Infinite Combinatorics in Function Spaces

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Abstract

The infinite combinatorics here give statements in which, from some sequence, an infinite subsequence will satisfy some condition – for example, belong to some specified set. Our results give such statements generically – that is, for 'nearly all' points, or as we shall say, for quasi all points – all off a null set in the measure case, or all off a meagre set in the category case. The prototypical result here goes back to Kestelman in 1947 and to Borwein and Ditor in the measure case, and can be extended to the category case also. Our main result is what we call the Category Embedding Theorem (CET), which contains the Kestelman-Borwein-Ditor Theorem (KBD) as a special case. Our main contribution is to obtain functionwise rather than pointwise versions of such results. We thus subsume results in a number of recent and related areas, concerning e.g. additive, subadditive, convex and regularly varying functions.

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1 Introduction

We shall be concerned here with both measure and category (cf. [Oxt]), and need concepts of smallness for each. On the measure side, we deal with the class $\mathcal{L}$ of (Lebesgue) measurable sets, and interpret small sets as (Lebesgue) null sets; on the category side we deal with the class $\mathcal{B}a$ of sets with the Baire property (briefly, Baire sets), and interpret small sets as meagre sets (those of the first category). We use *quasi everywhere (q.e.),* or for *quasi all points,* to mean *for all points off a meagre set.* For Γ $\mathcal{L}$ or $\mathcal{B}a$, we say that $P \in \Gamma$ holds *for generically all* t if $\{t : t \notin P\}$ is null/meagre according as Γ is $\mathcal{L}$ or $\mathcal{B}a$.

Our starting-point is the following result, due to Kestelman [Kes] and to Borwein and Ditor [BoDi]. This exemplifies the infinite combinatorics of the title, but concerns scalars, rather than functions.

Theorem (Kestelman-Borwein-Ditor Theorem). *Let $\{z_n\} \to 0$ be a null sequence of reals. If T is measurable and non-null/Baire and non-meagre, then for generically all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

This result (briefly, the KBD theorem) is a corollary of a topological result, the Category Embedding Theorem (CET), given in one form in Section 2 below and in another form in [BOst-bit]. The starting point there is that $h_n(t) := t + z_n$ is a sequence of self-homeomorphisms of the line which converge uniformly to the identity.

Results of this type are crucial in several recent studies by the present authors. First, one may study *additive functions* – solutions of the Cauchy functional equation. For these, one has a dichotomy – such functions are either very good or very bad. Regularity conditions discriminating between these two may be given in either measure or category forms; a unified treatment is given in [BOst-SteinOstr], including as special cases classical results of Steinhaus and Ostrowski. Next, results of Steinhaus-Ostrowski type are the key to the fundamental theorem of regular variation, the uniform convergence theorem (UCT: see e.g. [BGT], Section 1.2). A similarly unified treatment of the measure and category cases here is contained in a companion paper, [BOst-bit]. Additivity may be weakened to *subadditivity;* the subadditive case is treated along similar lines in [BOst5]. It may also be weakened to (mid-point) *convexity,* for which see [BOst6]. Furthermore, such results remain valid under smooth deformation; homotopy versions are given in [BOst10].

Our object here is to give a unified treatment of such infinite combinatorics on function spaces in general, thus providing a common perspective on all these results. In Section 2 below we give the CET, in what we call its *conjuction* form (the motivation being the need to handle bilateral shifts $t - z_m, t + z_m$). In Section 3 we work in normed groups, as in [BOst12], extending the bitopological approach of [BOst-bit] to this more general setting. What motivates such a broader context is the re-interpretation of a sequence of self-homeomorphisms $h_n(t)$ uniformly converging to the identity as giving rise to null function sequences $z_n(t) := h_n(t) - t$ (converging in supremum norm to zero) which need not be constant as in the KBD Theorem. In Section 4 we give generic forms of some results appearing in Kuczma [Kucz], Ch. IX, which we term reflection theorems. We close in Section 5 with a treatment in this vein of a genericity result, due to Császár [Csa], which makes explicit work in [Kucz], IX.7.

As in [BOst-bit] we will need the *density topology* (introduced in [HauPau], [GoWa], [Mar] and studied also in [GNN] – see also [CLO], and for textbook treatments [Kech], [LMZ]). Recall that for T measurable, t is a (metric) density point of T if $\lim_{\delta \to 0} |T \cap I_\delta(t)|/\delta = 1$, where $I_\delta(t) = (t - \delta/2, t + \delta/2)$. By the Lebesgue Density Theorem almost all points of T are density points ([Hal] Section 61, [Oxt] Th. 3.20, or [Goff]). A set U is d-open (open in the density topology) if each of its points is a density point of U. We mention three properties:

(i) The density topology (d-topology) is finer than (contains) the Euclidean topology ([Kech], 17.47(ii)).

