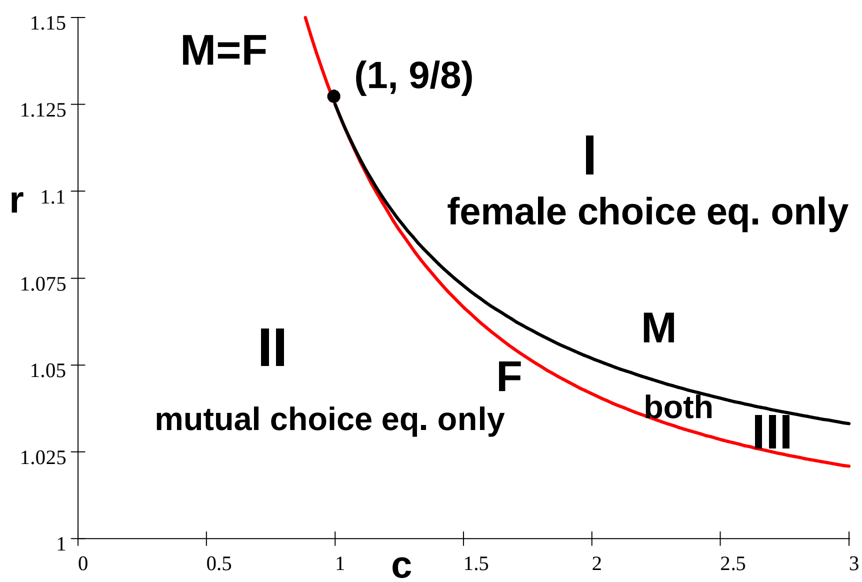

Figure 2. Illustration of Theorem 1

We use this numbering of the regions because: region I has a *one*-sided (female-choice) equilibrium, region II has a *two*-sided (mutual-choice) equilibrium, and region III has *three* equilibria (one female-choice and two mutual choice). The following is our main result for the two period games, and will be proved in the next section.

Theorem 1 *Consider the two period game $\Gamma_2(r, c)$, for $r \geq 1$ and $c \geq 0$, and let the regions I, II, and III be defined as in (13-15).*

1. *If $(r, c) \in I$, then there is a unique equilibrium and it is a female-choice equilibrium*
2. *If $(r, c) \in II$, then there is a unique equilibrium and it is a mutual-choice equilibrium*
3. *If $(r, c) \in III$, then there are three equilibria: one of them is a female-choice equilibrium, and the other two are mutual-choice equilibria.*

Figure 3 illustrates equilibria in regions I, II, III, given as the intersection of the female equilibrium condition (3) drawn in red (thin) and the male equilibrium condition (9) drawn in green (thick).

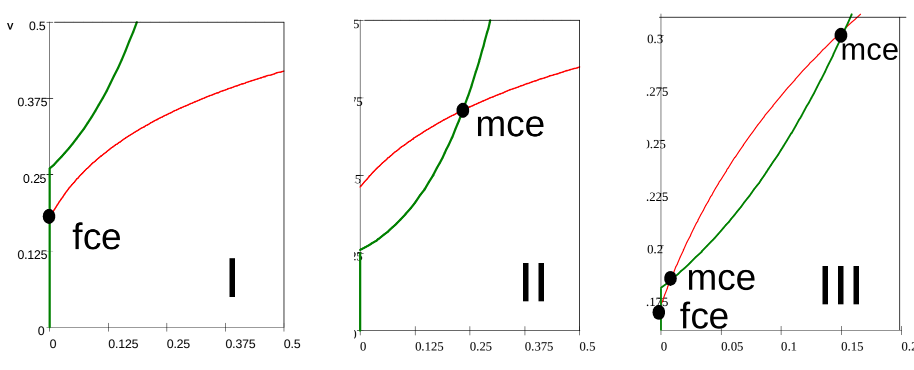

Figure 3. Male (green) and female (red) equilibrium conditions

2.2 Proof of Theorem 1

In Figure 3 we illustrated Theorem 1 by exhibiting the equilibria E symmetrically with respect to male and female strategies u and v, as the intersection of the male and female equilibrium conditions. However to prove Theorem 1, we now take an asymmetric approach in which we determine only the male cutoff strategies u which form half of an equilibrium pair (u, v). Of course if we know u, then v is uniquely determined by the female equilibrium condition (3). So for the time being we forget about v and concentrate only on u.

To establish Theorem 1, we show that the set $E = E_1$(one sided, f.c.e.'s) $\cup$ E_2 (two-sided, mce's) of equilibria is determined by the intersections of a certain cubic polynomial $q(u)$ with the disjoint union

$$\lceil \;=\; L_1 \cup L_2, \tag{16}$$

where L_1 is the negative $y-$axis $\{(y,0), -\infty < y \leq 0\}$ and L_2 is the open interval $\{(u,0) : 0 < u < 1\}$. Intersections with L_1 give fce's and those with L_2 give mce's. This is illustrated in Figure 4 for the three regions discussed in Theorem 1: For region I, q intersects only L_1; for region II, q only intersects L_2; for region III, q intersects L_1 and then intersects L_2 twice, once before and once after the relative maximum of q. Figure 4 should be compared with the earlier Figure 3, noting that the earlier one indicated both coordinates (u and v) of each equilibrium, while this figure indicates only the u coordinate.

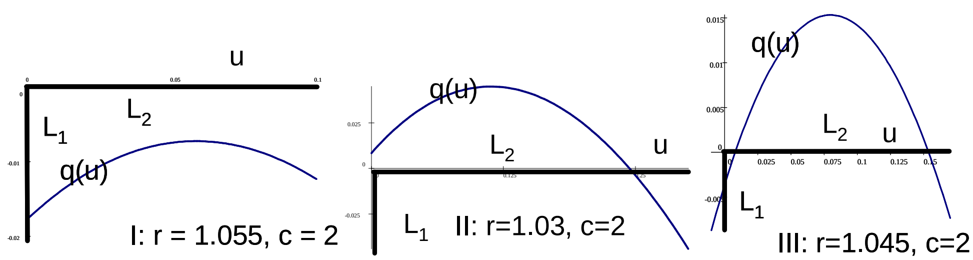

Figure 4: Intersection of q with $L_1 \cup L_2$, regions I,II,III

The following result establishes that the intersections shown in Figure 4 are indeed equilibria.

