ROBUST REPLICATION UNDER MODEL UNCERTAINTY

PAVEL V. GAPEEV, TOMMI SOTTINEN, AND ESKO VALKEILA

Abstract. We consider the robust hedging problem in which an investor wants to super-hedge an option in the framework of uncertainty in a model of a stock price process. More specifically, the investor knows that the stock price process is H-self-similar with $H \in (1/2, 1)$, and that the log-returns are Gaussian. This leads to two natural but mutually exclusive hypotheses both being self-contained to fix the probabilistic model for the stock price. Namely, the investor may assume that either the market is efficient, i.e. the stock price process is a semimartingale, or that the centred log-returns are stationary. We show that to be able to super-hedge a convex European vanilla-type option robustly the investor must assume that the markets are efficient. If it turns out that if the other hypothesis of stationarity of the log-returns is true, then the investor can actually super-hedge the option as well as receive a net hedging profit.

Mathematics Subject Classification 2000: Primary 91B28, 60G15, 60G18. Secondary 60G44, 60H05, 91B70.

JEL Classification: Primary G13. Secondary C52.

Key words and phrases: Arbitrage option pricing and hedging, fractional Brownian motion, self-similarity, long-range dependence, fractional Black-Scholes market model, forward pathwise integral, model uncertainty, average risk-neutral measure, net hedging profit, Wick-Itô-Skorohod integral.

1. Introduction

In the classical Black-Scholes model of financial market the logarithm of the stock price is modeled by a drifted Brownian motion. However, in some studies of real financial data it is concluded that the centred log-returns of the stock prices exhibit the so-called long-range dependency property (see, e.g., [12, Chapter IV]). This observation generates an intention to replace the driving Brownian motion with independent increments by another Gaussian process with long memory, or at least having the so-called H-self-similarity property, which is in many cases taken to be evidence for the long-range dependence (when H belongs to the interval $(1/2, 1)$). A natural candidate for the new driving process is the *fractional Brownian motion*, which is a Gaussian process characterized by being self-similar with stationary increments. This, what we call hypothesis **(H1)**, will result in a market model that exhibits arbitrage opportunities (see, e.g., [5, 10, 13]). Another natural candidate for the replacement of the Gaussian driving process is an H-self-similar Gaussian martingale, which would still be in the realm of H-self-similarity but would not generate arbitrage. This, what we

Date: October 1, 2007.
This preprint is CDAM Research Report LSE-CDAM-2007-28.

call hypothesis **(H2)**, does not exhibit long-range dependence but it resembles it statistically, at least through the H-self-similarity property.

We consider a model of financial market, where the investor issuing a European-type contingent claim assumes that the centred log-prices of the underlying risky asset are jointly Gaussian and self-similar with parameter H from the interval $(1/2, 1)$, that corresponds to the case of long-range dependence. We assume that the investor is not sure which one of the hypotheses, **(H1)** or **(H2)**, is actually realized. Then she looks for a so-called *robust hedging strategy* with possible consumption, which allows to super-hedge the given contingent claim independently of which one of the hypothesis is true. We define the *robust hedging price* of a contingent claim as the minimal initial capital required to construct a robust hedging strategy.

It turns out that in order to super-hedge a convex European vanilla-type option robustly the investor should assume that **(H2)** is true. So that, the *robust hedging price* is the hedging price under the hypothesis **(H2)**. If **(H1)** is true then the investor receives a *net hedging profit*.

The paper is organized as follows. In Section 2 we introduce the Gaussian market models and fix some notation. Then we introduce the uncertainty setting, i.e. the two competing hypothesis **(H1)** and **(H2)** on the model followed by a short technical review of forward integration. This is necessary since we need stochastic integration in order to define the self-financing condition, and we cannot use the classical Itô calculus to define stochastic integrals. We also give a short technical note on H-self-similar Gaussian processes, which provides us with a reformulation of the uncertainty setting. In Section 3, which is the core of the paper, we introduce and solve the robust hedging problem for convex vanilla-type European options under the model uncertainty. We end the paper with some remarks and discussion in Section 4. We comment how the robust option-pricing can be viewed through a concept called *average risk-neutral measure*. We also remark the connection between robust hedging and the Wick-Itô-Skorohod approach for option pricing. Finally, we comment on role of the Gaussianity assumption for the uncertainty setting.