(ii) A set is Baire in the density topology iff it is (Lebesgue) measurable ([Kech], 17.47(iv)).

(iii) A function is d-continuous iff it is approximately continuous in Denjoy's sense ([Den]; [LMZ], p.1, 149).

The reader unfamiliar with the density topology may find it helpful to think, in the style of Littlewood's First Principle, of basic opens sets as being intervals less some measurable set. See [Lit] Ch. 4, [Roy] Section 3.6 p.72.

2 Conjunction Category Embedding Theorem

We recall a definition from [BOst-bit] and then formulate two variants. The first two definitions refer to homeomorphisms which form a sequence of 'approximations' to the identity in the sense of (approx) below, while the third introduces a relaxation.

Definition (**weak category convergence**). A sequence of auto-homeomorphisms h_n of a space X satisfies the *weak category convergence* condition (wcc) if:

For any non-empty open set U, there is an non-empty open set $V \subseteq U$ such that, for each $k \in \omega$,

$$\bigcap_{n \geq k} V \backslash h_n^{-1}(V) \text{ is meagre.} \tag{wcc}$$

Equivalently, for each $k \in \omega$, there is a meagre set M such that, for $t \notin M$,

$$t \in V \Longrightarrow (\exists n \geq k)\ h_n(t) \in V. \tag{approx}$$

We say that the homeomorphisms h_n satisfy the weak category convergence *conjunctively* (wccc) if:

$$\bigcap_{n \geq k} V \backslash [h_{2n}^{-1}(V) \cup h_{2n+1}^{-1}(V)] \text{ is meagre.} \tag{wccc}$$

Finally, we formulate a local version of (wcc) which allows some rescaling of h_n. Say that the sequence of homeomorphisms h_n satisfies the *re-scaled weak category convergence* condition at u (rwcc) if for every open set U with $u \in U$ there is an open set V with $u \in V \subset U$ and $\eta = \eta_u > 0$ with

$$\bigcap_{n \geq k} \eta V \backslash h_n^{-1}(V) \text{ is meagre.} \tag{rwcc}$$

Remarks. 1. In the case of the line with Euclidean topology the functions $h_n(t) = t \pm z_n$, with sign selected according to parity, are autohomeomorphisms. The condition (wccc) is used to deduce the bilateral embedding result

$$\{t - z_m, t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

Multiple conjunction forms, k-fold ones, may also be considered by working modulo k rather than 2 in (wccc).

2. Taking $h_{2n+1} = h_{2n}$ reduces (wccc) to (wcc).
3. For (rwcc) the approximation condition (approx) becomes

$$s = \eta^{-1}t \in V \Longrightarrow (\exists n \geq k)\ h_n(\eta s) \in V.$$

4. Consider the affine homeomorphisms

$$A_n(t) = \alpha_n t + z_n$$

with $\alpha_n \geq 2\eta > 0$ and $z_n \to 0$. For any symmetric interval I_δ about the origin of radius δ, we have

$$\alpha_n I_\delta + z_n \supseteq 2\eta I_\delta + z_n = I_{2\eta\delta} + z_n.$$

For n large enough we have $z_n \in I_{\eta\delta}$, so

$$\alpha_n I_\delta + z_n \supseteq I_{\eta\delta},$$

i.e.

$$A_n(I_\delta) \supseteq I_{\eta\delta}, \text{ so that } \eta I_\delta \backslash A_n(I_\delta) \text{ is meagre.}$$

Thus A_n^{-1} satisfies the (rwcc) at the origin.

Note that if M is meagre then $T := I_\delta \backslash M$ is Baire non-null, and we have

$$A_n(T) = A_n(I_\delta \backslash M) \supseteq \eta I_\delta \backslash A_n(M),$$

so

$$\eta T \backslash A_n(T) \text{ is meagre.}$$

Theorem 1 (Category Embedding Theorem - Conjunction form). *Let X be a Baire space. Suppose given homeomorphisms $h_n : X \to X$ which satisfy the weak category convergence condition conjunctively* (wccc)*. Then, for any non-meagre Baire set T, for quasi all $t \in T$, there is an infinite set $\mathbb{M}_t$ such that*

$$\{h_m(t), h_{m+1}(t) : m \in \mathbb{M}_t\} \subseteq T.$$

Proof. Suppose T is Baire and non-meagre. We may assume that $T = U \backslash M$ with U non-empty and M meagre. Let $V \subseteq U$ satisfy (wccc).

