The Converse Ostrowski Theorem: aspects of compactness

N. H. Bingham
Mathematics Department, Imperial College London,
South Kensington, London SW7 2AZ
n.bingham@ic.ac.uk

A. J. Ostaszewski
Mathematics Department, London School of Economics,
Houghton Street, London WC2A 2AE
a.j.ostaszewski@lse.ac.uk

October 2007, BOstCDAM8-1rev.tex

In memoriam Rolf Trautner (1939-2005)

CDAM Research Report LSE-CDAM-2007-25 (revision)

Abstract

The Ostrowski theorem in question is that an additive function bounded (above, say) on a set T of positive measure is continuous. In the converse direction, recall that a topological space T is pseudocompact if every function continuous on T is bounded. Thus theorems of 'converse Ostrowski' type relate to 'additive (pseudo)compactness'. We give a different characterization of such sets, in terms of the property of 'generic subuniversality', arising from the Kestelman-Borwein-Ditor theorem and relate these to various new forms of compactness.

Classification: 26A03

Keywords: Ostrowski's theorem, additive function, pseudocompact, generically subuniversal, additively compact, shift-compact, compactly shift-covered

Usually in analysis, one distinguishes between ordinary, or Euclidean, analysis and functional analysis by dimension – finite or infinite – or by compactness properties (whether or not the unit ball is compact). By contrast, here we do real (or Euclidean) analysis, but rather than taking the real line $\mathbb{R}$ as a one-dimensional vector space over itself, instead consider it as an infinite-dimensional vector space over the rationals $\mathbb{Q}$. Although sets such as the unit ball are then non-compact, one can induce compactness by using the shift, that is, the additive group structure of the reals. It is this combination of compactness and additivity that underlies the paper. Indeed, in the light of our results, we are now able to interpret the important work of Kestelman [Kes], which partly motivates this paper and fully motivates [BOst3], as a contribution relating to compactness.

Let $\mathcal{A}dd$ denote the additive (real-valued) functions on $\mathbb{R}$ and for $T \subseteq \mathbb{R}$, let $\mathcal{A}dd(T) = \{f|T : f \in \mathcal{A}dd\}$ denote the family of their restrictions to T. Darboux's theorem of 1875 ([Dar], [AD] Section 21.6; see also [BOst12] Section 7 for a natural metric group setting) asserts that an additive function bounded above on some interval is continuous, i.e.

$$\mathcal{A}dd \cap \mathcal{B}^{+}_{\mathrm{loc}} \subseteq \mathcal{C},$$

where $\mathcal{B}^{+}_{\mathrm{loc}}$ denotes the functions on $\mathbb{R}$ which are locally bounded from above everywhere and $\mathcal{C}$ those which are continuous. (Note that for an additive function local boundedness from above in some neighbourhood implies local boundedness everywhere.) The stronger result is Ostrowski's theorem of 1929 [Ostr] that a (mid-point) convex, so *a fortiori* an additive, function bounded above on some set $T \subseteq \mathbb{R}$ of positive measure is continuous (this too has a natural metric group generalization, see [BOst12] Section 6), or, adapting the earlier notation, in symbols

$$\mathcal{A}dd \cap \mathcal{B}^{+}_{\mathrm{loc}}(T) \subseteq \mathcal{C}, \text{ with } |T| > 0.$$

We identify the natural converse of this (in relation to *additive* functions), but require first a stronger formulation still, establishing a connection with a weak (sequential) compactness notion of independent interest (additive pseudocompactness).

The proof of Ostrowski's theorem rests on two ideas: the *Darboux dichotomy* – for $f \in \mathcal{A}dd$, either f is continuous, or everywhere (locally) unbounded – and the *subuniversal property* of T ([Kes] Th. 2) – for any null sequence $\{z_n\} \to 0$, there is a point t and an infinite set $\mathbb{M}_t$ such that

$\{t+z_m : m \in \mathbb{M}_t\} \subseteq T$. (Thus $f(z_n) \to \infty$ makes $\{f(t+z_m) : m \in \mathbb{M}_t\}$ unbounded from above on T, a contradiction). Our strong form of Ostrowski's theorem requires merely the weaker hypothesis that $f \in \mathcal{A}dd$ be bounded above on a subuniversal T. In this form it also embraces the category analogue (cf. Mehdi's Theorem [Meh] on the continuity of a mid-point convex function bounded on a non-meagre Baire set); the two cases are viewed in [BOst11] as bi-topologically associated (under the association of the density topology to the Euclidean).