2. Gaussian market models with uncertainty

Gaussian market models. In this section we consider the classical pricing model of two assets: the riskless bond, or money-market, $S^0 = (S^0_t)_{t\in[0,T]}$, and the risky stock $S = (S_t)_{t\in[0,T]}$. Here $T > 0$ is the maturity time for the contingent claims. We assume that the stock price is already discounted, i.e. $S^0_t \equiv 1$.

Suppose $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbf{P})$ is a filtered probability space satisfying the usual conditions of completeness and right continuity of the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. The stock-price process is driven by an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted centred Gaussian process $X = (X_t)_{t\in[0,T]}$, which is normalized with $X_0 = 0$ and $\mathbf{Var}[X_1] = 1$, i.e.

$$S_t = S_0 e^{m(t) - \frac{\sigma^2}{2}\mathbf{Var}[X_t] + \sigma X_t}, \tag{2.1}$$

where $\sigma > 0$ is a model parameter, the *volatility* of the stock. On the mean function m we will assume that it is absolutely continuous with respect to the

Lebesgue measure, thus being of the form

$$m(t) = \int_0^t \mu(u)\,\mathrm{d}u$$

for some $\mu \in L^1([0,T])$.

2.2. *Remark.* Note that $\mu(t)\mathrm{d}t$ is the mean of the returns (relative changes) of the stock, but $\mu(t)\mathrm{d}t - (\sigma^2/2)\,\mathrm{d}\mathbf{Var}[X_t]$ is the mean of the log-returns. However, it is well-known that the actual mean function, as long as it is smooth, is irrelevant in option pricing. So that, we should not bother ourselves too much on the differences of the mean of returns and log-returns.

Gaussian self-similar processes. In addition to the normalization $X_0 = 0$ and $\mathbf{Var}[X_1] = 1$ we assume that the Gaussian noise X is H-self-similar for some $H \in (1/2, 1)$, i.e.

$$(X_t)_{t\in[0,T]} \stackrel{d}{=} (a^{-H} X_{at})_{t\in[0,T/a]},$$

where $\stackrel{d}{=}$ means equality of finite-dimensional distributions.

Since we assume that X is centred and Gaussian with $X_0 = 0$ the H-self-similarity can be written in terms of the covariance function as

$$\mathbf{Cov}\,[X_t, X_s] = a^{-2H}\mathbf{Cov}\,[X_{at}, X_{as}]$$

for all $a > 0$. In particular, since $\mathbf{Var}[X_1] = 1$, we must have

$$\mathbf{Var}[X_t] = t^{2H}.$$

But now, taking a process B centred with $B_0 = 0$ and $\mathbf{Var}[B_1] = 1$ fixed, it is easy to see that a Gaussian process B that is H-self-similar with stationary increments must have the covariance function

$$\begin{aligned}
\mathbf{Cov}\,[B_t, B_s] &= \frac{1}{2}\left(\mathbf{Var}[B_t] + \mathbf{Var}[B_s] - \mathbf{Var}[B_t - B_s]\right) \\
&= \frac{1}{2}\left(\mathbf{Var}[B_t] + \mathbf{Var}[B_s] - \mathbf{Var}[B_{|t-s|}]\right) \\
&= \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).
\end{aligned}$$

We see that the process B is uniquely determined. This process is called the *fractional Brownian motion.* It was introduced in [8] and given its name in [9].