Since the functions h_n are homeomorphisms, the set

$$M' := M \cup \bigcup_n h_n^{-1}(M)$$

is meagre. Put

$$W=\mathbf{h}(V):=\bigcap_{k\in\omega}\bigcup_{n\geq k}V\cap h_{2n}^{-1}(V)\cap h_{2n+1}^{-1}(V)\subseteq V\subseteq U.$$

Then $V\cap W$ is co-meagre in V. Indeed

$$V\backslash W=\bigcup_{k\in\omega}\bigcap_{k\geq n}V\backslash[h_{2n}^{-1}(V)\cup V\backslash h_{2n}^{-1}(V)],$$

which by assumption is meagre.

Let $t\in V\cap W\backslash M'$ so that $t\in T$. Now there exists an infinite set $\mathbb{M}_t$ such that, for $m\in\mathbb{M}_t$, there are points $v_{2m},v_{2m+1}\in V$ with $t=h_{2m}^{-1}(v_m)=h_{2m+1}^{-1}(v_{m+1})$. Since $h_{2m}^{-1}(v_{2m})=t\notin h_{2m}^{-1}(M)$, we have $v_{2m}\notin M$, and hence $v_{2m}\in T$; likewise $v_{2m+1}\in T$. Thus $\{h_{2m}(t),h_{2m+1}(t):m\in\mathbb{M}_t\}\subseteq T$ for t in a co-meagre set, as asserted. $\square$

The result above strengthens the Category Embedding Theorem of [BOst-bit] with almost the same proof. We close with a further strengthening obtained by reworking the proof so as replace (wccc) with (rwcc).

Corollary 1 (Locally rescaled CET). *Let $\mathbb{R}$ be given a Baire topology and let T be Baire non-meagre. Suppose that h_n are homeomorphisms satisfying* (rwcc) *at* 0. *Then, for quasi all $u\in T$ and quasi all $t\in T$ near u (i.e in some open set U with $u\in U$), there is an infinite $\mathbb{M}_{t,u}$ such that*

$$u+h_m(t-u)\in T,\text{ for all }m\in\mathbb{M}_{t,u}.$$

Proof. Let $T=U\backslash M\cup N$ with U open and M,N meagre. As our conclusions concern quasi all members of T, we may take $N=\varnothing$, which means that 'for quasi all $u\in T$' is synonymous with 'for all $u\in U\backslash M$'. Fix $u\in T$. Then $0\in U-u$; select V, with $u\in V\subset U$, and $\eta=\eta_u$ such that

$$V\subseteq U-u\text{ and }\bigcap_{n\geq k}\eta V\backslash h_n^{-1}(V)\text{ is meagre.}$$

Further, select $W\subset V$ with

$$\eta W\subseteq V\subseteq U-u.$$

Put

$$S = \eta W \cap \bigcap_{k\in\omega} \bigcup_{n\geq k} h_n^{-1}(T_u);$$

then

$$M' = \eta W \backslash S = \bigcup_{k\in\omega} \bigcap_{n\geq k} \eta W \backslash h_n^{-1}(T_u) \subset \bigcup_{k\in\omega} \bigcap_{n\geq k} \eta V \backslash h_n^{-1}(T_u)$$

is meagre. But $\eta W\backslash(M-u) \subseteq (U-u)\backslash(M\backslash u)$ so for $t \in (u+\eta W)\cap T$ with $t \notin (M'+u)\cup M$ we have $x := t-u \in (T_u \cap S)$ and so there is an infinite set $\mathbb{M}_{t,u}$ such that

$$t - u = x \in h_m^{-1}(T_u), \text{ for } m \in \mathbb{M}_{t,u}. \tag{equiv}$$

Thus

$$u + h_m(t-u) \in T, \text{ for } m \in \mathbb{M}_{t,u}. \qquad \square$$

3 Shift-embeddings

We now specialize Theorem 1 to a metric group setting in order to consider sequences of autohomeomorphisms generated as shifts $h_n(x) = xz_n$. Let T be a normed group with norm $||t|| := d(t, e_T)$, where d is right-invariant (see [BOst12] for background and references). Thus $d(x,y) = d(e, yx^{-1}) = ||yx^{-1}||$. The conjugate metric is $\tilde{d}(x,y) = ||xy^{-1}|| = d(e, xy^{-1}) = d(x^{-1}, y^{-1})$. Let $\mathcal{A} = Auth(T)$ denote the set of bounded autohomeomorphisms h from T to T (i.e. having $\sup_T d(h(t),t) < \infty$) with composition $\circ$ as group operation. We write ε for $e_{\mathcal{A}}$, so that $\varepsilon(t) := t$. Recall that $\mathcal{A}$ has the right-invariant metric

$$d_{\mathcal{A}}(h,h') = \sup_T d(h(t), h'(t)),$$

which generates the norm

$$||h||_{\mathcal{A}} := d_{\mathcal{A}}(h, \varepsilon) = \sup_T d(h(t), t).$$

Let $\mathcal{C} = \mathcal{C}_b(T)$ denote the set of continuous functions from T to T with norm-bounded range and with group operation pointwise multiplication:

$$x \cdot y(t) = x(t)y(t).$$

Here the identity element is the constant function $t \to e_T$, to be written e. Thus $e(t) := e_T$. We give $\mathcal{C}$ the supremum norm.