Lemma 2 *Fix any parameters $r \geq 1$ and $c \geq 0$. Then*

$$(u, g(u)) \in E \text{ if and only if } (u, q(u)) \in \lceil. \tag{17}$$

Furthermore

$$(u, g(u)) \in E_i \text{ if and only if } (u, q(u)) \in L_i. \tag{18}$$

Proof. Recall that $v = g(u)$ (10) is a form of the male equilibrium equation. In this analysis u is always in $[0,1]$. If we combine the two (female and male) equilibrium conditions in the form of (4) and (10), we can obtain all mutual-choice equilibria by seeking solutions $u \in (0,1)$ to the fixed point equation

$$\begin{aligned} u &= f(g(u)), \text{ or equivalently, solving} && (19) \\ h(u) &= f(g(u)) - u = 0. && (20) \end{aligned}$$

We can factor the rational function $h(u)$ in the form

$$h(u) = \frac{(1-u) \quad q(u)}{(1+2c-2cr-2ru)^2}, \text{ where } q = q(u) \text{ is the cubic} \tag{21}$$

$$q = ru^3 - 3ru^2 + \left(5r + 4cr - 4r^2 - 4cr^2\right) u - 4c^2r^2 - 4cr^2 + 8c^2r + 8cr + r - 4c^2 - 4c - 1$$

For $u > 0$, $q(u)$ is 0 if and only if $h(u)$ is 0, which is equivalent to (18) for $i = 2$. The condition $f(g(0)) < 0$ is equivalent to $(0, g(0))$ being a female-choice equilibrium, because $(0, g(0))$ always satisfies the male equilibrium condition (10) and $(f(g(0)), g(0))$ satisfies the female equilibrium condition (4). But a negative cutoff strategy $f(g(0))$ for the males is strategically equivalent to $u = 0$ (as there as no females of negative type y). Hence $(0, g(0))$ is an equilibrium. But the condition $f(g(0)) < 0$ is equivalent to $f(g(0)) - 0 < 0$, or $q(0) < 0$. Thus (18) holds for $i = 1$ as well, and hence the main condition (17) also holds. ■

Lemma 2 reduces the proof of Theorem 1 to the determination of the intersections of the cubic curve $q(u)$ with the set $\lceil$, for different values of the parameters r and c. The analysis of the cubic q is given in the following lemma. For Theorem 1 we will need information about the location α of the relative maximum and its height $q(\alpha)$.

Lemma 3 (analysis of cubic q) *The cubic $q(u)$ increases from $-\infty$ to it's relative maximum $q(\alpha)$ at α, then decreases until its relative minimum at β, from which point it increases to infinity. The numbers α and β (the two solutions of the quadratic equation $q'(u) = 0$) are given by*

$$\begin{aligned} \alpha &= 1 - \sqrt{2/3}\sqrt{D} < 1, \text{ and } \beta = 1 + \sqrt{2/3}\sqrt{D} > 1, \text{ where} && (22) \\ D &\equiv 2r - 1 + 2c(r-1) > 1. && (23) \end{aligned}$$

For all parameter values, we have $q(1) < 0$, *but the values of* $q(0)$ *and* $q(\alpha)$ *and* α *depend on the parameters* r *and* c *in that*

$$\begin{aligned} sign\,(q(0)) &= sign\,(r_F(c) - r)\,, & (24)\\ if\; r &< r_M(c)\,,\; c \geq 1,\; then\; q(\alpha) > 0 & (25)\\ if\; r &> r_M(c)\,,\; c \geq 1, then\; either\; q(\alpha) < 0\; or\; \alpha < 0 & (26)\\ sign\,(\alpha) &= sign\;(r_3(c) - r)\,,\; where\; r_3(c) = \frac{4c+5}{4c+4}. & (27)\end{aligned}$$

We can now use our two lemmas to give a simple proof of Theorem 1 which involves breaking up region I into two regions I_a (with $c \leq 1$) and I_b (with $c > 1$), as shown below in Figure 5.

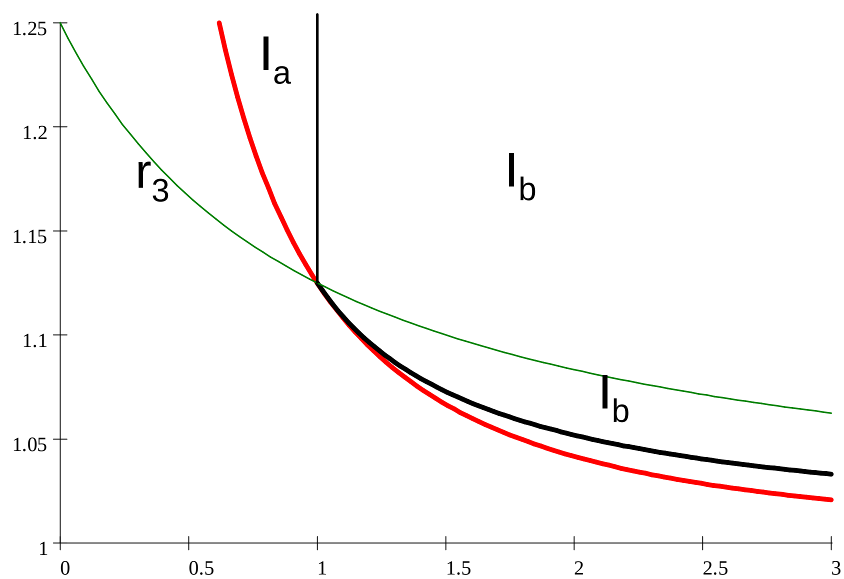

Figure 5: Partition of I into I_a and I_b

Proof of Theorem 1: The portions of the Theorem concerning female choice equilibria follow immediately from (18) for $i = 1$ and (24). That is, there is a female-choice equilibrium if $q(0) \leq 0$, which is equivalent to $r < r_F(c)$. So we need to consider only mutual-choice equilibria. Recall that u corresponds to a mutual-choice equilibrium if $0 < u < 1$ and $q(u) = 0$. We have shown in the previous lemma that $q(1)$ is always negative. We know that q is decreasing from α to $\beta > 1$. We now prove the cases in turn.