Assume then that M is a H-self-similar Gaussian martingale with $M_0 = 0$ and $\mathbf{Var}[M_1] = 1$. Since M is a martingale we see that

$$\begin{aligned}
\mathbf{Cov}[M_t, M_s] &= \mathbf{E}[M_t M_s] \\
&= \mathbf{E}\big[\mathbf{E}[M_t M_s|\mathcal{F}_s]\big] \\
&= \mathbf{E}\big[M_s \mathbf{E}[M_t|\mathcal{F}_s]\big] \\
&= \mathbf{E}[M_s^2] \\
&= \mathbf{Var}[M_s] \\
&= s^{2H}
\end{aligned}$$

for $s \leq t$. So that, in this case too the process M is uniquely defined.

Self-financing strategies and arbitrage. In order to define the notion of arbitrage, let us now fix some notation and recall some basic concepts of a financial market model. A *trading strategy* is a two-dimensional process

$$\pi_t = (\beta_t, \gamma_t), \qquad t \in [0, T],$$

where β_t denotes the number of bonds, and γ_t denotes the number of stocks owned by the investor at time t. The process π is assumed to be adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, which is assumed to be generated by the stock-price process S. The *wealth* process $V(\pi)$ associated to a trading strategy π is

$$V_t(\pi) = \beta_t + \gamma_t S_t, \qquad t \in [0, T].$$

We assume that π is admissible, i.e. $V(\pi)$ is bounded from below by some deterministic constant. Being based on the idea of the budget constraint on the change of the position of the portfolio on the time interval $[t, t + \Delta t]$, we assume that trading strategies are *self-financing*:

$$\beta_{t+\Delta t} + \gamma_{t+\Delta t} S_t = \beta_t + \gamma_t S_t.$$

Reorganizing the terms in the equation above we obtain the condition

$$V_{t+\Delta t}(\pi) = V_t(\pi) + \gamma_t (S_{t+\Delta t} - S_t).$$

From this it follows that the trading strategy is self-financing if its wealth satisfies

$$V_t(\pi) = V_0(\pi) + \int_0^t \gamma_u \, \mathrm{d}S_u, \tag{2.3}$$

where the integral is understood in a forward (pathwise) sense, and we give a short review of forward integration below.

A (self-financing) trading strategy π realizes an *arbitrage opportunity* if for $V_0(\pi) = 0$ we have $V_T(\pi) \geq 0$ $\mathbf{P}$-a.s. and $V_T(\pi) > 0$ with a positive $\mathbf{P}$-probability.

A (perfect) *hedge* of a contingent claim with an $\mathcal{F}_T$-measurable payoff G is a (self-financing) trading strategy π that replicates the claim, i.e.

$$V_T(\pi) \;=\; G \qquad \mathbf{P}\text{-a.s.}$$

A *super-hedge* of a claim with the payoff G is a trading strategy π such that

$$V_T(\pi) \;\geq\; G \qquad \mathbf{P}\text{-a.s.}$$

The *fair price* $P(G)$ of the claim G is then the minimal initial capital needed to super-hedge it:

$$P(G) \;=\; \inf\left\{V_0(\pi)\,;\ \text{there is } \pi \text{ such that } V_T(\pi) \geq G \quad \mathbf{P}\text{-a.s.}\right\}.$$

The uncertainty setting. The aforementioned properties for X are not sufficient to fix the probabilistic model for the risky asset S. Let us now give two natural but mutually exclusive assumptions that will fix the model:

(H1) The centred log-returns are stationary, i.e. the X has stationary increments.
(H2) The market is efficient in the sense that there are no arbitrage opportunities.

Since the hypothesis **(H1)** leads to a non-semimartingale model, the classical Itô integration theory is not at our disposal. However, there is an economically meaningful notion of integral, viz. the *forward integral*, that can be applied for non-semimartingales and, in particular, to the definition (2.3).

Forward integration. Actually, there are several slightly different versions of the forward integral. Here we use a simple approach introduced by [6]. For different kinds of forward integrals we refer to [11] and [18].

Let $\Pi_n = \{0 = t_{n,0} < \cdots < t_{n,K(n)} = T\}$ be a partition of $[0,T]$ such that

$$\mathrm{mesh}(\Pi_n) := \max_{t_{n,k}\in\Pi_n} |t_{n,k} - t_{n,k-1}| \to 0$$

as $n \to \infty$. Further, we cannot assume that our processes are properly integrable over the entire interval $[0,T]$. Thus, we define the integrals over the sub-intervals $[0,t]$, $t < T$. The integral over the interval $[0,T]$ will then be interpreted as an improper forward integral.