The key concept of universality (for which $\mathbb{M}_t$ needs to be co-finite) is implicit in Banach [Ban-I] and explicit although not in in name in Banach [Ban-II] in the proofs that a measurable/Baire additive function is continuous (see the commentary by Henryk Fast loc. cit. p. 314 for various one-way implications among related results); in name it goes back to Kestelman [Kes], where also subuniversality (a term coined in [BOst1]) originates but is given less prominence than, with hindsight, it deserves. We will write $T \in \mathfrak{S}$ (Gothic 'S' for 'subsequence') when T is subuniversal. Its connection with the Steinhaus Theorem (cf. e.g. [Com] Th. 4.6) is studied in [BOst12].

We will say T is *generically subuniversal*, written $T \in \mathfrak{S}_{\text{gen}}$, if for any null sequence $\{z_n\} \to 0$, there is a point t and an infinite set $\mathbb{M}_t$ such that $\{t+z_m : m \in \mathbb{M}_t\} \subseteq T$ and further $t \in T$. As $t+z_m$ converges to t through $\mathbb{M}_t$, an alternative usage might be to say that T is *additively compact* (see Note 5 at the end). The term 'shift-compact' refers to a similar property studied in Probability Theory (in the context of probabilities regarded as a semigroup under convolution, for which see [PRV], [Par] Sect. 3.2, or [BH] Sect. 5.1), and we borrow this term for our own related definition at the end of the paper. Denoting by $\overline{\mathfrak{S}}$ the closed members of $\mathfrak{S}$, we have $\overline{\mathfrak{S}} \subseteq \mathfrak{S}_{\text{gen}} \subseteq \mathfrak{S}$. Use of the term is justified by a result due in the measure case to Borwein and Ditor [BoDi], known earlier (in weaker form, for both measure and category) by Kestelman ([Kes] Th. 3), and rediscovered by Trautner [Trau]. Much more is true, see [BOst6], [BOst5], [BOst9], [BOst10], [BOst12] which examine consequences for various real-variable function classes and also metric group formulations. Recall that in the category setting, 'quasi everywhere' means 'except on a meagre set' (see e.g. [Kah]).

Theorem (Kestelman-Borwein-Ditor Theorem). *Let* $\{z_n\} \to 0$ *be a null sequence of reals. If* T *is measurable and non-null (resp. non-meagre), then, for almost all (resp. for quasi-all)* $t \in T$*, there is an infinite set* $\mathbb{M}_t$

such that

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

Recently understood to be the most significant ingredient for the fundamental theorems of regular variation, this theorem contains hidden topological aspects in the form of a notion of compactness, as we shall see below. Hitherto universality was researched by combinatorialists to capture its limitations (see [Mil] for the associated literature and for 'forcing' connections with genericity).

Adopting now a topological perspective, for any space T (not necessarily a subset of reals as before), let $\mathcal{B}(T)$, $\mathcal{B}^+(T)$, $\mathcal{C}(T)$ denote respectively the sets of real-valued functions that are bounded, bounded above, or continuous on T; recall (cf. [Eng] Section 3.10, or for a group perspective [Com] Section 6) that a space T is *pseudocompact* if $\mathcal{C}(T) \subseteq \mathcal{B}(T)$, equivalently $\mathcal{C}(T) \subseteq \mathcal{B}^+(T)$, i.e. every continuous function on T is bounded/bounded above. (In a separable metric space, in particular when T is a subset of $\mathbb{R}$ equipped with the usual topology, this property is of course equivalent to any of countable compactness, compactness, sequential compactness – again, see [Eng] Section 3.10.) For $T \subseteq \mathbb{R}$ with $T \in \mathfrak{S}$, the Ostrowski theorem mentioned above implies a 'reverse' inclusion: $\mathcal{A}dd(T) \cap \mathcal{B}^+(T) \subseteq \mathcal{C}(T)$, since for $f \in \mathcal{A}dd$, $f|T$ bounded implies in particular $f|T$ continuous. In the definition below we narrow the class of functions under the scope of pseudocompactness. This allows generic subuniversality to be viewed as an *additive (sequential) compactness* property and yields a natural converse to Ostrowski's theorem. See Note 5 for a further topological insight in relation to the Euclidean topology of $\mathbb{R}$.

