Automatic continuity via analytic thinning

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October 2007; This version June 2008
BOstCDAM7-rev.tex

To Roy Davies on the occasion of his 80th birthday

CDAM Research Report LSE-CDAM-2007-24 (revised)

Abstract

We use Choquet's analytic capacitability theorem and the Kestelman-Borwein-Ditor theorem (on the inclusion of null sequences by translation) to derive results on 'analytic automaticity' – for instance, a stronger common generalization of the Jones/Kominek theorems that an additive function, whose restriction is continuous/bounded on an analytic set T spanning $\mathbb{R}$ (e.g., containing a Hamel basis), is continuous on $\mathbb{R}$. We obtain results on 'compact spannability' – the ability of compact sets to span $\mathbb{R}$. From this, we derive Jones' Theorem from Kominek's. We cite several applications including the Uniform Convergence Theorem of regular variation.

Classification: 26A03

Keywords: Jones' theorem, Kominek's theorem, analytic set, Choquet capacity, Hamel basis, uniform convergence theorem, regular variation.

1 Introduction

This paper, on additive functions, is a sequel to [BOst], where we study subadditive functions, and in turn leads on to the companion paper [BOst6], on convex and related functions.

Darboux's theorem of 1875 ([Dar], [AD, Section 21.6]) asserts that, for additive functions, local boundedness implies continuity. Ostrowski's result of 1929 [Ostr], that a (mid-point) convex, so *a fortiori* an additive, function bounded above on some set T of positive measure is continuous, may be regarded as *thinning out* Darboux's assumed 'local' character from a property holding on an interval to the same property holding only on a set of positive measure. From this perspective Jones' theorem of 1942 [Jones2], that an additive function continuous on a set T which is *analytic* (for definition and background see [Rog2]) and contains a Hamel basis is continuous, may be seen as a further thinning out.

There is a closely related, more recent result of Z. Kominek in 1981 [Kom2], that an additive function *bounded* on a set T, which is analytic and contains a Hamel basis, is continuous. As boundedness is here limited to T, it is not immediately clear what the logical relationship between these theorems is, despite the almost identical proof structure, as elegantly derived by Kominek (from ideas which he attributes as implict in Jones). For convenience, let us say briefly that T is a *spanning set* when $\mathbb{R}$ regarded as a vector space over $\mathbb{Q}$ has T as a spanning set of vectors. (In the presence of the Axiom of Choice a spanning set contains a Hamel basis.)

The theorems of Jones and Kominek are results on automatic continuity (see Hoffmann-Jørgensen [THJ] for background and references) of a particular type: the set T, on which a property is assumed, is analytic (and 'big enough' – say, spanning a subset of positive measure or one that is co-meagre, and hence spanning all of $\mathbb{R}$, or containing a Hamel basis if the Axiom of Choice is assumed). We unify results of this type, which we dub theorems on *analytic automaticity*, by showing that both are instances of a stronger theorem of the same type. See [BOst6] for a range of such theorems touching convex functions (see Note 1 at the end of the paper) and making the connection with the uniform convergence theorem of regular variation (for which see e.g. [BGT]).

Our theorem identifies circumstances when a weak property, such as local boundedness, that has been given 'analytic thinning out' still implies a strong property, such as continuity. The theorem calls for three ingredients: an initial 'weak implies strong' hypothesis (for which the canonical example is Darboux's theorem), the sequential character of the weak property (to be defined), and a modicum of vector-space structure (given by the theorem; but see also the re-formulation of Section 5 making explicit the underlying sequential combinatorics, which we require for [BOst6]).

Definitions. For a family $\mathcal{F}$ of functions from $\mathbb{R}^d$ to $\mathbb{R}$, we denote by $\mathcal{F}(T)$ the family $\{f|T : f \in \mathcal{F}\}$ of functions in $\mathcal{F}$ restricted to $T \subseteq \mathbb{R}^d$. Let us denote a convergent sequence with limit $\mathbf{x}_0$, by $\{\mathbf{x}_n\} \to \mathbf{x}_0$. We say the property $\mathcal{Q}$ of functions (property being regarded set-theoretically, i.e. as a family of functions from $\mathbb{R}^d$ to $\mathbb{R}$) is *sequential on T* if

$$f \in \mathcal{Q} \text{ iff } (\forall\{\mathbf{x}_n : n > 0\} \subseteq T)[(\{\mathbf{x}_n\} \to \mathbf{x}_0) \implies f|\{\mathbf{x}_n : n > 0\} \in \mathcal{Q}(\{\mathbf{x}_n : n > 0\})].$$