1a If $(c,r) \in I_a$, then $r \geq r_F(c)$ and $r \geq r_3(c)$. It follows from (24) that $q(0) \leq 0$ and from (27) that $\alpha \leq 0$. Consequently q is decreasing between 0 and 1, and can have no root in that open interval. Hence there is no mutual-choice equilibrium.

1b If $(c,r) \in I_b$, $r > r_M(c)$ and hence also $r > r_F(c)$. The latter condition ensures by (24) that $q(0)$ is negative. Since $r > r_M(c)$ we have by (26) that $q(\alpha) < 0$ or $\alpha \leq 0$. If $\alpha \leq 0$, then as in the previous part, q is decreasing between 0 and 1. If $\alpha > 0$, q will increase until α, but $q(\alpha) < 0$, so q has no root less than α. From α to 1 it is decreasing and the result follows.

2 If $(c,r) \in II$, we have $r < r_F(c)$ and so by (24) we have $q(0) > 0$. Since $q(1) < 0$ for all parameters, the Intermediate Value Theorem guarantees at least one root of q between 0 and 1, hence at least one mutual choice equilibrium. If q had two roots between 0 and 1, then it would have a relative minimum between them. But $q(u)$ has only one relative minimum, at $\beta > 1$. Hence in this case there is exactly one mutual-choice equilibrium.

3 If $(c,r) \in III$, then $r > r_F(c)$, $r < r_2(c)$, and hence $r < r_3(c)$. So by (24) we have $q(0) < 0$, by (25) we have $q(\alpha) > 0$ and by (27) we have $\alpha > 0$. Hence by the Intermediate Value Theorem, q has a root between 0 and α and another root between α and 1. We have already explained above why q cannot have more than two roots between 0 and 1. Hence there are two mutual-choice equilibria.

2.3 Analysis of Equilibria

In Theorem 1 we determined the number and type of equilibria, as a discrete function (regions I, II, III) of the parameter values r and c. Here we obtain explicit formulae for these equilibria and analyze how they depend continuously on the parameters r and c. Our first observation is that when comparing equilibria, the level of choosiness (acceptance level) goes in the same direction for both males and females, the monotonicity lemma. In other word, one of the equilibria is choosier than the other (for both sexes). To see this, recall that any equilibrium pair (u,v) satisfies the female equilibrium equation (4) $u = f(v)$, so $f'(v) = 2vr/(1-v)^3 > 0$ implies that u is increasing in v, giving the following.

Lemma 4 (Monotonicity) *Given any two equilibria* (u,v) *and* (u',v'), *we have* $u \neq u'$ *and*

$$(u'-u)(v'-v) > 0 \tag{28}$$

As an application of this lemma, the three potential equilibria can be ordered in terms of *choosiness* as $e^i = (\bar{u}^i, \bar{v}^i)$, $i = 1,2,3$, where for $i < j$ we have both $\bar{u}^i < \bar{u}^j$ and $\bar{v}^i < \bar{v}^j$. We name these (where the latter two are mutual choice equilibria) as:

e^1, the *female choice equilibrium*, which exists on and above the F curve,

e^2, the *easy equilibrium*, which exists between the M and F curves, and

e^3, the *choosy equilibrium*, which exists on and below the M curve.

For the female choice equilibrium e^1, we have obviously $\bar{u}^1 = 0$, and can obtain $\bar{v}^1$ directly from the female equilibrium condition (4) $0 = f(v)$, or $0 = -r - 2v + 2rv + v^2 + 1$, with unique positive solution

$$\bar{v}_1 = 1 - r + \sqrt{r^2 - r}.$$

For the mutual choice equilibria $e^i = (\bar{u}^i, \bar{v}^i)$, $i = 2,3$, we obtain the the formula for $\bar{u}^i$ by explicitly solving the cubic equation $q(u) = 0$ for $u = \bar{u}^i$, getting the corresponding $\bar{v}^i$ from the formula $\bar{v}_i = g(\bar{u}^i)$ (10).

$$\bar{u}^2 = 2\sqrt{\gamma}\cos\left(\frac{t+2\pi}{3}\right) + 1, \quad \bar{u}^3 = 2\sqrt{\gamma}\cos(\frac{t+4\pi}{3}) + 1, \text{ where} \tag{29}$$

$$\begin{aligned} t &= \arccos\left((2-a_1-a_0)/2\sqrt{\gamma^3}\right), \gamma = (3-a_1)/3, \\ a_1 &= 5 - 4c(r-1) - 4r, a_0 = \left(-1 + 4c(r-1) + 4c^2(r-1)\right)(1-r)/r \end{aligned}$$

To see what the equilibria look like in u,v space, for a 4×5 grid of c, r parameters taken to cover points in all three regions I, II, and III (see Figure 6), we draw the equilibria in a similarly arranged array of boxes in Figure 7. In each box (with r and c fixed), we plot any female choice equilibria with a red triangle and any mutual choice equilibria with blue diamonds.