Definition. Say that $T \subseteq \mathbb{R}$ is *$\mathcal{A}dd$-pseudocompact* (additively pseudocompact) if every function of $\mathcal{A}dd(T)$ in $\mathcal{C}(T)$ is bounded, i.e. $\mathcal{A}dd(T) \cap \mathcal{C}(T) \subseteq \mathcal{B}(T)$, equivalently $\mathcal{A}dd(T) \cap \mathcal{C}(T) \subseteq \mathcal{B}^+(T)$.

Additive-pseudocompactness Theorem (The Converse Ostrowski Theorem). *Let $f \in \mathcal{A}dd(\mathbb{R})$ and let $T \in \mathfrak{S}_{\text{gen}}$ be bounded. If f is continuous on T, then f is bounded above on bounded intervals of $\mathbb{R}$. In particular, T is additively-pseudocompact.*

The conclusions remain valid for the more general class of functions f satisfying

$$\bar{k}_v(t) := \inf_{\delta > 0} \sup_{|z| < \delta} |f(t + v + z) - f(v + z)| < \infty, \text{ for all } u, t. \tag{1}$$

Proof. If not, we may take f unbounded above; suppose that $f(u_n) \to \infty$ for $f \in \mathcal{A}dd(\mathbb{R})$ and $\{u_n\}$ bounded. Suppose without loss of generality that $u_n \to u$. Then $z_n := u_n - u \to 0$. As $T \in \mathfrak{S}_{\text{gen}}$, for some $t \in T$ and some infinite $\mathbb{M}_t$ we have $\{t + z_m : m \in \mathbb{M}_t\} \subseteq T$. Hence $f(t) + f(z_m) = f(t + z_m) \to f(t)$, for $m \in \mathbb{M}_t$, because f is continuous on T. Thus, taking limits with $m \in \mathbb{M}_t$, we have

$$f(u) - f(t) = \lim f(u + z_m) - f(t + z_m) = \infty,$$

a contradiction. With the more general assumption on $f(.)$, the fact that $z_m \to 0$ yields the conclusion

$$\inf_{\delta>0} \sup_{|z|<\delta} |f(u+z) - f(t+z)| = \infty,$$

again a contradiction to (1) with $v = u - t$. $\square$

We include for completeness the strong Ostrowski theorem (see the opening paragraphs) in this language:

Converse Additive-pseudocompactness Theorem (Strong Ostrowski Theorem). *For $f \in \mathcal{A}dd(\mathbb{R})$ and $T \in \mathfrak{S}$ bounded, if f is bounded above on T, then f is continuous on $\mathbb{R}$.*

Notes. 1. The above analysis, mutatis mutandis, may be repeated in a Euclidean space $\mathbb{R}^d$, or for homomorphisms between metric groups.

2. There is a formal connection here to the Ger-Kuczma investigation of 'sets of automatic continuity' (see [GerKucz] or [Kucz]): for f additive, a subuniversal T is a 'set of automatic continuity of f' given boundedness from above. That is, f bounded above on T implies f is continuous – or in Ger-Kuczma notation, $T \in \mathfrak{B}$ (Gothic 'B' for boundedness). In symbols, $\mathfrak{S} \subseteq \mathfrak{B}$. See [BOst6] for more on this.

3. In the measure, or category, setting, behaviour is 'generic' if it holds almost everywhere, or quasi-everywhere; thus one may speak of generic properties, and thence of generic points (as with points in general position in geometry). In mathematical logic, genericity in the sense of forcing is linked in the case of Cohen's original forcing argument to genericity in the sense of Baire category. See Mostowski's book [Most], Ch. IX (especially p. 132, and p.127, where Mostowski attributes this link to Ryll-Nardzewski) for the earliest exposition here, and also [Kech], I.8B, II.16D for a modern textbook

treatment. There is a similar link between Solovay's 'random forcing' argument in the measure context – the general framework is provided in [Zap] (see also the ramifications considered in the recent paper [FarZap]). The terms 'generic', or 'typical', are also widely used in ergodic theory; cf. cite: [AlpPras-1], [AlpPras-2].