2.4. **Definition.** Let $t < T$ and let $Z = (Z_u)_{u\in[0,T]}$ be a continuous process. The *forward integral* of a process $Y = (Y_u)_{u\in[0,T]}$ with respect to Z *along the* *sequence* $(\Pi_n)_{n=1}^{\infty}$ is

$$\int_0^t Y_u \,\mathrm{d}Z_u := \lim_{n\to\infty} \sum_{\substack{t_{n,k}\in\Pi_n \\ t_{n,k}\le t}} Y_{t_{n,k-1}} \left(Z_{t_{n,k}} - Z_{t_{n,k-1}}\right),$$

where the limit is assumed to exist $\mathbf{P}$-a.s. The forward integral over the whole interval $[0,T]$ is the improper forward integral

$$\int_0^T Y_u \,\mathrm{d}Z_u := \lim_{t\uparrow T} \int_0^t Y_u \,\mathrm{d}Z_u,$$

where the limit is again understood in the $\mathbf{P}$-a.s. sense.

A priori there is nothing that ensures the existence of the forward integral. However, we can show that if the integrator is a quadratic variation process and the integrand is a smooth function of the integrator then the forward integral exists.

2.5. **Definition.** A process $Z = (Z_t)_{t\in[0,T]}$ is a *quadratic variation process along the sequence* $(\Pi_n)_{n=1}^{\infty}$ if the limit

$$\langle Z\rangle_t := \sum_{\substack{t_{n,k}\in\pi_n \\ t_{n,k}\le t}} \left(Z_{t_{n,k}} - Z_{t_{n,k-1}}\right)^2$$

exists $\mathbf{P}$-a.s. for all $t \le T$, and it is continuous in t.

2.6. *Example.*

(i) For a standard Brownian motion W we have $\mathrm{d}\langle W\rangle_t = \mathrm{d}t$ if the sequence (Π_n) is refining. This follows from the Borel-Cantelli lemma.

(ii) For a fractional Brownian motion B with $H \in (1/2, 1)$ we have $\mathrm{d}\langle B\rangle_t = 0$. This follows from the Hölder continuity of the fractional Brownian motion.

(iii) If A is a continuous process with zero quadratic variation along (Π_n) and X is a continuous quadratic variation process along (Π_n) then $\mathrm{d}\langle Z + A\rangle_t = \mathrm{d}\langle Z\rangle_t$. This follows from the Cauchy-Schwartz inequality.

(iv) If X is a quadratic variation process along (Π_n) and $f \in \mathcal{C}^1(\mathbb{R})$ then $f\circ Z$ is also a quadratic variation process along (Π_n). Indeed,

$$\mathrm{d}\langle f\circ Z\rangle_t = f'(Z_t)\,\mathrm{d}\langle Z\rangle_t$$

(cf. [6, p. 148]).

In what follows the sequence (Π_n) will be a fixed refining partition of $[0,T]$ that is omitted in the notation.

The following Itô formula for the forward integral is a simple generalization of the theorem that can be found in [6, p. 144]. The proof is based on a second order two-dimensional Taylor expansion. Actually, it is basically the same as in the semimartingale case.

2.7. **Lemma** (Itô formula). *Let Z be a continuous quadratic variation process. Suppose $f \in \mathcal{C}^{1,2}([0,T) \times \mathbb{R})$. Then*

$$
\begin{aligned}
f(t, Z_t) &= f(s, Z_s) + \int_s^t \frac{\partial}{\partial t} f(u, Z_u)\,\mathrm{d}u + \frac{1}{2}\int_s^t \frac{\partial^2}{\partial z^2} f(u, Z_u)\,\mathrm{d}\langle Z\rangle_u \\
&\quad + \int_s^t \frac{\partial}{\partial z} f(u, Z_u)\,\mathrm{d}Z_u \qquad (2.8)
\end{aligned}
$$

for $0 \le s \le t < T$. In particular, the forward integral (2.8) exists and has a modification, which is continuous in t.

