Bitopology and measure-category duality

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In memoriam Caspar Goffman (1913 - 2006)

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Abstract

We re-examine measure-category duality by a bitopological approach, using both the Euclidean and the density toplologies of the line. We give a topological result (on convergence of homeomorphisms to the identity) obtaining as a corollary results on infinitary combinatorics due to Kestelman and to Borwein and Ditor. As a by-product we give a unified proof of the measure and category cases of Uniform Convergence Theorem for slowly varying functions.

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1 Introduction

In a topological space one has one space and one topology. One often needs to have one space and two comparable topologies, one stronger and one weaker (as in functional analysis, where one may have the strong and weak topologies in play, or the weak and weak-star topologies). The resulting setting is that of a *bitopological space,* formalized in this language by Kelly [Kel].

Measure-category duality is the theme of the well-known book by Oxtoby [Oxt]. Here one has on the one hand measurable sets or functions, and small sets are null sets (sets of measure zero), and on the other hand sets or functions with the Baire property (briefly, Baire sets or functions), where small sets are meagre sets (sets of the first category).

In some situations, one has a dual theory, which has a measure-theoretic formulation on the one hand and a topological (or Baire) formulation on the other. We present here as a unifying theme the use of two topologies, each of which gives one of the two cases.

Our starting point is the *density topology* (introduced in [HauPau], [GoWa], [Mar] and studied also in [GNN] – see also [CLO], and for textbook treatments [Kech], [LMZ]). Recall that for T measurable, t is a (metric) density point of T if $\lim_{\delta\to 0}|T\cap I_\delta(t)|/\delta = 1$, where $I_\delta(t) = (t-\delta/2, t+\delta/2)$. By the Lebesgue Density Theorem almost all points of T are density points ([Hal] Section 61, [Oxt] Th. 3.20, or [Goff]). A set U is d-open (open in the density topology) if each of its points is a density point of U. We mention three properties:

(i) The density topology (d-topology) is finer than (contains) the Euclidean topology ([Kech], 17.47(ii)).

(ii) A set is Baire in the density topology iff it is (Lebesgue) measurable ([Kech], 17.47(iv)).

(iii) A function is d-continuous iff it is approximately continuous in Denjoy's sense ([Den]; [LMZ], p.1, 149).

The reader unfamiliar with the density topology may find it helpful to think, in the style of Littlewood's First Principle, of basic opens sets as being intervals less some measurable set. See [Lit] Ch. 4, [Roy] Section 3.6 p.72.

Both measurability and the Baire property have been used as regularity conditions, to exclude pathological situations. A classic instance is that of *additive functions,* satisfying the *Cauchy functional equation* $f(x+y) = f(x)+f(y)$. Such functions are either very good – continuous, and so linear, $f(x) = cx$ for some c – or very bad (one can construct such functions from

Hamel bases, so this is called the *Hamel pathology);* see [BOst-SteinOstr] for details. Another, related instance is that of the theory of *regular variation* [BGT], where each may be used as a regularity condition to prove the basic result of the theory, the Uniform Convergence Theorem (UCT). In such situations, the theory is usually developed in parallel, with the measure case regarded as primary and the Baire case as secondary. Here, we develop the two cases together. Our new viewpoint gives the interesting insight that it is in fact the Baire case that is the primary one.

In Section 2 below we give our main result, the Category Embedding Theorem (CET); the natural setting is a *Baire space*, i.e. a topological space in which the Baire Category Theorem holds (see e.g. [Eng] 3.9): $\mathbb{R}$ is a Baire space under both the Euclidean and density topologies. In Section 3 we give our unified treatment of the UCT. We close in Section 4 with some remarks.

2 Category Embedding Theorem (CET)

The three results of this section (or four, as Theorem 3 below has two cases) develop a new aspect of measure-category duality. This has powerful applications: see Section 3 below for the Uniform Convergence Theorem (UCT) of regular variation, [BOst5] for subadditive functions, [BOst10] for homotopy versions, and [BOst-SteinOstr] for applications to classical theorems of Steinhaus and Ostrowski.

The topological Theorem 1 below is a topological version of the Kestelman-Borwein-Ditor (KBD) Theorem given at the end of this section (see also [BOst1], [BOst10], [BOst-SteinOstr]). The latter is a (homeomorphic) *embedding* theorem (see e.g. [Eng] p. 67); Trautner uses the term covering principle in [Trau]. We need the following definition.