If we further require the limit point to be enumerated in the sequence, we call $\mathcal{Q}$ *completely sequential on T* if

$$f \in \mathcal{Q} \text{ iff } (\forall\{\mathbf{x}_n\} \subseteq T)[(\{\mathbf{x}_n\} \to \mathbf{x}_0) \implies f|\{\mathbf{x}_n\} \in \mathcal{Q}(\{\mathbf{x}_n\})].$$

Our interest rests on properties that are completely sequential; our theorem below contains a condition referring to completely sequential properties, that is, the condition is required to hold on convergent sequences with limit included (so on a compact set), rather than on arbitrary sequences.

Note that if $\mathcal{Q}$ is (completely) sequential then $f|\{\mathbf{x}_n\} \in \mathcal{Q}(\{\mathbf{x}_n\})$ iff $f|\{\mathbf{x}_n : n \in \mathbb{M}\} \in \mathcal{Q}(\{\mathbf{x}_n : n \in \mathbb{M}\})$, for every infinite $\mathbb{M}$. The theorem below gives conditions for the following *analytic thinning principle*: if

$$\mathcal{Q}(\mathbb{R}^d) \Longrightarrow \mathcal{P}(\mathbb{R}^d)$$

holds, then

$$\mathcal{Q}(T) \Longrightarrow \mathcal{P}(\mathbb{R}^d)$$

holds for analytic T.

Main Theorem (Analytic Automaticity Theorem). *Suppose that*

(a) *functions of $\mathcal{F}$ having the property $\mathcal{Q}$ on $\mathbb{R}^d$ have a property $\mathcal{P}$ on $\mathbb{R}^d$, where $\mathcal{Q}$ is a property of functions from $\mathbb{R}^d$ to $\mathbb{R}$ that is completely sequential on $\mathbb{R}^d$, and*

(b) *$\mathcal{F}$ preserves $\mathcal{Q}$ under vector addition and subtraction on compact sets and also under shift, that is:*

(i) *for compact sets* S *and* T, *functions of* $\mathcal{F}$ *having* $\mathcal{Q}$ *on* S *and* T *have* $\mathcal{Q}$ *on* $S \pm T$;

(ii) *functions of* $\mathcal{F}$ *having* $\mathcal{Q}$ *on any* $T \subseteq \mathbb{R}^d$ *have* $\mathcal{Q}$ *on* $\tau + T := \{\tau + t : t \in T\}$, *for any* $\tau \in \mathbb{R}^d$.

Then, for any analytic set T *spanning* $\mathbb{R}^d$ *as a vector space over* $\mathbb{Q}$ *(e.g. containing a Hamel basis), functions of* $\mathcal{F}$ *having* $\mathcal{Q}$ *on* T *have* $\mathcal{P}$ *on* $\mathbb{R}^d$.

Remark. In applications, as in the two examples that follow, the conditions (i) and (ii) need only be verified on compact sets arising as convergent sequences with limit points included, in view of the properties needing to be completely sequential. This is indeed the form that is relevant in examples, but we have not included an extra assertion here along these lines to avoid overburdening the statement of the theorem. See, however, Section 5.

Example 1. The class of additive functions, $\mathcal{A}dd$, *preserves* $\mathcal{C}$, *the continuous functions, under vector sums and differences on compact domains*, i.e. for $f \in \mathcal{A}dd$ and S, T compact, if $f|S \in \mathcal{C}(S)$ and $f|T \in \mathcal{C}(T)$ then $f|S \pm T \in \mathcal{C}(S \pm T)$. Indeed let $u_n = s_n \pm t_n \in S \pm T$. Then $\{s_n : n \in \omega\}$ and $\{t_n : n \in \omega\}$ are precompact sets. By compactness of S and T, without loss of generality we may assume that $s_n \to s \in S$ and $t_n \to t \in T$. Then, by additivity, $\lim f(s_n \pm t_n) = \lim[f(s_n) \pm f(t_n)]$, and by continuity $\lim[f(s_n) \pm f(t_n)] = f(s) \pm f(t)$. Thus f is continuous on $S \pm T$.

Example 2. The class of additive functions, $\mathcal{A}dd$, *preserves* $\mathcal{B}_{\mathrm{loc}}$, *the locally bounded functions, under vector sums and differences on compact domains*, i.e. for $f \in \mathcal{A}dd$ and S, T compact, if $f|S \in \mathcal{B}_{\mathrm{loc}}(S)$ and $f|T \in \mathcal{B}_{\mathrm{loc}}(T)$ then $f|S \pm T \in \mathcal{B}_{\mathrm{loc}}(S \pm T)$. The proof is similar to but simpler than that above.