Definition. Say that $z_n \in \mathcal{C}$ is a *null sequence in* $\mathcal{C}$ or simply that z_n is *uniformly null*, if $z_n \to e$, in sup norm, i.e.

$$||z_n|| := \sup d_T(z_n(t), e_T) \to 0.$$

Thus z_n is a null sequence in $\mathcal{C}$ iff z_n^{-1} is a null sequence in $\mathcal{C}$ (where $z_n^{-1}(t) := z_n(t)^{-1}$). Put

$$\theta_n(t) = z_n(t)t;$$

then

$$||\theta_n||_{\mathcal{A}} := \sup d_T(\theta_n(t), t) = \sup d_T(z_n(t)t, t) = \sup d_T(z_n(t), e_T) = ||z_n||_{\mathcal{C}}.$$

One thus has the following result.

Lemma. *For z_n in $\mathcal{C}$, the sequence θ_n converges to the identity in $\mathcal{A}$ iff z_n is a uniformly null sequence (in $\mathcal{C}$).*

The next two theorems correspond to Theorem 3E and 3D of [BOst-bit] for the (wcc), extended from the reals to normed groups.

Theorem 2N (Norm topology shift theorem). *If ψ_n in $\mathcal{A}$ converges to the identity, then ψ_n satisfies the weak category convergence condition* (wcc). *Indeed the sequence satisfies* (wccc).

Proof. It is more convenient to prove the equivalent statement that ψ_n^{-1} satisfies the category convergence condition.

Put $z_n = \psi_n(z_0)$, so that $z_n \to z_0$. Let k be given.

Suppose that $y \in B_\varepsilon(z_0)$, i.e. $r = d(y, z_0) < \varepsilon$. For some $N > k$, we have $\varepsilon_n = d(\psi_n, id) < \frac{1}{3}(\varepsilon - r)$, for all $n \geq N$. Now

$$\begin{aligned} d(y, z_n) &\leq d(y, z_0) + d(z_0, z_n) \\ &= d(y, z_0) + d(z_0, \psi_n(z_0)) \leq r + \varepsilon_n. \end{aligned}$$

For $y = \psi_n(x)$ and $n \geq N$,

$$\begin{aligned} d(z_0, x) &\leq d(z_0, z_n) + d(z_n, y) + d(y, x) \\ &= d(z_0, z_n) + d(z_n, y) + d(x, \psi_n(x)) \\ &\leq \varepsilon_n + (r + \varepsilon_n) + \varepsilon_n < \varepsilon. \end{aligned}$$

So $x \in B_\varepsilon(z_0)$, giving $y \in \psi_n(B_\varepsilon(z_0))$. Thus

$$y \notin \bigcap_{n \geq N} B_\varepsilon(z_0) \backslash \psi_n(B_\varepsilon(z_0)) \supseteq \bigcap_{n \geq k} B_\varepsilon(z_0) \backslash \psi_n(B_\varepsilon(z_0)).$$

It now follows that

$$\bigcap_{n \geq k} B_\varepsilon(z_0) \backslash \psi_n(B_\varepsilon(z_0)) = \varnothing,$$

giving (wcc) as required; similarly for (wccc). $\square$

Theorem 2D (Density topology shift theorem). *Let T be a normed locally compact group with left-invariant Haar measure m. Let V be m-measurable and non-null. For any null sequence z_n in $\mathcal{C}(T)$ let $h_n(t) := tz_n^{-1}(t)$. Then for each $k \in \omega$,*

$$H_k = \bigcap_{n \geq k} V \backslash [h_{2n}^{-1}(V) \cup h_{2n+1}^{-1}(V)] \text{ is of } m\text{-measure zero, so meagre in the } d\text{-topology.}$$

That is, the sequence $h_n(t) = tz_n^{-1}(t)$ satisfies the weak category convergence condition (wccc)