Definition (**weak category convergence**). A sequence of homeomorphisms h_n satisfies the *weak category convergence* condition (wcc) if:

For any non-empty open set U, there is an non-empty open set $V \subseteq U$ such that, for each $k \in \omega$,

$$\bigcap_{n \geq k} V \backslash h_n^{-1}(V) \text{ is meagre.} \tag{wcc}$$

Equivalently, for each $k \in \omega$, there is a meagre set M such that, for $t \notin M$,

$$t \in V \Longrightarrow (\exists n \geq k)\ h_n(t) \in V.$$

We will see below in Theorem 2 that this is a weak form of convergence to the identity and indeed Theorems 3E and D verify that, for $z_n \to 0$, the homeomorphisms $h_n(x) := x + z_n$ satisfy (wcc) in the Euclidean and in the density topologies. However, it is not true that $h_n(x)$ converges to the identity pointwise in the sense of the density topology; furthermore, whereas addition (a two-argument operation) is not d-continuous (see [HePo]), translation (a one-argument operation) is. We write 'quasi all' for 'all off a meagre set' and, for P a set of reals (or property) that is measurable/Baire, we say that 'P holds for generically all t' to mean that $\{t : t \notin P\}$ is null/meagre.

Theorem 1 (Category Embedding Theorem, CET). *Let X be a Baire space. Suppose given homeomorphisms $h_n : X \to X$ for which the weak category convergence condition* (wcc) *is met. Then, for any non-meagre Baire set T, for quasi all $t \in T$, there is an infinite set $\mathbb{M}_t$ such that*

$$\{h_m(t) : m \in \mathbb{M}_t\} \subseteq T.$$

Proof. Suppose T is Baire and non-meagre. We may assume that $T = U \backslash M$ with U non-empty and M meagre. Let $V \subseteq U$ satisfy (wcc).

Since the functions h_n are homeomorphisms, the set

$$M' := M \cup \bigcup_n h_n^{-1}(M)$$

is meagre. Put

$$W = \mathbf{h}(V) := \bigcap_{k \in \omega} \bigcup_{n \geq k} V \cap h_n^{-1}(V) \subseteq V \subseteq U.$$

Then $V \cap W$ is co-meagre in V. Indeed

$$V \backslash W = \bigcup_{k \in \omega} \bigcap_{n \geq k} V \backslash h_n^{-1}(V),$$

which by assumption is meagre.

Let $t \in V \cap W \backslash M'$ so that $t \in T$. Now there exists an infinite set $\mathbb{M}_t$ such that, for $m \in \mathbb{M}_t$, there are points $v_m \in V$ with $t = h_m^{-1}(v_m)$. Since $h_m^{-1}(v_m) = t \notin h_m^{-1}(M)$, we have $v_m \notin M$, and hence $v_m \in T$. Thus $\{h_m(t) : m \in \mathbb{M}_t\} \subseteq T$ for t in a co-meagre set, as asserted. $\square$

Clearly the result relativizes to any open subset of T on which T is non-meagre; that is, the embedding property is a *local* one. The following lemma sheds some light on the significance of the category convergence condition (wcc). The result is capable of improvement, for instance by replacing the countable family generating the coarser topology by a σ-discrete family (which is then metrizable by Bing's Theorem, given regularity assumptions – see [Eng] Th. 4.4.8).

Theorem 2 (Convergence to the identity). *Assume that the homeomorphisms $h_n : X \to X$ satisfy the weak category convergence condition* (wcc) *and that X is a Baire space. Suppose there is a countable family $\mathcal{B}$ of open subsets of X which generates a (coarser) Hausdorff topology on X. Then, for quasi all (under the original topology) t, there is an infinite $\mathbb{N}_t$ such that*

$$\lim_{m \in \mathbb{N}_t} h_m(t) = t.$$

Proof. For U in the countable base $\mathcal{B}$ of the coarser topology and for $k \in \omega$ select open $V_k(U)$ so that $M_k(U) := \bigcap_{n \geq k} V_k(U) \backslash h_n^{-1}(V_k(U))$ is meagre. Thus

$$M := \bigcup_{k \in \omega} \bigcup_{U \in \mathcal{B}} M_k(U)$$

is meagre. Now $\mathcal{B}_t = \{U \in \mathcal{B} : t \in U\}$ is a basis for the neighbourhoods of t. But, for $t \in V_k(U) \backslash M$, we have $t \in h_m^{-1}(V_k(U))$ for some $m = m_k(t) \geq k$, i.e. $h_m(t) \in V_k(U) \subseteq U$. Thus $h_{m_k(t)}(t) \to t$, for all $t \notin M$. $\square$

We now deduce the category and measure cases of the Kestelman-Borwein-Ditor Theorem (stated below) as two corollaries of the above theorem by applying it first to the usual and then to the density topology on the reals, $\mathbb{R}$.