Corollary (Theorems of Jones and Kominek). *Let* f *be additive and either have a continuous restriction, or a bounded restriction,* $f|T$, *where* T *is some analytic set spanning* $\mathbb{R}$. *Then* f *is continuous.*

Proof. Applying the Main Theorem, let $\mathcal{F}$ be $\mathcal{A}dd$, the family of additive functions, and $\mathcal{P}$ be $\mathcal{C}$, the family of continuous functions. Thus with $\mathcal{Q} = \mathcal{C}$ we obtain Jones' Theorem and with $\mathcal{Q} = \mathcal{B}_{\mathrm{loc}}$, the locally bounded functions, we obtain a sharpened form of Kominek's theorem. □

The use of spanning sets here is natural, since an additive function is specified by its values on a spanning set. This idea can be made precise.

Kominek ([Kom2, Th. 1]) also shows that nothing less than a spanning set will do here. This – 'Kominek's other theorem' – may be regarded as a converse to the Kominek theorem above.

We will need the following result, due in the measure case in this form to Borwein and Ditor [BoDi], but already known much earlier albeit in somewhat weaker form by Kestelman ([Kes, Th. 3]), and rediscovered by Trautner [Trau]. Much more is true, see [BOst6], [BOst9], [BOst11]. Following J-P Kahane [Kah], the term 'quasi all' below refers to 'all off some meagre set'.

Theorem (Kestelman-Borwein-Ditor Theorem). *Let $\{z_n\} \to 0$ be a null sequence of reals. If T is measurable and non-null/ Baire non-meagre, then for almost all/quasi all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

2 Some lemmas: expansion and contraction

For clarity's sake we work in $\mathbb{R}$ rather than $\mathbb{R}^d$. We begin with the common proof of the Jones and Kominek theorems as it is short, illuminating and depends on three simple lemmas, the second and third of which we need elsewhere (for the Souslin operation therein, see [Rog2]). The stronger result that Jones' Theorem implies Kominek's Theorem is deduced in the subsequent section on spannability.

Analytic Covering Lemma ([Kucz, p. 227], cf. [Jones2, Th. 11]). *Let T be analytic and let $f : \mathbb{R} \to \mathbb{R}$ have continuous restriction $f|T$. Then T is covered by a countable family of bounded analytic sets on each of which f is bounded.*

Proof. For $k \in \omega$ define $T_k := \{x \in T : |f(x)| < k\} \cap (-k, k)$. Now $\{x \in T : |f(x)| < k\}$ is relatively open and so takes the form $T \cap U_k$ for some open subset U_k of $\mathbb{R}$, giving the result since U_k is analytic. $\square$

Analytic Dichotomy Lemma (Spanning). *Suppose that an analytic set $T \subseteq \mathbb{R}$ spans a set of positive measure or a non-meagre set. Then T spans $\mathbb{R}$.*

Proof. If T spans $P \subseteq \mathbb{R}$, then

$$P \subseteq S := \bigcup_{m,h \in \omega, m>0} \bigcup_{\mathbf{r} \in \mathbb{Z}^h} \left(\frac{r_1}{m} T + ... + \frac{r_h}{m} T \right).$$

In the measure case, if P is non-null, it follows that, for some $h, m \in \mathbb{N}$ and $\mathbf{r} \in \mathbb{Z}^h$, the set

$$\frac{r_1}{m} T + ... + \frac{r_h}{m} T$$

has positive measure, and hence so does $S' = r_1 T + ... + r_h T$. By Steinhaus' Theorem ([St], [BGT, Th. 1.1.1], [BOst3]), $S' - S'$ contains an interval around the origin, and so T spans an interval, say $(-\frac{1}{k}, \frac{1}{k})$ for some $k \in \mathbb{N}$. Hence T spans $(-\frac{n}{k}, \frac{n}{k})$ for any $n \in \mathbb{N}$, i.e. T spans $\mathbb{R}$.