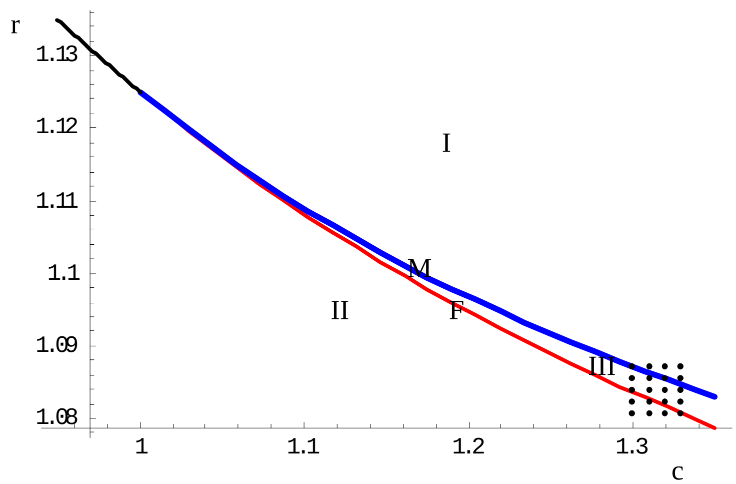

Figure 6. The 20 grid points for equilibrium analysis

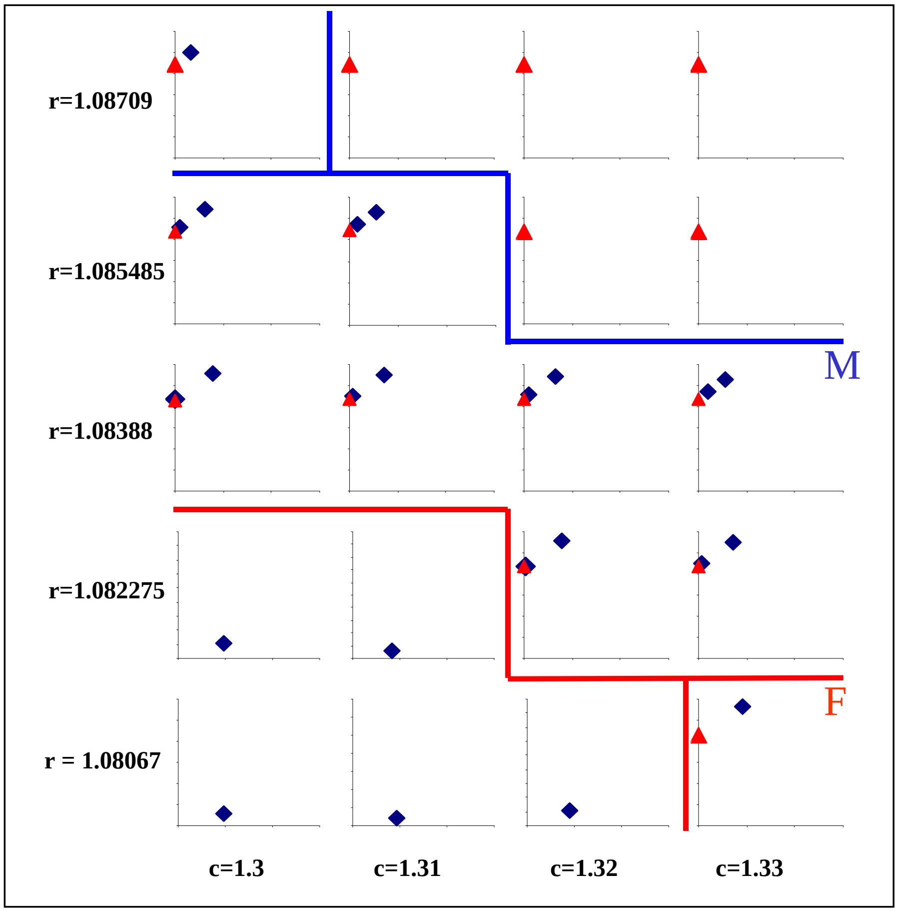

Figure 7: Equilibria at the 20 c, r grid points

We have drawn the M line in blue and the F line in red. Note that the grid points were chosen so that the former goes through the upper left box and the latter goes through the lower right box. For that reason we have drawn the lines as splitting around these boxes. Observe that in all cases in Figure 7, a line between two equilibria is always upwards sloping, as follows from the Monotonicity Lemma. Note that as we go up (increasing r) the column of boxes corresponding to $c = 1.32$, we start with one mutual choice equilibrium (which is the choosy one e^3), then get all three, and finally get only the female choice equilibrium e^1. A better way of seeing these transitions is to consider the bifurcation diagram drawn in Figure 8 with the sex ratio r increasing to the right, and the male and female acceptance levels drawn in the vertical axis. The lower (black) curves describe the male equilibrium acceptance levels u, while the top (red) lines describe the female levels v. In region III, the equilibrium values for each sex appear, from top to bottom, in the order e^3, e^2, e^1.

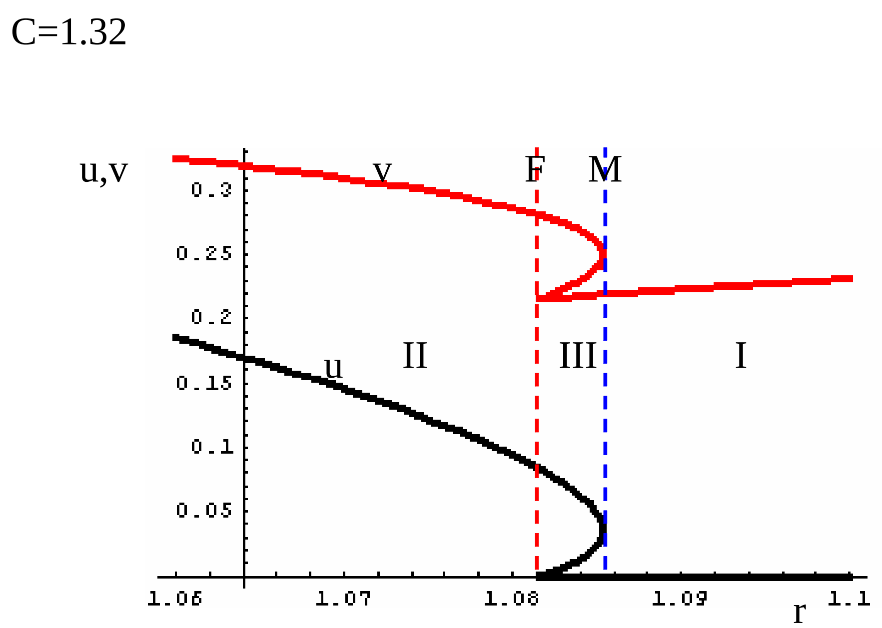

Figure 8: Bifurcation in r of equilibria for $c = 1.32$

2.4 Dynamical Stability of Equilibria

The equilibria e^i are solutions to both the male and female equilibrium conditions (9,3), or equivalently are fixed points of the mapping T given by