For our first application we take $X = \mathbb{R}$ with the usual *Euclidean* topology, a Baire space. We let $z_n \to 0$ be a null sequence. It is convenient to take

$$h_n(x) = x - z_n, \text{ so that } h_n^{-1}(x) = x + z_n.$$

This is a homeomorphism. We verify (wcc). If U is non-empty and open, let $V = (a, b)$ be any interval contained in U. For later use we identify our result thus (with E here for Euclidean and D below for density).

Theorem 3E (Translation Theorem – E). *Let V be an open interval. For any null sequence $\{z_n\} \to 0$ and each $k \in \omega$,*

$$H_k = \bigcap_{n \geq k} V \backslash (V + z_n) \text{ is empty.}$$

Proof. Assume first that the null sequence is positive. Then, for all n so large that $a + z_n < b$, we have

$$V \cap h_n^{-1}(V) = (a, a + z_n),$$

and so, for any $k \in \omega$,

$$\bigcap_{n \geq k} V \backslash h_n^{-1}(V) \text{ is empty.}$$

The same argument applies if the null sequence is negative, but with the end-points exchanged. □

For our second application we enrich the topology of $\mathbb{R}$, retaining the functions h_n. We consider instead the *density* topology. This is translation-invariant, and so each h_n continues to be a homeomorphism. The intervals remain open. A set is d-nowhere dense iff it is measurable and null.

Lemma. *$\mathbb{R}$ under the density topology is a Baire space.*

Proof. Let U be a non-empty d-open set. Suppose that A_n is a sequence of sets d-nowhere dense in U. Since this means that each set A_n has measure zero, their union has measure zero and so the complement in U is non-empty, since U has positive measure. □

To verify the weak category convergence of the sequence h_n, consider U non-empty and d-open; then consider any measurable non-null $V \subseteq U$, for instance V of the form $U \cap (a, b)$ for some finite interval. To verify (wcc) in relation to V, it now suffices to prove the following result, which is of independent interest (cf. Littlewood's First Principle, as above).

Theorem 3D (Translation Theorem – D). *Let V be measurable and non-null. For any null sequence $\{z_n\} \to 0$ and each $k \in \omega$,*

$$H_k = \bigcap_{n \geq k} V \backslash (V + z_n) \text{ is of measure zero, so meagre in the } d\text{-topology.}$$

Proof. Suppose otherwise.

Then for some k, $|H_k| > 0$. Write H for H_k. Since $H \subseteq V$, we have, for $n \geq k$, that $\emptyset = H \cap h_n^{-1}(V) = H \cap (V + z_n)$ and so a fortiori $\emptyset = H \cap (H + z_n)$.

Let u be a density point of H. Thus for some interval $I_\delta(u) = (u-\delta/2, u+\delta/2)$ we have

$$|H \cap I_\delta(u)| > \frac{3}{4}\delta.$$

Let $E = H \cap I_\delta(u)$. For any z_n, we have $|(E+z_n) \cap (I_\delta(u)+z_n)| = |E| > \frac{3}{4}\delta$. For $0 < z_n < \delta/4$, we have $|(E+z_n) \backslash I_\delta(u)| \leq |(u+\delta/2, u+3\delta/4)| \leq \delta/4$. Put $F = (E+z_n) \cap I_\delta(u)$; then $|F| > \delta/2$.

But $\delta \geq |E \cup F| = |E| + |F| - |E \cap F| \geq \frac{3}{4}\delta + \frac{1}{2}\delta - |E \cap F|$. So

$$|H \cap (H+z_n)| \geq |E \cap F| \geq \frac{1}{4}\delta,$$

contradicting $\emptyset = H \cap (H+z_n)$. This establishes the claim. $\square$

An immediate first corollary of the theorems above is the following result, due in this form in the measure case to Borwein and Ditor [BoDi], but already known much earlier albeit in somewhat weaker form by Kestelman ([Kes] Th. 3), and rediscovered by Trautner [Trau] (see [BGT] p. xix and footnote p. 10).

Theorem (Kestelman-Borwein-Ditor Theorem). *Let $\{z_n\} \to 0$ be a null sequence of reals. If T is Lebesgue, non-null/Baire, non-meagre, then for generically all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

Furthermore, for any density point u of T, there is $t \in T$ arbitrarily close to u for which the above holds.