In the category case, S' is non-meagre and so, by the Pettis-Piccard Theorem ([Pic1], [Pic2], [Pet1], [BGT, Th. 1.1.1], [BOst3]), $S' - S'$ contains an interval around the origin, hence the similar result. $\square$

In the category case, the result may also be derived from the Banach-Kuratowski Dichotomy Theorem ([Ban, Satz 1], [Kur1, Ch. VI. 13. XII], [Kel, Ch. 6 Prob. P p. 211]) by considering S, the subgroup generated by T; since T is analytic, S is Baire and, being non-meagre, is clopen and hence all of $\mathbb{R}$, as the latter is a connected group.

Expansion Lemma ([Jones2, Th. 4], [Kom2, Th. 2], and [Kucz, p. 215]). *Suppose that S is Souslin-$\mathcal{H}$, i.e. of the form*

$$S = \bigcup_{\alpha \in \omega^\omega} \cap_{n=1}^{\infty} H(\alpha|n),$$

with each $H(\alpha|n) \in \mathcal{H}$, for some family of analytic sets $\mathcal{H}$ on which f is bounded. If S spans $\mathbb{R}$ as a vector space over $\mathbb{Q}$ (e.g. contains a Hamel basis), then for each n there are sets $H_1, ..., H_k$ each of the form $H(\alpha|n)$, such that for some integers $r_1, ..., r_k$

$$T = r_1 H_1 + ... + r_k H_k$$

has positive measure/ is non-meagre, and so $T - T$ contains an interval.

Proof. For any $n \in \omega$ we have

$$S \subseteq \bigcup_{\alpha \in \omega^\omega} H(\alpha|n).$$

Enumerate the countable family $\{H(\alpha|n) : \alpha \in \omega^n\}$ as $\{T_h : h \in \omega\}$. Since S spans $\mathbb{R}^d$, we have

$$\mathbb{R}^d = \bigcup_{h\in\omega} \bigcup_{\mathbf{k},\mathbf{s}\in\mathbb{N}^h} \bigcup_{\mathbf{r}\in\mathbb{Z}^h} \left(\frac{r_1}{s_1}T_{k_1} + ... + \frac{r_h}{s_h}T_{k_h}\right) = \bigcup_{m,h\in\omega,m>0} \bigcup_{\mathbf{k}\in\mathbb{N}^h} \bigcup_{\mathbf{r}\in\mathbb{Z}^h} \left(\frac{r_1}{m}T_{k_1} + ... + \frac{r_h}{m}T_{k_h}\right).$$

As each T_k is analytic, so too is the continuous image

$$\frac{r_1}{m}T_{k_1} + ... + \frac{r_h}{m}T_{k_h},$$

which is thus measurable. Hence, for some $h, m \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{N}^h$ and $\mathbf{r} \in \mathbb{Z}^h$ the set

$$\frac{r_1}{m}T_{k_1} + ... + \frac{r_h}{m}T_{k_h}$$

has positive measure/ is non-meagre, and hence $r_1T_{k_1} + ... + r_hT_{k_h}$ does the same. $\square$

The above results will be used to prove the Main Theorem; but we may now also obtain, as a direct corollary, a reformulation and strengthening of the two theorems which motivate this paper:

Theorem JK (Theorems of Jones and Kominek). *Let f be additive and either have a continuous restriction, or a bounded restriction $f|T$, where T is some analytic set spanning a set of positive measure/ a non-meagre set. Then f is continuous.*

Proof. By the Analytic Dichotomy Lemma, T spans $\mathbb{R}$. By the Expansion Lemma, in all cases, f is bounded on a set of the form $(r_1H_1 + ... + r_kH_k) - (r_1H_1 + ... + r_kH_k)$, which contains an interval. So by Darboux's Theorem f is continuous. $\square$

In particular, one has the theorems of Jones and Kominek in their original formulation:

Corollary (Theorems of Jones and Kominek). *Let f be additive and either have a continuous restriction, or a bounded restriction, $f|T$, where T is some analytic set spanning $\mathbb{R}$. Then f is continuous.*

3 On spannability theory

Our proof of the Analytic Automaticity Theorem relies on the Expansion Lemma of the last section and on an analysis of spanning properties of analytic sets. Here we view $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and for $S \subseteq \mathbb{R}$ denote by $\mathrm{Lin}_{\mathbb{Q}}(S)$ the linear span of S regarded as a set of vectors in $\mathbb{R}$ as a space over $\mathbb{Q}$. One might expect by analogy with classical theorems asserting that a 'large' analytic set contains a 'large' compact subset (cf. [Rog2, Part 1 Sect. 3.5], and [Kech, Ch. III 29.E]) that perhaps for analytic S the span $\mathrm{Lin}_{\mathbb{Q}}(S)$ is equal to $\mathrm{Lin}_{\mathbb{Q}}(F)$ with F either compact or σ-compact. We examine this intuition and find below that enough of this is true to enable a deduction of Jones' Theorem from Kominek's Theorem.