Proof. Suppose otherwise. We write Vz for $V \cdot z$, etc. so that $t \in h_n^{-1}(V)$ iff $h_n(t) \in V$ iff $t \in Vz_n(t)$. Now, for some k, $m(H_k) > 0$. Write H for H_k. Since $H \subseteq V$, we have, for $n \geq k$, that $\emptyset = H \cap h_n^{-1}(V)$ and so a fortiori $h \notin Hz_n(h)$ for $h \in H$. Let u be a metric density point of H. Thus, for some bounded (Borel) neighbourhood $U_\nu u$ we have

$$m[H \cap U_\nu u] > \frac{3}{4} m[U_\nu u].$$

Fix U_ν and put

$$\delta = m[U_\nu u].$$

Let $E = H \cap U_\nu u$. For any $z_n(t)$, we have $m[(Ez_n(t)) \cap U_\nu u z_n(t)] = m[E] > \frac{3}{4}\delta$. By Theorem A of [Hal] p. 266, for all large enough n, we have

$$m(U_\nu u \triangle U_\nu u z_n(t)) < \delta/4.$$

Hence, for all n large enough we have $|(Ez_n(t)) \backslash U_\nu u| \leq \delta/4$. Put $F = (EB_{||z_n||}(e)) \cap U_\nu u$; then $m[F] > \delta/2$ for all large enough n. But $\delta \geq m[E \cup F] = m[E] + m[F] - m[E \cap F] \geq \frac{3}{4}\delta + \frac{1}{2}\delta - m[E \cap F]$. So for $h \in H$ we have

$$m[H \cap (Hz_n(h))] \geq m[E \cap F] \geq \frac{1}{4}\delta,$$

contradicting $h \notin Hz_n(h)$ for $h \in H$. This establishes the claim. $\square$

Remark. The only fact about h_n used in the proof above is that, for some sequence of radii $r(n)$ tending to zero, $h_n(t) \in B_{r(n)}(t)$. One may thus verify the (rwcc) condition in the following context.

Corollary 2. *For $A_n(t) := \alpha_n t + z_n$, with $\alpha_n \to \alpha > 0$ and z_n uniformly null, and for V bounded and of finite positive measure,*

$$\bigcap_{n \geq k} \alpha V \backslash A_n(V) \text{ is of } m\text{-measure zero, so meagre in the } d\text{-topology.}$$

Proof. Put $\alpha_n = \alpha + \varepsilon_n$, so that $\varepsilon_n \to 0$, and let

$$W_n := (\varepsilon_n + z_n)(V) := \{\varepsilon_n v + z_n(v) : v \in V\}$$

so that

$$(\alpha_n + z_n)(V) \subseteq \alpha V + W_n.$$

Now $m[W_n] \to 0$ and $diam(W_n) \to 0$, so since αV is of finite positive measure Theorem 2D yields that

$$\bigcap_{n \geq k} \alpha V \backslash A_n(V) \text{ is null,}$$

as required. $\square$

As an immediate corollary of Theorems 1 and 2N we obtain the following special case of Theorem 1.

Corollary 3. *If $\mathcal{X}$ is a Baire non-meagre subset of functions $x(.)$ in $\mathcal{C}[0,1]$ and $f_n \to f$ in $\mathcal{C}[0,1]$ in sup-norm, then for quasi all $x \in \mathcal{X}$ there is an infinite set* $\mathbb{M}_x$ such that

$$\{x + f_m - f : m \in \mathbb{M}_x\} \subseteq \mathcal{X}.$$

Proof. Let $z_n = f_n - f$; then $z_n \to 0$. Since $\mathcal{C}[0,1]$, a complete metric space, is a Baire space, and $x \to x + z_n$ is a sequence of homeomorphisms, Theorem 2N applies. $\square$

We may now deduce two strengthened forms of the Kestelman-Borwein-Ditor embedding theorem. Putting $h_n(t) = tz_n(t)$ we obtain the following corollary.

Theorem 3 (Functionwise Embedding Theorem). *Let T be a normed locally compact group, z_n a null sequence in $\mathcal{C}_b(T)$ such that $t \to tz_n(t)$ is, for each n, an autohomeomorphism. If S is Haar measurable and non-null, resp. Baire and non-meagre, then for generically all $t \in S$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{tz_m(t) : m \in \mathbb{M}_t\} \subseteq S.$$

Next let z_n and w_n be null sequences in $\mathcal{C}_b(T)$. Put $h_{2n}(t) = tz_n(t)$ and $h_{2n+1}(t) = tw_n(t)$; then the merged sequence $z_0(t), w_0(t), z_1(t), w_1(t), ...$ is a null sequence in $\mathcal{C}_b(T)$. Thus one has

Theorem 4 (Functionwise Conjunction Embedding Theorem). *Let T be a normed locally compact group, z_n and w_n null sequences in $\mathcal{C}_b(T)$ such that $t \to tz_n(t)$ is, for each n, an autohomeomorphism. If S is Haar measurable and non-null, resp. Baire and non-meagre, then for generically all $t \in S$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{tz_m(t), tw_m(t) : m \in \mathbb{M}_t\} \subseteq T.$$

This includes the result on bi-lateral shifts mentioned earlier.