3 The Uniform Convergence Theorem

A second corollary of our results is the fundamental theorem of regular variation below, the UCT. This has traditionally been proved in the measure and Baire cases separately (see [BGT] Section 1.2 for details and references). This raises the question of finding the minimal common generalization of the measurability and Baire-property assumptions, an old question raised in [BGT] p. 11 and answered in [BOst4]. We content ourselves here with proving the two cases together. The brief proof, inspired by one due to Csiszár and Erdős

[CsEr] (called the 'fourth proof' in [BGT]), appeals only to the Kestelman-Borwein-Ditor Theorem. That was discovered in [BOst1]; we quote it here for completeness – shortened (indeed, "from 3ε to 2ε"), and abstracted from a wider combinatorial context. For another proof, albeit for the measurable case only, see [Trau], where Trautner employs also a theorem of Egorov (cf. Littlewood's Third Principle, see [Lit] Ch. 4, [Roy] Section 3.6 and Problem 31, or [Hal] Section 55 p. 243). Recall (see [BGT]) that a function $h : \mathbb{R} \to \mathbb{R}$ is *slowly varying* (in additive notation) if for every sequence $\{x_n\} \to \infty$ and each $u \in \mathbb{R}$

$$\lim_{n\to\infty} h(u + x_n) - h(x_n) = 0.$$

Theorem (Uniform Convergence Theorem). *If h is slowly varying and measurable, or Baire, then uniformly in u on compacts:*

$$\lim_{n\to\infty} h(u + x_n) - h(x_n) = 0.$$

Proof. Suppose otherwise. Then for some measurable/Baire slowly varying function h and some $\varepsilon > 0$, there is $\{u_n\} \to u$ and $\{x_n\} \to \infty$ such that

$$|h(u_n + x_n) - h(x_n)| \geq 2\varepsilon. \tag{1}$$

Now, for each point y, we have

$$\lim_{n} |h(y + x_n) - h(x_n)| = 0,$$

so there is $k = k(y)$ such that, for $n \geq k$,

$$|h(y + x_n) - h(x_n)| < \varepsilon.$$

For $k \in \omega$, define the measurable/Baire set

$$T_k := \bigcap_{n\geq k} \{y : |h(y + u + x_n) - h(x_n)| < \varepsilon\}.$$

Since $\{T_k : k \in \omega\}$ covers $\mathbb{R}$, it follows that, for some $k \in \omega$, the set T_k is non-null/non-meagre. Writing $z_n := u_n - u$, we have, for some $t \in T_k$ and for some infinite $\mathbb{M}_t$, that

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T_k.$$

Thus

$$|h(t + u_m + x_m) - h(x_m)| < \varepsilon.$$

Since $u_m + x_m \to \infty$, we have, for m large enough and in $\mathbb{M}_t$, that

$$|h(t + u_m + x_m) - h(u_m + x_m)| < \varepsilon.$$

The last two inequalities together imply, for m large enough and in $\mathbb{M}_t$, that

$$\begin{aligned} |h(u_m + x_m) - h(x_m)| &\leq |h(u_m + x_m) - h(t + u_m + x_m)| \\ &\quad + |h(t + u_m + x_m) - h(x_m)| \\ &< 2\varepsilon, \end{aligned}$$

and this contradicts (1). □

4 Remarks

1. *Topology and category.* Regular variation is a continuous-variable theory, and so refers to an uncountable setting. In general topology also, countability conditions may be used, but are not assumed in general. By contrast, roughly speaking, Baire category methods apply in that part of general topology in which some degree of countability is present (hence its affinity with measure theory, in which countability is intrinsic). That this suffices here – e.g., in the proof of the UCT – is a result of the use of proof by contradiction, in which we work with a *sequence* witnessing the supposed failure of the conclusion. See also [BGT], Section 1.9 for more on sequential aspects.

2. Q*ualitative and quantitative measure theory.* When measure-category duality applies, one passes from the Baire to the measure case by changing 'meagre/non-meagre' to 'null/non-null'. This is *qualitative* measure theory (where all that counts is whether measure is zero or positive), rather than *quantitative* measure theory, where the numerical value of measure counts. Quantitative measure theory is used in the first proof of the UCT in [BGT] (p.6-7 – due to Delange in 1955, [Del]). This is the only direct proof; the other proofs are indirect, by contradiction (see Remark 1 above). The place that quantitative measure theory is used here is in the proof of Theorem 3D, establishing the applicability of the CET to the density topology. It seems that some use of quantitative measure theory, somewhere, is needed here.