Our first lemma follows directly from Choquet's Capacitability Theorem [Ch4] (see especially [Del2, p. 186], and [Kech, Ch. III 30.C]). For completeness, we include the brief proof. Incidentally, the argument we employ goes back to Choquet's theorem, and indeed further, to [ROD] (see e.g. [Del1, p. 43]).

Compact Contraction Lemma. *For T analytic, if $T+T$ has positive Lebesgue measure, then for some compact subset S of T, $S+S$ has positive measure.*

Proof. We present a direct proof (see below for our original inspiration in Choquet's Theorem). As T^2 is analytic, we may write ([Rog2, p. 11]) $T^2 = h(H)$, for some continuous h and some $\mathcal{K}_{\sigma\delta}$ subset of the reals, e.g. the set H of the irrationals, so that $H = \bigcap_i \bigcup_j d(i,j)$, where $d(i,j)$ are compact and, without loss of generality, the unions are each increasing: $d(i,j) \subseteq d(i,j+1)$. The map $g(x,y) := x+y$ is continuous and hence so is the composition $f = g \circ h$. Thus $T+T = f(H)$ is analytic. Suppose that $T+T$ is of positive measure. Hence, by the capacitability argument for analytic sets ([Ch4], or [Si, Th.4.2 p. 774], or [Rog1, p. 90], there referred to as an 'Increasing sets lemma'), for some compact set A, the set $f(A)$ has positive measure. Indeed if $|f(H)| > \eta > 0$, then the set A may be taken in the form $\bigcap_i d(i,j_i)$, where the indices j_i are chosen inductively, by reference to the increasing union, so that $|f[H \cap \bigcap_{i<k} d(i,j_i)]| > \eta$, for each k. (Thus $A \subseteq H$ and $f(A) = \bigcap_i f[H \cap \bigcap_{i<k} d(i,j_i)]$ has positive measure, cf. [EKR]).

The conclusion follows as $S = h(A)$ is compact and $S+S = g(S) = f(A)$. □

Note. The result may be deduced indirectly from the Choquet Capacitability Theorem by considering the capacity $I : \mathbb{R}^2 \to \mathbb{R}$, defined by $I(X) = |g(X)|$, where, as before, $g(x,y) := x + y$ is continuous and $|.|$ denotes Lebesgue measure on $\mathbb{R}$ (on this point see [Del2, Section 1.1.1, p. 186]). Indeed, the set T^2 is analytic ([Rog2, Section 2.8, p. 37-41]), so $I(T^2) = \sup I(K^2)$, where the supremum ranges over compact subsets K of T. Actually, the Capacitability Theorem says only that $I(T^2) = \sup I(K_2)$, where the supremum ranges over compact subsets K_2 of T^2, but such a set may be embedded in K^2 where $K = \pi_1(K) \cup \pi_2(K)$, with π_i the projections onto the axes of the product space.

Corollary. *For T analytic and $\varepsilon_i \in \{\pm 1\}$, if $\varepsilon_1 T + ... + \varepsilon_d T$ has positive measure (measure greater than η) or is non-meagre, then for some compact subset S of T, the compact set $K = \varepsilon_1 S + ... + \varepsilon_d S$ has $K + K$ of positive measure (measure greater than η).*

Proof. In the measure case the same approach may be used based now on the continuous function $g(x_1, ..., x_d) := \varepsilon_1 x_1 + ... + \varepsilon_d x_d$ ensuring that K is of positive measure (measure greater than η). In the category case, if $T' = \varepsilon_1 T + ... + \varepsilon_d T$ is non-meagre then, by the Steinhaus Theorem ([St], or [BGT, Cor. 1.1.3]), $T' + T'$ contains an interval. The measure case may now be applied to T' in lieu of T. (Alternatively one may apply the Pettis-Piccard Theorem, as in the Analytic Dichotomy Lemma.) $\square$

Theorem (Compact Spanning Approximation). *For T analytic, if the linear span of T is non-null or is non-meagre, then there exists a compact subset of T which spans all the reals. If T is symmetric about the origin, then the compact spanning subset may be taken symmetric.*

Proof. If T is non-null or non-meagre, then T spans all the reals (by the Analytic Dichotomy Lemma); then for some $\varepsilon_i \in \{\pm 1\}$, $\varepsilon_1 T + ... + \varepsilon_d T$ has positive measure/ is non-meagre. Hence for some K compact $\varepsilon_1 K + ... + \varepsilon_d K$ has positive measure/ is non-meagre. Hence K spans some and hence all reals.