4. The condition (1) is motivated by regular variation (for which see [BGT], or [BOst1]). Indeed, we may say (following [BGT], Ch. 2, OR, Ch. 3, $O\Pi$) that $f(.)$ is O-regularly varying at u, if $\bar{k}_u(s) < \infty$, for all s. Thus (1) asserts that $f(.)$ is O-regularly varying at all points u. A stronger form still of this, appropriate to a Baire space which is a metric group, is explored in [BOst12] Section 10.

5. In the proof above, the step from the bounded sequence $\{u_n\}$ to the shifted subsequence $t + z_m = t - u + u_m$ justifies the alternative usage for a generically subuniversal set $T \subseteq \mathbb{R}$ being instead called *additively compact.* The theorem and that step in its proof may be regarded as facilitating the main contribution of this paper, which is the identification of these two, apparently different, properties as being of interest, and of their being one and the same. In turn this suggests a narrowing of the concept (to fewer sequences), as follows.

Definition. Let us say that a set A in $\mathbb{R}^d$ is *shift-compact* if, for any sequence of points a_n in A, there is a point t and a subsequence $\{a_n : n \in \mathbb{M}_t\}$ such that $t + a_n$ converges through $\mathbb{M}_t$ to a point $t + a_0$ in A. Thus a shift-compact set A is bounded.

Lemma. *If A is bounded and additively-compact then A is shift-compact.*

Proof. Let a_n is a sequence in A with limit a_0, then $z_n := a_n - a_0$ is a null sequence, hence for some $t \in A$ and infinite $\mathbb{M}_t$ we have $t + z_n$ in A for $n \in \mathbb{M}_t$; thus with $s := t - a_0$ we have $s + a_n = t + z_n$ in A converging through $\mathbb{M}_t$ to $s + a_0 = t$, also in A. Thus A is shift-compact. $\square$

Evidently, finite products of shift-compact sets are shift-compact. Countable products exhibit a more general form of shift-compactness. The following theorem asserts that, modulo shifts, a covering property is satisfied by bounded shift-compact sets. The Kestelman-Borwein-Ditor Theorem (which may also be established in a group setting, [BOst12]) thus identifies 'large sets' (non-null measurable sets and non-meagre Baire sets) as having these properties locally, or globally when bounded; as to their scope, reference may be made to models of set-theory where all sets are measurable, or all are Baire (Solovay's model [So], or Shelah's model [She]).

It will be convenient to make the following.

Definitions. 1. Say that $\mathcal{D}:= \{D_1, ..., D_h\}$ *shift-covers* A, or is a *shifted-cover* of A if, there are $d_1, ..., d_h$ in $\mathbb{R}$ such that

$$(D_1 - d_1) \cup ... \cup (D_h - d_h) \supseteq A.$$

Say that A is *compactly shift-covered* if every open cover $\mathcal{U}$ of A contains a finite subfamily $\mathcal{D}$ which shift-covers A.

2. Say that $\mathcal{D}:= \{D_1, ..., D_h\}$ *strongly shift-covers* A, or is a *strong shifted-cover* of A if, there are *arbitrarily small* $d_1, ..., d_h$ in $\mathbb{R}$ such that

$$(D_1 - d_1) \cup ... \cup (D_h - d_h) \supseteq A.$$

Say that A is *compactly strongly shift-covered* if every open cover $\mathcal{U}$ of A contains a finite subfamily $\mathcal{D}$ which strongly shift-covers A.

Example. Note that $A \subseteq \mathbb{R}$ is a dense-open (open in the density topology) iff each point of A is a density point of A. Suppose a_0 is a limit point of such a set A in the usual topology; then, for any $\varepsilon > 0$, we may find a point $\alpha \in A$ to within $\varepsilon/2$ of a_0 and hence some $t \in A$ within $\varepsilon/2$ of the point α such that some subsequence $t + a_m$ is included in A, with limit $t + a_0$ and with $|t| < \varepsilon$. That is, a dense-open set is strongly shift-compact.