2.9. *Remark.*
(i) If the process Z has zero quadratic variation then the Itô formula is the classical change-of-variables formula.
(ii) In the remainder of the paper we choose continuous modifications of forward integrals.
(iii) The forward integral with non-semimartingale integrator does not satisfy a dominated convergence theorem.

The uncertainty setting revisited. Recall the *Fundamental Theorem of Asset Pricing*: The market is free of arbitrage if and only if there is a probability measure equivalent to the original measure such that under it the stock-price process is a local martingale. From this it follows that **(H2)** is equivalent to the assumption that X is a martingale. We now also know that an H-self-similar stationary-increment Gaussian process must be a fractional Brownian motion. So that, we may equivalently rewrite our hypotheses **(H1)** and **(H2)** as:

(H1') The driving process X is the fractional Brownian motion B.
(H2') The driving process X is the Gaussian martingale M.

2.10. *Remark.*
(i) By using the Kolmogorov continuity criterion we see that both M and B are continuous processes.
(ii) The Gaussian martingale M can be realized by using a standard Brownian motion W. Indeed, it is easy to see that the process

$$W_t = \frac{1}{\sqrt{2H}} \int_0^t \frac{1}{u^{H-1/2}}\,\mathrm{d}M_u$$

is a standard Brownian motion. So that, we have

$$M_t = \sqrt{2H} \int_0^t u^{H-1/2}\,\mathrm{d}W_u.$$

In particular, this yields that the hypothesis **(H2')**, or equivalently the hypothesis **(H2)**, means that we are dealing with a non-homogeneous Black-Scholes model

$$\frac{\mathrm{d}S_t}{S_t} = \mu(t)\,\mathrm{d}t + \sigma\sqrt{2H}t^{H-1/2}\,\mathrm{d}W_t.$$

It can be read from the equation above that M is a quadratic variation process with

$$\mathrm{d}\langle M\rangle_t = 2Ht^{2H-1}\,\mathrm{d}t,$$

and S is a quadratic variation process with

$$\mathrm{d}\langle S\rangle_t \;=\; 2Ht^{2H-1}S_t^2\,\mathrm{d}t.$$

We also note that the market model under **(H2)** is complete in the sense that all claims can be hedged with admissible self-financing strategies.

(iii) Let us also observe that the hypothesis **(H1)** or, equivalently, the hypothesis **(H1')** corresponds to the model

$$\frac{\mathrm{d}S_t}{S_t} = \left(\mu(t) - \sigma^2 Ht^{2H-1}\right)\mathrm{d}t + \sigma\,\mathrm{d}B_t.$$

The reason for the local drift to be $\mu(t) - \sigma^2 Ht^{2H-1}$ is that the fractional Brownian motion with $H \in (1/2, 1)$ has zero quadratic variation. Hence, the Itô formula with respect to it takes the form of a classical change-of-variables formula. Moreover, in this case S also has zero quadratic variation, and thus, the market model under **(H1)** admits arbitrage opportunities.

3. Robust replication

The problem. In this section we consider the robust hedging problem for an investor, who does not know whether **(H1)** or **(H2)** is true, but who must super-hedge a European contingent claim.

3.1. **Definition.** (i) A (self-financing) strategy π is a *robust hedge* for the claim G under the uncertainty **(H1)** versus **(H2)** if it super-hedges a contingent claim with the payoff G under the both hypotheses **(H1)** and **(H2)**.

(ii) A robust hedge π is *minimal* if it is a perfect hedge under **(H1)** or **(H2)**.

(iii) The *robust hedging price* $P(G)$ of the claim with the payoff G is

$$P(G) \;=\; \inf\left\{\beta_0 + \gamma_0 S_0\,;\, \pi = (\beta,\gamma) \text{ is a robust hedge for } G\right\}.$$

We now find the solution to the robust hedging problem in the case where the claim G is a convex European vanilla-type option, i.e. if it is of the form $G = F(S_T)$ for some convex function F.