4 Generic Reflection Theorem

In this section, working again in the context of $T = \mathbb{R}$, we begin by formulating simple conditions ensuring that various null sequences $z_n \to 0$ in $C_b(\mathbb{R})$ lead to autohomeomorphisms $h_n(t) := t + z_n(t)$ of $\mathbb{R}$ in the usual or in the density topology. This will enable us to apply the functionwise embedding theorems.

Definition. Say that $h : \mathbb{R} \to \mathbb{R}$ is *bi-Lipschitz* (a notion implicit in [Br]) if, for some α, β,

$$0 < \alpha \leq \frac{h(u) - h(v)}{u - v} \leq \beta, \text{ for } u \neq v.$$

In particular, h is continuous and strictly increasing, and so is invertible with continuous and strictly increasing inverse, also bi-Lipschitz, and differentiable, except possibly for at most countably many points. The bi-Lipschitz functions preserve density points – in particular images and preimages of null/meagre sets are null/meagre (see [Br], or [CL1] and [CL2]) – and so are homeomorphisms in the d-topology on $\mathbb{R}$.

Definition. Call a null sequence z_n in C_b *bi-Lipschitz* if the mappings $u \to u + z_n(u)$ are bi-Lipschitz uniformly in n, i.e. for some α, β and all n we have

$$0 < \alpha \leq 1 + \frac{z_n(u) - z_n(v)}{u - v} \leq \beta, \text{ for } u \neq v. \tag{1}$$

In particular $\mathbf{z}'_n$ is bounded away from -1, except perhaps at countably many points.

Definition. For z_n a sequence in $\mathcal{C}_b$, the *f-conjugate sequence* $\bar{z}_n$ is defined as follows:

$$\bar{z}_n(t), \text{ or } z_n^f(t), := f(t + z_n(t)) - f(t).$$

Lemma. *For f Lipschitz, the f-conjugate sequence is null in $\mathcal{C}_b$. If $z_n(t)$ satisfies* (1) *and the derivative $f'(t)$ is continuous near $z = u$ and satisfies*

$$1 + (\alpha - 1)f'(u) > 0,$$

and is bounded above in a neighbourhood of $t = u$, then the f-conjugate sequence $\{\bar{z}_n(t)\}$ is locally bi-Lipschitz near $t = u$. In particular for z_n differentiable this is so if

$$1 + f'(u)z'_n(u) > 0, \text{ for all } n.$$

Proof. For f with Lipschitz constant β_f we have $||\bar{z}_n|| \leq \beta_f ||z_n||$, as

$$|\bar{z}_n(t)| = |f(t + z_n(t)) - f(t)| \leq \beta_f |z_n(t)|.$$

For f differentiable, we may write $f(u) - f(v) = f'(w(u, v))(u - v)$ and

$$f(u + z_n(u)) - f(v + z_n(v)) = f'(w_n(u, v))[z_n(u) - z_n(v) + (u - v)].$$

Thus we have

$$\frac{\bar{z}_n(u) - \bar{z}_n(v)}{u - v} = f'(w_n(u, v))\frac{z_n(u) - z_n(v)}{u - v} + [f'(w_n(u, v)) - f'(w(u, v))].$$

Hence

$$
\begin{aligned}
1+\frac{\bar{z}_n(u)-\bar{z}_n(v)}{u-v} &= 1+f'(w_n)\frac{z_n(u)-z_n(v)}{u-v}+[f'(w_n(u,v))-f'(w(u,v))] \\
&\geq 1+(\alpha-1)f'(w_n)+[f'(w_n(u,v))-f'(w(u,v))]
\end{aligned}
$$

and the latter term is positive for v in a small enough neighbourhood of $t=u$. To obtain the differentiable case we note that in the preceeding line

$$
1+f'(w_n)\frac{z_n(u)-z_n(v)}{u-v}>0
$$

for v in a small enough neighbourhood of $t=u$. $\square$

As an immediate corollary of the above Lemma and the CET and the two shift theorems, we have:

Theorem 5 (Generic Reflection Theorem). *Let T be measurable/Baire, $f(.)$ be continuously differentiable and non-stationary at generically all points, $z_n \to 0$ in supremum norm be a null sequence that is bi-Lipschitz with*

$$
1+f'(t)z_n'(t)>0, \text{ for all } n, \tag{2}
$$

for generically all $t \in T$. Then, for generically all $t \in T$, there is an infinite set $\mathbb{M}_t$ such that

$$
t+f(t+z_n(t))-f(t) \in T, \text{ for all } n \in \mathbb{M}_t. \tag{3}
$$

In particular, if in addition f is linear and $f(t)=\alpha t$ with $\alpha \neq 0$, then for generically all $u \in T$, there is an infinite set $\mathbb{M}_u$ such that

$$
\alpha u_n+(1-\alpha)u \in T \text{ for all } n \in \mathbb{M}_u, \text{ where } u_n=u+z_n(u). \tag{4}
$$

For our closing results we need the following.