3. *Other proofs of the UCT.* Several other proofs of the UCT are given in [BGT]. The second, third (due to Matuszewska) and fourth (due to Csiszár

and Erdős) are indirect, use qualitative measure theory, and have immediate category translations. The fifth proof, due to Elliott, uses Egorov's theorem but covers the measure case only. A sixth proof, due to Trautner [Trau], also uses Egorov's theorem so again applies only to the measure case, cf. [Oxt] Chapter 8. (Trautner was unaware both of Kestelman's work and that of Borwein and Ditor.) Our (seventh) proof here is based on the fourth, and on the insights gained in our earlier papers, cited below. A further proof via the Bounded Equivalence Principle is given in [BOst1].

4. *Limitations of measure-category duality.* Measure and category are explored at textbook length in [Oxt]; see Ch. 19 for duality (including the Sierpiński-Erdős Duality Principle under the Continuum Hypothesis), Ch. 17 (in ergodic theory, duality extends to some but not all forms of the Poincaré recurrence theorem) and Ch. 21 (in probability theory, duality extends as far as the zero-one law but not to the strong law of large numbers). Duality also fails to extend to the theory of random series [Kah]. For further limitations of duality, see [DoF], [Bart], [BGJS]. For Wilczyński's theory of a.e.-convergence associated with σ-ideals, see [PWW]. For a set-theoretic explanation of the duality in regular variation in terms of forcing see [BOst1] Section 5, [Mil1] Section 6.

5. *Convergence concepts.* In the second-countable case (wcc) implies a form of $\mathcal{I}$-*a.e. convergence to the identity*, with $\mathcal{I}$ the σ-ideal of meagre sets. See [PWW] for further information here. Note also the possibility of convergence down a sub-subsequence of a given subsequence, as occurs in the notion of *convergence with respect to* $\mathcal{I}$ [PWW] – in this connection see [Mil2] regarding circumstances (and their dependence on the Souslin Hypothesis) when the two modes of convergence relative to $\mathcal{I}$ are equivalent.

The result just cited is the abstract form of the relationship between convergence in probability and almost-sure convergence. The latter implies the former, but not conversely in general, although a sequence converging in probability has a subsequence converging almost surely. The former is metric, the latter not even topological in general (see [Dud] Section 9.2 Problem 2). But the latter reduces to the former in exceptional circumstances (when the measure space is purely atomic, when there are no non-trivial null sets); again, see [Mil2] and [WW].

6. *Regular variation and Tauberian theory.* For a textbook account of the extensive applications of regular variation in Tauberian theory, see [Kor] Ch. IV (and also [BGT], Ch. 4,5).

References

[Bart] T. Bartoszyński, *A note on duality between measure and category*, Proc. Amer. Math. Soc. 128 (2000), 2745–2748 (electronic).

[BGJS] T. Bartoszyński, M. Goldstern, H. Judah and S. Shelah, *All meagre filters may be null*, Proc. Amer. Math. Soc. 117 (1993), 515–521.

[BGT] N. H. Bingham, C.M. Goldie, J.L. Teugels, *Regular variation,* 2nd edition, Encycl. Math. Appl. 27, Cambridge University Press, Cambridge, 1989 (1st edition 1987).

[BOst1] N. H. Bingham and A. J. Ostaszewski, *Infinite combinatorics and the foundations of regular variation*, CDAM Research Report Series, LSE-CDAM-2006-22.

[BOst4] N. H. Bingham and A. J. Ostaszewski, *Beyond Lebesgue and Baire: generic regular variation*, Colloquium Mathematicum, to appear (LSE-CDAM Report, LSE-CDAM-2007-16).

[BOst5] N. H. Bingham and A. J. Ostaszewski, *Generic subadditive functions*, Proc. Amer Math. Soc. 136 (2008), 4257-4266.

[BOst10] N. H. Bingham and A. J. Ostaszewski, *Homotopy and the Kestelman Borwein Ditor Theorem*, Canadian Math. Bull., to appear, LSE-CDAM Report, LSE-CDAM-2007-27.

[BOst-SteinOstr] N. H. Bingham and A. J. Ostaszewski, *Infinite combinatorics and the theorems of Steinhaus and Ostrowski*, LSE-CDAM Report, LSE-CDAM-2007-15 (revised)

[BoDi] D. Borwein and S. Z. Ditor, *Translates of sequences in sets of positive measure*, Canadian Mathematical Bulletin 21 (1978) 497-498.