Compactness Theorem (Compactness modulo shift). *Let A be a shift-compact subset of $\mathbb{R}$. Then A is compactly shift-covered, i.e. for any Euclidean-open cover $\mathcal{U}$ of A, there is a finite subset $\mathcal{V}$ of $\mathcal{U}$ and arbitrarily small translations, one for each member of $\mathcal{V}$, such that the corresponding translates of $\mathcal{V}$ cover A.*

Proof. Let $\mathcal{U}$ be an open cover of A and let $\varepsilon > 0$. Since $\mathbb{R}$ is second-countable we may assume that $\mathcal{U}$ is a countable family. Write $\mathcal{U} = \{U_i : i \in \omega\}$. Let $Q = \{q_j : j \in \omega\}$ enumerate the rationals. Suppose, contrary to the theorem, that there is no finite subset $\mathcal{V}$ of $\mathcal{U}$ such that translates of elements $\mathcal{V}$, each translated by one element of Q, do not cover A. For each n, choose $a_n \in A$ not covered by $\{U_i - q_j : i, j < n\}$. As A is bounded, we may assume, by passing to a subsequence (if necessary), that a_n converges to some point a_0, and also that, for some t, the sequence $t + a_n$ lies entirely in A. Let U_i in $\mathcal{U}$ cover $t + a_0$. Without loss of generality we may assume that $t + a_n \in U_i$ for all n. Thus $a_n \in U_i - t$ for all n. Thus we may select $V := U_i - q_j$ to be a translation of U_i such that $a_n \in V = U_i - q_j$ for all n. But this is a contradiction, since a_n is not covered by $\{U_{i'} - q_{j'} : i', j' < n\}$ for $n > \max\{i, j\}$. $\square$

The above proof of the compactness theorem for shift-covering may be improved to strong shift-covering, with only a minor modification (replacing Q with a set $Q^\varepsilon = \{q_j^\varepsilon : j \in \omega\}$ which enumerates, for given $\varepsilon > 0$, a dense subset of the ε-interval about 0), yielding the following.

Strong Compactness Theorem (Strong Compactness modulo shift). *Let A be a strongly right-shift compact subset of $\mathbb{R}$. Then A is compactly strongly shift-covered, i.e. for any Euclidean-open cover $\mathcal{U}$ of A, there is a finite subset $\mathcal{V}$ of $\mathcal{U}$ and arbitrarily small translations, one for each member of $\mathcal{V}$, such that the corresponding translates of $\mathcal{V}$ cover A.*

Thus by re-engaging with problems in classical real analysis, we have been able to identify a new topological scheme for various notions of weak compactness, based on families that cover a space only after their elements have been subjected to a small translation, as above (or more generally a uniformly small topological transformation).

References

[AD] J. Aczél and J. Dhombres, *Functional equations in several variables*, Encycl. Math. Appl. 31, Cambridge University Press, Cambridge, 1989

[AlpPras-1] S. Alpern and V. S. Prasad, *Typical dynamics of volume preserving homeomorphisms*, Cambridge Tracts in Mathematics, 139. Cambridge University Press, Cambridge, 2000.

[AlpPras-2] S. Alpern and V. S. Prasad, *Properties generic for Lebesgue space automorphisms are generic for measure-preserving manifold homeomorphisms*, Ergodic Theory Dynam. Systems 22:6 (2002), 1587–1620.

[Ban-I] S. Banach, *Sur l'équation fonctionelle* $f(x+y) = f(x) + f(y)$, Fund. Math. 1(1920), 123-124, , reprinted in collected works vol. I, 47-48, PWN, Warszawa, 1967, (Commentary by H. Fast p. 314).

[Ban-II] S. Banach, *Théorie des opérations linéaires*, reprinted in collected works vol. II, 401-411, PWN, Warszawa, 1979, (1st. edition 1932).

[BGT] N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular variation,* 2nd edition, Encycl. Math. Appl. 27, Cambridge University Press, Cambridge, 1989 (1st edition 1987).

[BOst1] N. H. Bingham and A. J. Ostaszewski, *Infinite combinatorics and the foundations of regular variation*, CDAM Research Report Series, LSE-CDAM-2006-22rev.

[BOst3] N. H. Bingham and A. J. Ostaszewski, *Infinite combinatorics and the theorems of Steinhaus and Ostrowski*, LSE-CDAM Report, LSE-CDAM-2007-15rev.

[BOst5] N. H. Bingham and A. J. Ostaszewski, *Generic subadditive functions*, Proc. Amer. Math. Soc. 136 (2008), 4257-4266, LSE-CDAM Report, LSE-CDAM-2007-23.

[BOst6] N. H. Bingham and A. J. Ostaszewski, *New automatic properties: convexity, uniformity, genericity*, LSE-CDAM Report, LSE-CDAM-2007-23.

[BOst9] N. H. Bingham and A. J. Ostaszewski, *Infinite combinatorics in function spaces*, LSE-CDAM Report, LSE-CDAM-2007-26rev.