The solution. Let us first consider the case when hypothesis **(H2)** is true. It is well-known how one can hedge claims in such non-homogeneous Black-Scholes model (see, e.g., [12, Chapter VII]). Indeed, let $v(t, S_t)$ be the price of the option $F(S_T)$ at time t. Since

$$\mathrm{d}\langle S\rangle_t \;=\; \sigma^2 2Ht^{2H-1}S_t^2\,\mathrm{d}t,$$

by using the Itô formula we get

$$\begin{aligned} v(t,S_t) &= v(0,S_0) + \int_0^t \frac{\partial}{\partial s} v(u,S_u)\,\mathrm{d}S_u \\ &\quad + \int_0^t \left(\frac{\partial}{\partial t} v(u,S_u) + \sigma^2 H u^{2H-1} S_u^2 \frac{\partial^2}{\partial s^2} v(u,S_u) \right) \mathrm{d}u. \end{aligned}$$

Thus, we see that $v(t,s)$ satisfies the *fractional-type backward Black-Scholes partial differential equation*

$$\frac{\partial}{\partial t} v(t,s) + \sigma^2 H t^{2H-1} s^2 \frac{\partial^2}{\partial s^2} v(t,s) = 0, \tag{3.2}$$

$$v(T,s) = F(s). \tag{3.3}$$

Hence, by applying the *Feynman-Kac formula* we obtain

$$v(t,S_t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F\left(S_t e^{\sigma\sqrt{T^{2H}-t^{2H}}y - \frac{\sigma^2}{2}\left(T^{2H}-t^{2H}\right)} \right) e^{-\frac{y^2}{2}}\,\mathrm{d}y, \tag{3.4}$$

and, in particular, that

$$v(0,S_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F\left(S_0 e^{\sigma T^H y - \frac{\sigma^2}{2} T^{2H}} \right) e^{-\frac{y^2}{2}}\,\mathrm{d}y. \tag{3.5}$$

We also note that the strategy

$$\pi_t = \left(v(t,S_t) - \frac{\partial}{\partial s} v(t,S_t) S_t, \frac{\partial}{\partial s} v(t,S_t) \right) \tag{3.6}$$

is a perfect hedge for the option $F(S_T)$ under **(H2)**. So that, the robust price $P(F(S_T))$ must be at least $v(0,S_0)$.

Let us then consider the case when the hypothesis **(H1)** is true. We shall show that the strategy (3.6) with initial wealth (3.5) is actually a super-hedge. Then (3.6) is the minimal robust hedge and that (3.5) is the robust hedging price.

We shall now use the fact that the option payoff $F(S_T)$ is convex.

3.7. **Lemma.** *Let v be the function solving the fractional-type backward Black-Scholes partial differential equation (3.2)–(3.3) with a convex boundary function F. Then*

$$\frac{\partial}{\partial t} v(t,s) \le 0$$

for almost all $t \le T$ and $s > 0$.

Proof. Observe that (3.2) directly implies

$$\frac{\partial}{\partial t} v(t,s) = -\sigma^2 H t^{2H-1} s^2 \frac{\partial^2}{\partial s^2} v(t,s),$$

and thus, it is enough to show that

$$\frac{\partial^2}{\partial s^2} v(t,s) \;\geq\; 0$$

for almost all $t \leq T$ and $s > 0$. Now, by the Feynman-Kac formula (3.4) we have that

$$\begin{aligned}
&\frac{\partial^2}{\partial s^2} v(t,s) \\
&= \frac{\partial^2}{\partial s^2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F\left(s e^{\sigma\sqrt{T^{2H}-t^{2H}}y-\frac{\sigma^2}{2}(T^{2H}-t^{2H})}\right) e^{-\frac{y^2}{2}}\,\mathrm{d}y\right) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial^2}{\partial s^2} F\left(s e^{\sigma\sqrt{T^{2H}-t^{2H}}y-\frac{\sigma^2}{2}(T^{2H}-t^{2H})}\right) e^{-\frac{y^2}{2}}\,\mathrm{d}y.
\end{aligned}$$