Definitions.

1. Say that f is *smooth* for z_n if (2) holds.

2. More generally, say that the sequence f_n of function from $\mathbb{R}$ to $\mathbb{R}$ is *smooth* for z_n if:

(i) $\bar{z}_n(t) := f_n(t+z_n(t))-f_n(t)$ is a null sequence, and

(ii) $h_n(t) := t + \bar{z}_n(t)$ is an autohomeomorphism.

Example 1. Here the linear case $f(t) = \alpha t$ is of particular interest. Here

$$h_n(t) := t + f(t + z_n(t)) - f(t) = t + \alpha z_n(t).$$

For $\alpha > 0$, the derivative condition for h_n to be increasing reads

$$1 + \alpha z_n'(t) \geq 0, \text{ or } z_n'(t) \geq -1/\alpha.$$

So, if the null function sequence is constant (as in Kestelman-Borwein-Ditor Theorem), with $z_n(t) \equiv z_n$, the condition is satisfied, as it reduces simply to $0 \geq -1/\alpha$.

Example 2. Let λ_n be a sequence of non-zero reals and z_n a null sequence in C. Put

$$f_n(t) = \lambda_n f(t),$$

where $f(.)$ is continuously differentiable. Thus

$$|\bar{z}_n(t)| = |f_n(t + z_n(t)) - f_n(t)| = \lambda_n |z_n(t)||f'(v_n(t))|,$$

for some $v_n(t)$. Thus $|\bar{z}_n(t)| \to 0$ on compacts if λ_n is bounded. Now

$$\begin{aligned} \frac{d}{dt}\left(t + \lambda_n f(t + z_n(t)) - \lambda_n f(t)\right) &= 1 + \gamma_n(f'(t + z_n(t))[1 + z_n'(t)] - f'(t)) \\ &= 1 + \gamma_n f'(t + z_n(t)) z_n'(t) + \lambda_n[f'(t + z_n(t)) - f'(t)]. \end{aligned}$$

Thus, for λ_n bounded, a condition such as

$$1 + \lambda_n f'(t) z_n'(t) > 0$$

ensures that $t + \bar{z}_n(t)$ is a Euclidean homeomorphism. This will be so when $z_n(t) \equiv z_n$ (constant).

For $f(t) = t$ we have

$$\bar{z}_n(t) = \lambda_n z_n(t).$$

Thus if (1) holds for z_n, then, for u, v distinct and $\lambda_n > 0$,

$$1 - \lambda_n < 1 + \lambda_n(\alpha - 1) \leq 1 + \lambda_n \frac{z_n(u) - z_n(v)}{u - v} \leq 1 + \lambda_n(\beta - 1).$$

So, for $0 < \lambda_n < 1$, we conclude that $\bar{z}_n$ is bi-Lipschitz. If $z_n(t) = z_n$ (constant) then the only condition that needs to be in place is that $\lambda_n ||z_n|| \to 0$. This can be easily be arranged by replacing z_n by a subsequence $\hat{z}_n = z_{k(n)}$ such that $\lambda_n ||z_{k(n)}|| \to 0$.

Theorem 6 (Smooth Image Theorem). *Let f and g both be smooth for $z_n \in \mathcal{C}$ which is differentiable and bi-Lipschitz. Then, for generically all $t \in T$, there is an infinite set $\mathbb{M}_t$ such that*

$$t + z_n^f \in T, \text{ and } t + z_n^g \in T \text{ for all } n \in \mathbb{M}_u. \tag{5}$$

In particular, for f smooth and $g(t) = t$ the identity map we obtain the simultaneous embedding:

$$t + z_n^f \in T, \text{ and } t + z_n \in T \text{ for all } n \in \mathbb{M}_t.$$

Furthermore, if f and g are smooth and linear and $f(t) = \alpha t$ with $\alpha \neq 0$, $g(t) = \beta t$ with $\beta \neq 0$, then for generically all $t \in T$, there is an infinite set $\mathbb{M}_t$ such that

$$t + \alpha z_n \in T, \text{ and } t + \beta z_n \in T \text{ for all } n \in \mathbb{M}_t.$$

For instance, taking $\alpha = 1, \beta = -1$ we obtain generic bilateral embedding:

$$t + z_n \in T, \text{ and } t - z_n \in T \text{ for all } n \in \mathbb{M}_t.$$

For $\alpha_n = 2^n$ and $z_n(t) = z_n$ constant, the following result (though not its proof) appears implicitly in the proof of Császár's Non-separation theorem (of a mid-point convex function and its lower hull by a measurable function); see [BOst6] for applications.