$$T(u,v) = \left(\psi^{+}(u,v), \phi(u,v)\right). \tag{30}$$

In this section we determine the dynamical stability of the equilibria e^i as fixed points of the mapping T. That is, a fixed point is stable if iterations of T applied to nearby points converge back to it. To do this, we must determine the matrix norm

$$N^{j}\left(u,v\right)=\left\|\begin{pmatrix}\frac{\partial\psi^{+}}{\partial u} & \frac{\partial\psi^{+}}{\partial v}\\ \frac{\partial\phi}{\partial u} & \frac{\partial\phi}{\partial v}\end{pmatrix}^{j}\right\|. \tag{31}$$

A fixed point (u,v) will be dyamically stable if for some j, $N^{j}(u,v)<1$. Since we found the formulae for the equilibria $e^{i}(r,c)$ in the previous section, we can evaluate the Jacobian matrix at these u,v values. We find that $N^{2}(e^{1}(r,c))<1$ where e^{1} exists (on and above the F curve) and $N^{2}(e^{3}(r,c))<1$ where e^{3} exists (on and below the M curve). Furthermore $N(e^{3}(r,c))<1$ below the F curve and $N^{1}(e^{1}(r,c))$ above the M curve. On the other hand both eigenvalues of the Jacobian of (ψ^{+},ϕ) at the easy equilibrium e^{2} have absolute values larger than 1. Summarizing these numerical results, we have the following.

Proposition 5 *Let e^{1},e^{2} and e^{3} be the female, easy and choosy equilibria. The equilibria e^{1} and e^{3} are dynamically stable and the equilibrium e^{2} is unstable.*

2.5 Marital Stability σ of Equilibria

Suppose we look at the distribution of couples over the (x,y) square that arises at the end of the play of our game, or indeed that arises in any way. We ignore in this analysis the unmated males. For the moment, suppose that agent preferences are arbitrary. We say that a pair of couples (x_1,y_1) and (x_2,y_2) is *unstable* if a male from one couple and a female from the other both prefer each other to their current partner. In our common preference model, where type equals utility to the opposite sex, this means that the better (higher type) male and the better female belong to distinct couples, or that $(x_2-x_1)(y_2-y_1)<0$. If a pair of couples is not unstable, we say it is *stable.* We define the *Stability* σ of a given distribution to be the probability that a randomly and independently chosen pair of couples is stable. This definition is similar in spirit to that proposed by K. Eriksson and P. Strimling in (2004). In our two period game, every strategy pair (u,v) (not only equilibrium pairs) leads to a couple distribution that is uniform (with some constant density) on each of the four subrectangles rectangles R_k of the unit square in x,y

space determined by the lines $x = v$ and $y = u$ drawn in Figure 9.

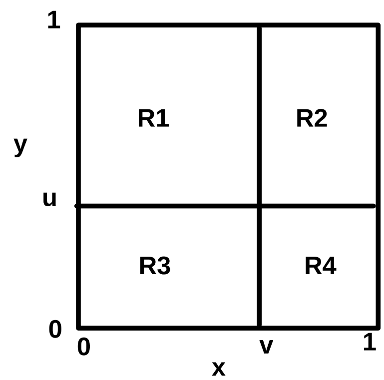

Figure 9. Couple distribution on the type square.

Let π_k denote the probability that a couple belongs to R_k. Note that $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$ and that $\pi_1 + \pi_2 = 1 - u$, while $\pi_1 + \pi_3$ depends on r. Define a symmetric 4×4 matrix S so that $s_{i,j}$ is the probability that a pair of couples is stable, given that the couples belong to R_i and R_j. It is easy to see that two couples belonging to R_1 and R_4 form an unstable pair, while a pair belonging to R_3 and R_2 form a stable pair. Otherwise, the couples belong to two rectangles whose union is a rectangle R, and which are each preserved under a symmetry transformation θ of R. Observe that θ transposes pairs of such couples in such a way that if one is stable then the other is unstable. Hence for all these cases, $s_{ij} = 1/2$. For example, Figure 10 illustrates how

$\theta(y) = u - y$ transposes stable with unstable couple pairs R_3 and R_4.

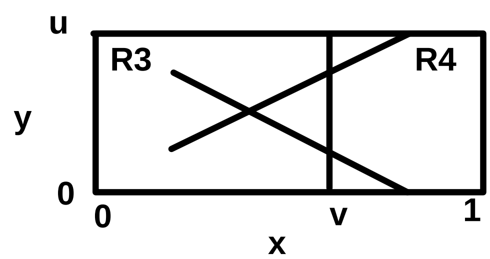

Figure 10: $s_{3,4} = 1/2$

Thus

$$S = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 \\ 1 & 1/2 & 1/2 & 1/2 \end{pmatrix}.$$

So if the the distribution over the rectangles is $\pi = \pi(u, v) = (\pi_1, \pi_2, \pi_3, \pi_4)$, the stability σ is given by

$$\sigma = \sigma(u, v) = \pi S \pi = \frac{1 + 2\pi_1\pi_4 - 2\pi_2\pi_3}{2}. \tag{32}$$

Note that at any female choice equilibrium we have $u = 0$, hence $\pi_3 = \pi_4 = 0$, so by (32) we have $\sigma = 1/2$. More generally, we calculated σ at the three equilibria in III, observing that

Proposition 6 *For any r and c, in region III, the choosy equilibrium e^3 is the most stable one and the female choice equilibrium e^1 is the most unstable one. That is,*

$$0.5 = \sigma(e^1(r, c)) \leq \sigma(e^2(r, c)) \leq \sigma(e^3(r, c)).$$

Furthermore, $\sigma(e^2(r, c)) \leq 0.54$ and $\sigma(e^3(r, c)) < 0.59$.