[BOst10] N. H. Bingham and A. J. Ostaszewski, *Homotopy and the Kestelman-Borwein-Ditor Theorem*, Canadian Bull. Math., to appear, LSE-CDAM Report, LSE-CDAM-2007-27.

[BOst11] N. H. Bingham and A. J. Ostaszewski, *Bitopology and measure-category duality*, LSE-CDAM Report, LSE-CDAM-2007-29rev.

[BOst12] N. H. Bingham and A. J. Ostaszewski, *Normed groups: dichotomy and duality*, LSE-CDAM Report, LSE-CDAM-2008-10.

[BH] W. Bloom and H. Heyer, *Harmonic analysis of probability measures on hypergroups*, de Gruyter Studies in Mathematics, 20. Walter de Gruyter & Co., Berlin, 1995.

[BoDi] D. Borwein and S. Z. Ditor, *Translates of sequences in sets of positive measure*, Canadian Mathematical Bulletin 21 (1978) 497-498.

[Com] W. W. Comfort, *Topological groups*, in *Handbook of set-theoretic topology*, K. Kunen and J. E. Vaughan, eds., North-Holland Publishing Co., Amsterdam, 1984.

[Dar] G. Darboux, *Sur la composition des forces en statiques*, Bull. des Sci. math. (1) 9 (1875) 281-288.

[Eng] R. Engelking, *General Topology,* Heldermann Verlag, Berlin 1989.

[FarZap] I. Farah, J. Zapletal, *Four and more*, Ann. Pure Appl. Logic 140 (2006), no. 1-3, 3–39.

[GerKucz] R. Ger and M. Kuczma, *On the boundedness and continuity of convex functions and additive functions,* Aeq. Math. 4 (1970), 157-162.

[H] H. Heyer, *Probability measures on locally compact groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 94. Springer-Verlag, Berlin-New York, 1977.

[Kah] J.-P. Kahane, *Probabilities and Baire's theory in harmonic analysis*, in: Twentieth Century Harmonic Analysis, A Celebration, editor: J. S. Byrnes, 57-72, Kluwer, Netherlands, 2001.

[Kech] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics 156, Springer 1995.

[Kel] J. L. Kelley, *General Topology*, Van Nostrand, 1955.

[Kes] H. Kestelman, *The convergent sequences belonging to a set*, J. London Math. Soc. 22 (1947), 130-136.

[Kucz] M. Kuczma, *An introduction to the theory of functional equations and inequalities. Cauchy's functional equation and Jensen's inequality,* PWN, Warsaw, 1985.

[Meh] M.R. Mehdi, *On convex functions*, J. London Math. Soc. 39 (1964), 321-326.

[Mil] A.W. Miller, *Special sets of reals*, 415-432, in H. Judah (ed.), *Set Theory of the Reals*, Israel Mathematical Conference Proceedings vol. 6, Proceedings of the Winter Institute held at Bar-Ilan University, Ramat Gan, 1993. (or see web version at : http://www.math.wisc.edu/~miller/res/).

[Most] A. Mostowski, *Constructible sets with applications,* PWN and North-Holland, 1969.

[Ostr] A. Ostrowski, *Mathematische Miszellen XIV: Über die Funktionalgleichung der Exponentialfunktion und verwandte Funktionalgleichungen*, Jahresb. Deutsch. Math. Ver. 38 (1929) 54-62 (reprinted in *Collected papers of Alexander Ostrowski*, Vol. 4, 49-57, Birkhäuser, Basel, 1984).

[Par] K. R. Parthasarathy, *Probability measures on metric spaces.* Reprint of the 1967 original. AMS Chelsea Publishing, Providence, RI, 2005.

[PRV] K. R. Parthasarathy, R. Ranga Rao, S.R.S Varadhan, *Probability distributions on locally compact abelian groups*, Illinois J. Math. 7 1963 337–369.

[She] S. Shelah, *Can you take Solovay's inaccessible away?*, Israel J. Math. 48 (1984), 1-47.

[So] R. M. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. of Math. 92 (1970), 1-56.

[Trau] R. Trautner, *A covering principle in real analysis*, Quart. J. Math. Oxford, (2), 38 (1987), 127-130.

[Zap] J. Zapletal, *Descriptive set theory and definable forcing*, Mem. Amer. Math. Soc. 167 (2004), no. 793.