Thus, the desired assertion follows from this since the second (distributional) derivative of a convex function is positive. □

Recall that under the hypothesis **(H1)** we have $\mathrm{d}\langle S\rangle_t = 0$. Consequently, by applying the Itô formula to the value function v defined by (3.4), we have

$$v(t,S_t) \;=\; v(0,S_0) + \int_0^t \frac{\partial}{\partial t} v(u,S_u)\,\mathrm{d}u + \int_0^t \frac{\partial}{\partial s} v(u,S_u)\,\mathrm{d}S_u. \tag{3.8}$$

Lemma 3.7 now guarantees that the *consumption* process

$$C_t \;:=\; -\int_0^t \frac{\partial}{\partial t} v(u,S_u)\,\mathrm{d}u \tag{3.9}$$

is positive and increasing. So that, once we notice that the formula (3.8) represents hedging with consumption, or super-hedging, we have proved the following result:

3.10. **Theorem.** *The robust price of the contingent claim with the convex payoff $F(S_T)$ is given by (3.5), and the minimal robust hedge is given by (3.6).*

If **(H2)** *is true then the investor hedges the claim perfectly. If* **(H1)** *is true then the investor super-hedges the claim, and on the time interval $[0,t]$ she could consume her net hedging profit C_t given by (3.9).*

3.11. *Remark.* Of course, consuming C_t, $t \in [0,T]$, progressively is out of the question for the investor, since she is not sure if **(H1)** is true. So that, if **(H1)** turns out to be true the investor will enjoy (consume) her net hedging profit C_T at time T when the option is exercised.

4. Concluding remarks and discussion

Average risk-neutral measure and Wick-Itô-Skorohod approach. The pricing of options and arbitrage in the fractional Black-Scholes model, i.e. under the hypothesis **(H1)**, has been studied in, e.g., [1, 2, 3, 4, 7, 14, 15, 16]. Since the fractional Black-Scholes model admits arbitrage opportunities, there is no risk-neutral or martingale measure to be used in pricing. A most likely analogue to the martingale measure is the so-called *average risk-neutral measure*. Since it is impossible to ask for an equivalent measure under which S is a martingale, one asks merely for an equivalent measure $\mathbf{Q}$ such that S_t is log-normal with

$$\mathbf{E}_{\mathbf{Q}}[S_t] \;\; = \;\; S_0.$$

Such a measure, which was introduced in [17], exists and it is unique. Another approach, which was taken in [7], is to use the so-called *Wick-Itô-Skorohod integrals* to define the wealth of a self-financing strategy. In [16] the connection of this and the more economically sounded forward integration approach was investigated.

Being economically not well-justified, both the Wick-Itô-Skorohod approach and the approach based on the average risk-neutral-measure surprisingly give the same pricing (and hedging) formulas as does the hypothesis **(H2)**. So that, they actually correspond to the case when the stock price process is driven by a Gaussian self-similar martingale. Let us also note that the consumption process (3.9) is the difference of the wealth of the forward (pathwise) and the Wick-Itô-Skorohod values of the replicating self-financing strategy studied in [16].

On the Assumption of Gaussianity. It is interesting to note that in deriving Theorem 3.10 we did not actually use the fact that the stock price process is Gaussian. The Gaussian pricing function in (3.4) was due to the Feynman-Kac formula, but the pricing argument was derived straight from the Itô formula. So that, Theorem 3.10 remains true under the more general uncertainty setting:

The driving process X in (2.1) is continuous with $X_0 = 0$, $\mathbf{E}[X_t] = 0$, $\mathbf{Var}[X_1] = 1$, and there is uncertainty between the following hypotheses:

(H1") $\mathrm{d}\langle X\rangle_t = 0$.
(H2") $\mathrm{d}\langle X\rangle_t = 2Ht^{2H-1}\,\mathrm{d}t$.