[CLO] K. Ciesielski, L. Larson, K. Ostaszewski, *$\mathcal{I}$-density continuous functions*, Mem. Amer. Math. Soc. 107 (1994), no. 515.

[CsEr] I. Csiszár and P. Erdős, *On the function* $g(t) = \limsup_{x\to\infty}(f(x+t)-f(x))$, Magyar Tud. Akad. Kut. Int. Közl. A 9 (1964) 603-606.

[Del] H. Delange, *Sur un théorème de Karamata*, Bull. Sci. Math. 79 (1955), 9-12.

[Den] A. Denjoy, *Sur les fonctions dérivées sommable*, Bull. Soc. Math. France 43 (1915), 161-248.

[DoF] R. Dougherty and M. Foreman, *Banach-Tarski decompositions using sets with the Baire property*, J. Amer. Math. Soc. 7 (1994), 75-124.

[Dud] R. M. Dudley, *Real analysis and probability,* Cambridge University Press 2002 (1st. ed. Wadsworth, 1989).

[Eng] R. Engelking, *General Topology,* Heldermann Verlag, Berlin 1989.

[Goff] C. Goffman, *On Lebesgue's density theorem,* Proc. AMS 1 (1950), 384-388.

[GoWa] C. Goffman, D. Waterman, *Approximately continuous transformations*, Proc. Amer. Math. Soc. 12 (1961), 116–121.

[GNN] C. Goffman, C. J. Neugebauer and T. Nishiura, *Density topology and approximate continuity*, Duke Math. J. 28 1961 497–505.

[Hal] P. R. Halmos, *Measure Theory*, Van Nostrand, 1969.

[HauPau] O. Haupt and C. Pauc, *La topologie approximative de Denjoy envisagée comme vraie topologie*, Comptes Rendus. Acad. Sci. Paris 234 (1952), 390-392.

[HePo] R.W. Heath, T. Poerio, *Topological groups and semigroups on the reals with the density topology*, Conference papers, Oxford 2006 (Conference in honour of Peter Collins and Mike Reed).

[Kah] J.-P. Kahane, *Probabilities and Baire's theory in harmonic analysis*, in: Twentieth Century Harmonic Analysis, A Celebration, editor: J. S. Byrnes, 57-72, Kluwer, Netherlands, 2001.

[Kech] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics 156, Springer, 1995.

[Kel] J. C. Kelly, *Bitopological spaces*, Proc. London. Math. Soc. 13 (1963) 71-89.

[Kes] H. Kestelman, *The convergent sequences belonging to a set*, J. London Math. Soc. 22 (1947), 130-136.

[Kor] J. Korevaar, *Tauberian Theory: A Century of Developments,* Grundl. math. Wiss. vol. 329, Springer-Verlag, Berlin, 2004.

[Lit] J.E. Littlewood, *Lectures on the theory of functions,* Oxford University Press, 1944.

[LMZ] J. Lukeš, J. Malý, L. Zajíček, *Fine topology methods in real analysis and potential theory*, Lecture Notes in Mathematics, 1189, Springer, 1986.

[Mar] N.F.G. Martin, *A topology for certain measure spaces*, Trans. Amer. Math. Soc. 112 (1964) 1–18.

[Mil1] A.W. Miller, *Infinite combinatorics and definability*, Ann. of Pure and Applied Mathematical Logic 41(1989), 179-203 (see also updated web version at : http://www.math.wisc.edu/~miller/res/).

[Mil2] A.W. Miller, *Souslin's hypothesis and convergence in measure*, Proc. Amer. Math. Soc. 124 (1996), no. 5, 1529–1532.

[Oxt] J. C. Oxtoby, *Measure and category,* 2nd ed., Grad. Texts Math. 2, Springer, New York, 1980.

[PWW] W. Poreda, E. Wagner-Bojakowska, W. Wilczyński, *A category analogue of the density topology,* Fund. Math. 125 (1985), 167-173.

[Roy] H.L. Royden, *Real Analysis*, Prentice-Hall, 1988 (3rd ed.)

[Trau] R. Trautner, *A covering principle in real analysis*, Quart. J. Math. Oxford (2), 38 (1987), 127- 130.

[WW] E. Wagner and W.Wilczyński, *Convergence of sequences of measurable functions,* Acta. Math. Acad. Sci. Hungar. 36 (1980), 125-128.

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