Figure 11 plots the marital stability σ of the three equilibria as a function of r, for fixed $c = 1.32$. The red line is $\sigma(e^1)$, the black line $\sigma(e^2)$ and the green line $\sigma\left(e^{3}\right)$.

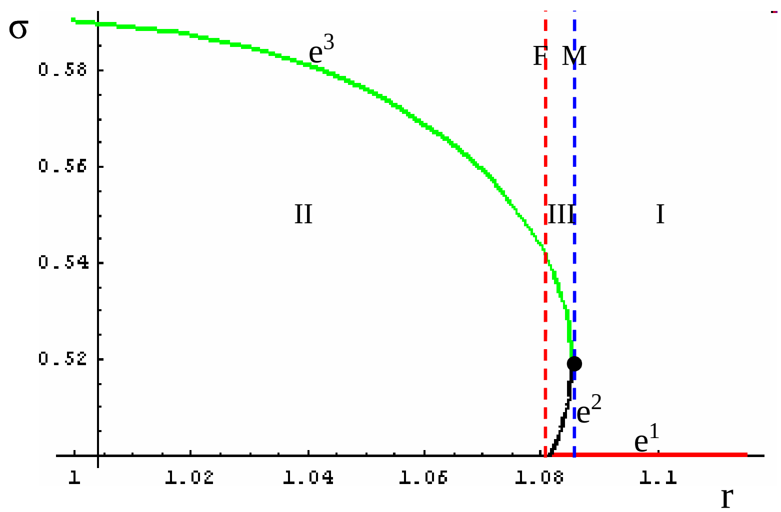

Figure 11. Marital Stability σ of the e^{i}.

3 n-Period Model, $n > 2$.

In the previous section, we were able to obtain a complete analytical description of the equilibria for the two period model, with explicit formulae. Due to the complexities involved, this will no longer be possible for models with $n > 2$ periods. However, we are still able to obtain some analytical solutions for the continuous (uniform distribution) model and fairly complete numerical solutions for models where both sexes come in m discrete types. We have extensive results for $n = 3$ and very partial results for $n = 4$.

3.1 Female-Choice Equilibrium for $n = 3$

This section presents our only analytical result for $n > 2$, the determination of the unique female choice equilibrium for $n = 3$. We assume that any male accepts any female in any period. For females, we denote by v_1 and v_2, $v_2 < v_1$, the lowest male type that a top female $(y = 1)$ will accept in periods 1 and 2. Let $p_i = 1/r_i$ be the inverse sex ratio, the ratio of females to males at the beginning of period i. This is the probability that a male entering period i will be matched in that period. Let $q_i = 1 - p_i$ be the complementary probability of not being matched in period i, so that

$$q_1 = 1 - 1/r, \quad q_2 = 1 - v_1/\left(v_1 + r - 1\right).$$

There are three type-classes of male: High, $H = [v_1, 1]$, with initial probability $P_H = 1 - v_1$ and mean $\mu_H = (1 + v_1)/2$; Medium, $M = [v_2, v_1]$ with initial probability $P_M = v_1 - v_2$ and mean $\mu_M = (v_1 + v_2)/2$; Low $L = [0, v_2]$ with initial probability $P_L = v_2$ and mean $\mu_L = v_2/2$. The probabilities that males of these types reach the final period unmated are given by

$$\bar{P}_L = 1, \bar{P}_M = q_2, \bar{P}_H = q_1 q_2.$$

The expected payoff e_3 to a female entering the final period is simply the mean type of the final period male distribution, and hence given by

$$e_3 = \frac{P_L s_3(L)\mu_L + P_M s_3(M)\mu_M + P_H s_3(H)\mu_H}{P_L s_3(L) + P_M s_3(M) + P_H s_3(H)}. \tag{33}$$

Thus

$$e_3 = \frac{r^2 - v_1^2 - 2r + v_1^2 r + v_1 v_2^2 r + 1}{2v_1 r - 4r - 2v_1 + 2v_1 v_2 r + 2r^2 + 2}. \tag{34}$$

The expected utility e_2 for a female entering period 2 unmated is calculated as follows: If she meets a Low male, she goes into final period and gets e_3; if she meets a Middle male she accepts and gets on average $(v_1 + v_2)/2$; if she meets a High male, she accepts and gets on average $(1 + v_1)/2$. Hence her expected payoff is given by

$$e_2 = \frac{v_2 r}{\kappa} e_3 + \left(\left(v_1 - v_2\right) r/\kappa\right)\left(\frac{v_1 + v_2}{2}\right) + \frac{(1 - v_1)(r - 1)}{\kappa}\left(\frac{1 + v_1}{2}\right), \tag{35}$$

where $\kappa = r - (1 - v_1)$ is the male population in period 2. Hence the female equilibrium condition is given by the two equations,

$$v_2 = e_3 \text{ and } v_1 = e_2 \tag{36}$$

The solution $v_1(r)$ and $v_2(r)$ to the female equilibrium equations is drawn (using *Mathematica*) in the following figure. Of course, these will be equilibria only if c is sufficiently large so that males will always accept, that is, if the male equilibrium equations are also satisfied.