Theorem 7 (Császár's Genericity Theorem, [Csa], or [Kucz] p 223-226**).** *Let T be measurable, non-null or Baire, non-meagre.*
(i) *Let $\{\alpha_n\}$ be bounded from below by unity and let $\{z_n\} \to 0$ be uniformly null. For generically all $t \in T$, there are points $t_n \in T$ such that, along some subsequence of n,*

$$t = \alpha_n t_n + (1 - \alpha_n) u_n(t), \text{ where } u_n(t) = t + z_n(t).$$

(ii) *Let $\{\alpha_n\}$ be positive and bounded away from zero and let $\{z_n\} \to 0$ be a null sequence of reals. For generically all $u \in T$ and generically all t near u, there are points $t_n \in T$ such that, along some subsequence of n,*

$$t = \alpha_n t_n + (1 - \alpha_n) u_n, \text{ where } u_n = u + z_n.$$

Proof. The conclusions concern subsequences; so we may divide the argument according as α_n tends to infinity or is convergent. Suppose first that $a_n \to \infty$, and so also that, for all n, $\alpha_n > 1$. For $\gamma_n := 1/\alpha_n$ and $\lambda_n = 1 - \gamma_n$, we have $0 < \lambda_n < 1$. Taking $f_n(t) = \lambda_n t = (1 - \gamma_n)t$, we conclude from Example 2 above that for generically all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that

$$t_n = t + (1 - \gamma_n)z_n(t) \in T, \text{ for } n \in \mathbb{M}_t.$$

So

$$t_n = \gamma_n t + (1 - \gamma_n)[t + z_n(t)] \in T,$$

and equivalently

$$t = \alpha_n t_n + (1 - \alpha_n)u_n(t).$$

Now suppose that $\alpha_n \to \alpha > 0$. Thus $(1 - \alpha_n)z_n \to 0$. Take $h_n^{-1}(t) = A_n(t) = \alpha_n t + (1 - \alpha_n)z_n(t)$. Since (rwcc) holds at 0 in the Euclidean case (by Remark 4 of Section 2), and also in the density case by Corollary 2, we conclude that there is an infinite set $\mathbb{M}_{t,u}$ such that

$$t - u = x \in h_m^{-1}(T_u), \text{ for } m \in \mathbb{M}_{t,u}.$$

Thus, as in (equiv), we have

$$t - u = h_n^{-1}(t_n - u) = \alpha_n(t_n - u) + (1 - \alpha_n)z_n,$$

or again

$$t = \alpha_n t_n + (1 - \alpha_n)(u + z_n). \qquad \square$$

Remarks. 1. Theorem 5 applies also to sequences z_n which converge to zero on compacts. This is because all our results are local and because of the procedure of *capping* which follows. Suppose $z_n(t)$ only converges to zero on compacts and that $t + z_n(t)$ is is a Euclidean homeomorphism (i.e. is strictly increasing and continuous). For any interval (a, b) in $\mathbb{R}$, the capped sequence:

$$\hat{z}_n(t) = \begin{cases} z_n(a), & \text{for } t \leq a, \\ z_n(t), & \text{for } a < t < b, \\ z_n(b) & \text{for } t \geq b, \end{cases}$$

has $\hat{z}_n \to 0$ in supremum norm, and the substitution of $\hat{z}_n$ for z_n preserves the homeomorphism property (i.e. $t + \hat{z}_n(t)$ is strictly increasing and continuous) as well as equality with $t + z_n(t)$ on (a, b).

For instance, consider $f(t) = t^2$ and a given null sequence of constants $w_n \to 0$. Here its f-conjugate sequence is $z_n(t) := w_n(2t + w_n)$ and

$$h_n(t) := t + z_n(t) = t(1 + 2w_n) + w_n^2$$

is increasing for n large enough; however $z_n \to 0$ uniformly only on compacts. Nevertheless, by the capping procedure, here too, for T Baire non-meagre/measurable non-null, for generically all t in T there is an infinite set $\mathbb{M}_t$ such that

$$\{t + z_n(t) : m \in \mathbb{M}_t\} \subset T.$$

2. Other examples of smooth generation of null sequences are

$$\bar{z}_n(t) := f(\varphi(t) + z_n(t)) - f(\varphi(t)),$$

where φ is homeomorphism. Thus if $\psi = \varphi^{-1}$, then $t + \bar{z}_n(t)$ becomes, under the substitution $u = \varphi(t)$

$$\psi(u) + f(u + z_n(\psi(u))) - f(u).$$

The special case $\psi = f$ then leads to the embedding of the sequence

$$f(u + z_n(\psi(u))).$$

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