Note that under hypothesis **(H1")** the market model admits arbitrage opportunities. Under hypothesis **(H2")** the market model may still have arbitrage strategies, but at least all European vanilla-type claims can be hedged (see [1]).

Acknowledgments. The paper was partially written when the first author was visiting Helsinki University of Technology and University of Helsinki in October 2006. The warm hospitality and financial support from European Science Foundation (ESF) through the grant number 1114 of the programme Advanced Mathematical Methods for Finance (AMaMeF) are gratefully acknowledged.

References

[1] Bender, C., Sottinen, T. and Valkeila, E. (2005). Pricing and hedging and no-arbitrage beyond semimartingales. *WIAS Preprint* **1110**.

[2] Bender, C., Sottinen, T. and Valkeila, E. (2007). Arbitrage with fractional Brownian motion? *Theory of Stochastic Processes* **13**(29) (23–34).

[3] Cheridito, P. (2003). Arbitrage in fractional Brownian motion models. *Finance and Stochastics* **7**(4) (533–553).

[4] Cheridito, P. (2001). *Regularizing fractional Brownian motion with a view towards stock price modeling.* Ph.D. dissertation, ETH Zürich.

[5] Dasgupta, A. and Kallianpur, G. (2000). Arbitrage opportunities for a class of Gladyshev processes. *Appl. Math. Opt.* **41**, 377–385.
[6] Föllmer, H. (1981). Calcul d'Itô sans probabilités. *Lecture Notes in Mathematics* **850**, Springer, 143–150.
[7] Hu, Y. and Øksendal, B. (1999). Fractional white noise calculus and applications to finance. *Inf. Dimens. Anal. Quantum Probab. Relat. Top.* **6**(1), 1–32.
[8] Kolmogorov, A. N. (1940). Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. *Comptes Rendus (Doklady) Acad. Sci. USSR* **26**, 115–118.
[9] Mandelbrot, B. and van Ness, J. (1968). Fractional Brownian motions, fractional noises and applications. *SIAM Review* **10**, 422–437.
[10] Rogers, L. C. G. (1997). Arbitrage with fractional Brownian motion. *Mathematical Finance* **7**(1), 95–105.
[11] Russo, F. and Vallois, P. (1993). Forward, backward and symmetric stochastic integration. *Probab. Theory Related Fields* **97**, 403–421.
[12] Shiryaev, A. N. (1999). *Essentials of Stochastic Finance.* World Scientific, Singapore.
[13] Sottinen, T. (2001). Fractional Brownian motion, random walks, and binary market models. *Finance and Stochastics.* **5**, 343–355.
[14] Sottinen, T. (2003). *Fractional Brownian motion in finance and queueing.* Ph.D. dissertation, University of Helsinki.
[15] Sottinen, T. and Valkeila, E. (2001). Fractional Brownian motion as a model in finance. University of Helsinki, Department of Mathematics. Preprint 302, 16 p.
[16] Sottinen, T. and Valkeila, E. (2003). On arbitrage and replication in the fractional Black-Scholes pricing model. *Statistics and Decisions* **21**, 93–107.
[17] Valkeila, E. (1999). On some properties of geometric fractional Brownian motions. University of Helsinki, Department of Mathematics. Preprint 224, 12 p.
[18] Zähle, M. (1997). Integration with respect to fractal functions and stochastic calculus. *Probability Theory and Related Fields* **111**, 333–374.

Pavel V. Gapeev, London School of Economics, Department of Mathematics, Houghton Street, London WC2A 2AE, United Kingdom

E-mail address: `p.v.gapeev@lse.ac.uk`

Tommi Sottinen, Reykjavik University, School of Science and Engineering and School of Business, Kringlan 1, 103 Reykjavik, Iceland

E-mail address: `tommi@ru.is`

Esko Valkeila, Helsinki University of Technology, Institute of Mathematics, P.O. Box 1100, FI-02015 Helsinki University of Technology, Finland

E-mail address: `esko.valkeila@tkk.fi`