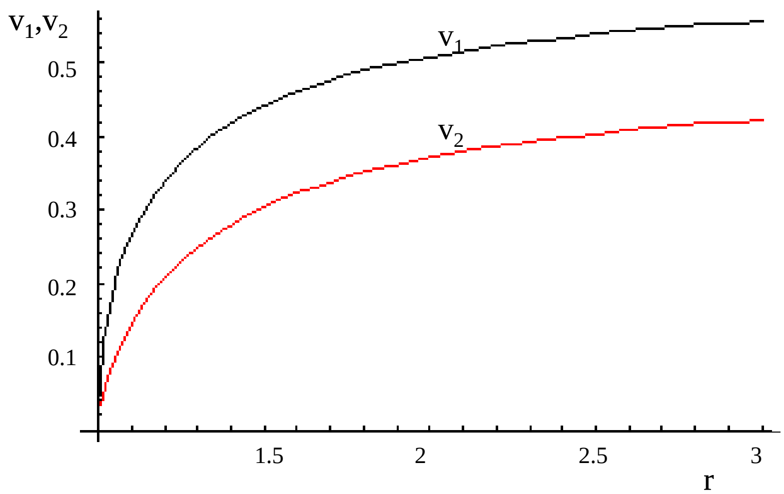

Figure 12. Female Choice Equilibrium, $n = 3$.

To determine when v_1, v_2 forms an equilibrium with males always accepting ($u_1 = u_2 = 0$) we must find when a male of type 1 will accept a female of type 0 in period 1. Clearly he will do this only if his expected payoff w_2, if he goes into period 2 unmated, is not positive. A type 1 male will be accepted if matched, and will on average be matched with a type 1/2 female. If he is not matched in either period, he gets $-c$. Thus

$$w_2 = w_2\left(r, c\right) = \left(p_2\right)\frac{1}{2} + \left(q_2\ p_3\right)\frac{1}{2} + \left(q_2\ q_3\right)\left(-c\right). \tag{37}$$

Solving the equation $0 = w_2\left(r, c\right)$ for r as a function of c gives the line (which we again call the F line) above which we have one female choice equilibrium and below which we have none. This line is drawn as $F = F^3$ (again using Mathematica) in Figure 13, alongside the F and M lines of the two period model of Figure 2.

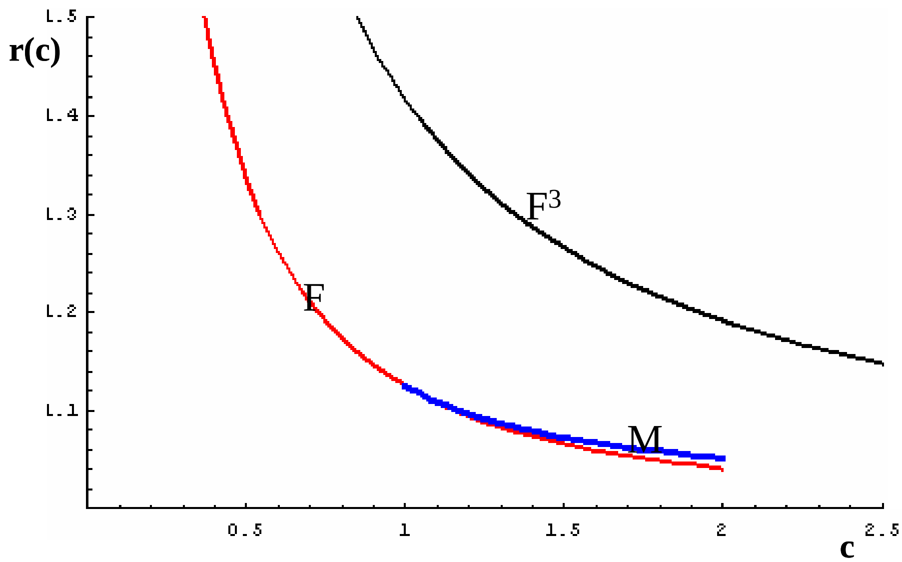

Figure 13. The F line for $n = 3$.

3.2 Discrete Type Model for $n = 3$

Although it was possible in the previous section to obtain the female choice equilibrium for $n = 3$ and the uniform distribution by analytical means, the mutual choice equilibria cannot be obtained in this way. For this reason we now turn to the game $\Gamma_{n,m}(r, c)$ in which both sexes are initially distributed with equal amounts of types $0, 1, 2, \ldots, m - 1$. To align our results with the continuous model where types belong to the interval $[0, 1]$, the cost to a male of not mating will be given as $c\ m$. For $n = 3$ and small m we can obtain *all* the equilibria by a modified exhaustive search technique developed by Katrantzi. Figure 14 charts, for $m = 8$, the qualitative aspects of this search, for r and c in the grid. Here, F represents just a female choice equilibrium, M just a male choice equilibrium, and B the presence of both types. The 'F' line is drawn in red, the 'M' line in blue, and the portion M=F is drawn in black. One can easily detect the same qualitative partitioning of c, r space into the regions I, II, and III of Figure 14. Note that, compared with the two period problem, region III (B's) is smaller and the 'M' line is lower.

	c=0.6	c=0.8	c=1	c=1.2	c=1.4
r=1.8	F	F	F	F	F
r=1.7	F	F	F	F	F
r=1.6	M	F	F	F	F
r=1.5	M	B	F	F	F
r=1.4	M	B	B	F	F
r=1.3	M	M	M	F	F
r=1.2	M	M	M	M	M
r=1.1	M	M	M	M	M
r=1	M	M	M	M	M

Figure 14. Regions I, II, III for $n = 3$, $m = 8$

To obtain a more quantitative analysis of the equilibria, as functions of r and c, we describe in Figure 15 the equilibria corresponding to a grid of r and c values. This is analogous to Figure 7 for $n = 2$, except that for $n = 2$ an equilibrium could be represented by a single point (u, v), whereas for $n = 3$ we represent each equilibrium by a line segment between the lower male equilibrium values (u_1, u_2) and the higher female values (v_1, v_2). The grid lines correspond to $u, v \in \{0, 1, 2, 3, 4\}$. Female choice equilibria, which have lower point $(u_1, u_2) = (0, 0)$, are drawn in red. Note that most of the mutual choice equilibria start at the bottom ($u_2 = 0$), so in these the males are only choosy in the first period. In fact, only in the two boxes corresponding to $r = 1.1$ and $c = .6$ and $.8$ is there a mutual choice equilibrium where the males are choosy in both periods.

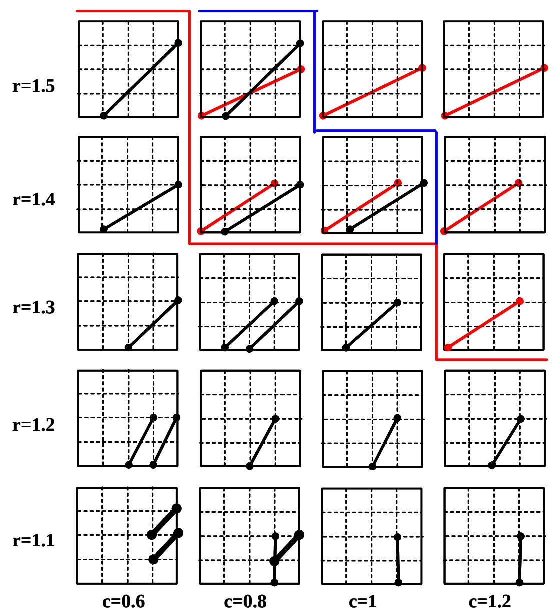

Figure 15. Equilibria for discrete uniform distribution, $m = 8$.

The red line between the boxes is the F line and the blue line is the M line.

3.3 Equilibria for $n = 4$

When there are $n = 4$ periods, it is not possible to do an exhaustive search for all equilibria, even for small numbers of types. So we adopt the iterative methods employed by Alpern and Reyniers (1999) in which an analog of the T mapping of (30) is iterated to find attracting fixed points. Of course in this case T acts on 6-dimensional space,

$$T\left(\left(u_1, u_2, u_3\right), \left(v_1, v_2, v_3\right)\right) = \left(\left(w_2, w_3, w_4\right), \left(e_2, e_3, e_4\right)\right),$$

where w_j (resp. e_j) is the expected payoff for a male (resp. female) entering period j unmated, given the strategies u_i, v_i. For each pair of r and c, we start the iteration at a number of different points, and note the fixed points (all orbits of T appear to converge). In the cases where only one type of equilibrium is observed (female or mutual choice), we indicate this by an F or M; if both appear we write down a B. Of course any F or M might become a B if we added the right additional starting point.

	c=0.6	c=0.7	c=0.8	c=0.9	c=1
r=2	M	F	F	F	F
r=1.9	M	F	M	F	F
r=1.8	M	**B**	**B**	F	F
r=1.7	M	M	M	F	F
r=1.6	M	M	M	M	F
r=1.5	M	M	M	M	M
r=1.4	M	M	M	M	M
r=1.3	M	M	M	M	M
r=1.2	M	M	M	M	M
r=1.1	M	M	M	M	M
r=1	M	M	M	M	M

Figure 16. Equilibria for $n = 4$, $m = 8$

The pattern is similar for the case $n = 3$ shown in Figure 14, except for the M between an F and a B at the top of column $c = .08$. Possibly the M is really a B.

4 Conclusions

This paper generalized the earlier matching model of Alpern and Reyniers (2005) by considering unequal sized groups to be matched. Calling the larger group 'males', and letting $r \geq 1$ denote the 'sex ratio' of males to females, we observed that a fraction $1/r$ of the males will end the game unmated. We set the utility of this eventuality to an unmated male as a cost (negative utility) $c \geq 0$. We then analyzed the equilibria of the resulting n-period game $\Gamma_n(r, c)$. We analytically determined the equilibria in terms of the parameters r and c for $n = 2$. We found two regions with unique equilibria (one-sided and two-sided) and a more interesting region with three simultaneous equilibria: a choosy equilibrium (both groups with high acceptance standards), an easy equilibrium (both groups have low standards) and a one-side equilibrium (males accept anyone). It is an interesting question as to which equilibrium one would expect to find in practice. If the process is one that is repeated each season (hiring season, mating season), one might expect that the equilibrium is determined in an evolutionary manner, in which case we would expect either the choosy or one-sided equilibria, which are dynamically stable (and the choosy one has a larger basin of attraction). If equilibria are chosen by society to be stable with respect to deviations after the couple formation (e.g. divorce), then we would also expect to see the choosy equilibrium, because it has the highest marital stability index σ. Our preliminary investigations (to be carried further in a subsequent article) indicate that different quality individuals (bands of types) fare unequally in the three equilibria, and so we might expect that the power of these groups (expressed through their numbers or otherwise) might be a determinant of the equilibrium that occurs.

In this paper we have taken the usual route of not explicitly modeling the process that pairs unmated individuals at the start of each period. Presumably, in order to be matched, the pairs must come into spatial contiguity by some process. A good candidate for this process, assuming individuals *want* to be matched, is what is known in the literature as *rendezvous search*, which temporally optimizes the search for a partner. Originally posed by the first author in 1976, this problem has been extensively studied (mostly, but not exclusively, for two searchers). See, for example, Alpern and Gal (2003), Alpern (2002),Gal (1999), and Howard (1999).. With more work in this area for multiple searchers, we might have to modify our assumption of *random* pairing in each period. For example in the housing market some real estate agents may cater mostly for expensive houses and rich buyers (a rendezvous focal point), giving some degree of assortative matching even before choice is taken into account. Similarly, in the biological setting, Cronin (1991) has suggested that assortative pairing may arise due to non-random arrival times at the breeding ground, another non-choice factor.

Another aspect of our model that might be varied is the assumption, as in the secretary problem, that one cannot 'go back' to someone you have met but rejected in a previous period. It is well known that in practice this behavior sometimes is seen (e.g. housing and dating markets). This would present an interesting problem of modelling. We are also planning to consider other initial distribution of type (x and y) – initial results in this direction are qualitatively quite similar when we adopt a truncated normal distribution. Of course the uniform distribution that we consider here has a specially nice interpretation in that utility of a mate is equated with mate's relative position in their group.

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