Let T be symmetric. If T spans the reals, then so does $T_+ = T \cap \mathbb{R}_+$. Choose a compact $K_+ \subseteq T_+$ to span the reals. Then $K := K_+ \cup (-K_+) \subseteq T$ is compact, symmetric and spans the reals. $\square$

As a corollary, we deduce the relation between the theorems of Jones and Kominek.

Corollary. *Kominek's Theorem implies Jones's Theorem.*

Proof. If T is an analytic spanning set, then it contains a compact spanning set K. If f is continuous on T, then f is bounded on the compact set K. By Kominek's Theorem, as f is additive and bounded on a compact spanning set, f is continuous. $\square$

We continue with regard to the question of whether, for T analytic, there is a compact $K \subseteq T$ such that $\mathrm{Lin}_{\mathbb{Q}}(T) = \mathrm{Lin}_{\mathbb{Q}}(K)$. Evidently, the question really relates to analytic sets T with $\mathrm{Lin}_{\mathbb{Q}}(T)$ (also analytic) of measure zero or meagre (as T has the Baire property), since the case where $\mathrm{Lin}_{\mathbb{Q}}(T)$ has positive measure, or is non-meagre, is settled positively by the Compact Spanning Approximation Theorem. Note that if $W := \mathrm{Lin}_{\mathbb{Q}}(T) = \mathrm{Lin}_{\mathbb{Q}}(K)$ for some compact K, then $\mathrm{Lin}_{\mathbb{Q}}(T)$ is σ-compact. Laczkovich [Lacz] shows that any proper analytic subgroup of the reals is covered by a null σ-compact set, so the σ-compact structure is not surprising. Indeed, as Roy Davies has observed (private communication), there are σ-compact proper additive subgroups A of the reals which can be covered by the sums $F, F+F, F+F+F, ...$ for some closed set F with $F = -F$, none of which contains a non-empty interval. For example A may be the subgroup generated by F. The earliest such example is due to Sierpiński ([Sierp1]). We are thus led to:

Theorem (Compact Spanning Theorem). *If the subspace W of $\mathbb{R}$ is both σ-compact and $\mathcal{G}_\delta$ there exists a compact set K in W such that $W = \mathrm{Lin}_{\mathbb{Q}}(K)$.*

Proof. This result follows from the generalized Piccard theorem (see e.g. [Kom1]), since W is completely metrizable ([Eng, Section 4.3]). Indeed, if $W = \bigcup_{n\in\omega} K_n$ with each K_n compact, then for some $n \in \omega$ the set K_n is non-meagre in W. So by Piccard's Theorem $K_n + K_n$ has non-empty interior in the topological space W. Let us suppose that W contains the relatively open interval of points $J \cap W$. Note that for any $w \in W$, $(w + J) \cap W = w + (J \cap W) \subseteq W$, as W is a vector space. Hence, for any $w \in J \cap W$, putting $I = J - w = (-a, +b)$, we have $0 \in I \cap W$ and $\mathbb{R} = \bigcup_{m\in\omega} mI$. Thus $W = \bigcup_{m\in\omega} m(I \cap W)$. We deduce that $J \cap W$ spans W, and the theorem follows since $K_n + K_n$ is compact. $\square$

Of course, the above theorem addresses not only the 'small case' of null and meagre subspaces of ambiguous Borel class one ([Kech, II.11.A]), but also the 'large case', e.g. the case when $W = \mathbb{R}$. Indeed, combining Th.6.3.3 and Th.6.3.4 of Jayne and Rogers in [Rog2], uncountable $\mathcal{G}_\delta$ sets that are σ-compact are characterized up to first-level Borel isomorphism uniquely as copies of $[0,1]$. Interest in the $\mathcal{G}_\delta$ case here is motivated by Solecki's analytic dichotomy theorem below.

The above result is perhaps the best one may hope for – for two reasons. First of all, we refer to a further result of Laczkovich in [Lacz], where he gives an example of a null Borel subgroup G of $\mathbb{R}$ with the property that, for any σ-compact cover $\{K_n : n \in \omega\}$, there is n such that $K_n + K_n$ contains an interval. By a simple modification G may be assumed to be a vector subspace (e.g. $W := \bigcup_{m\in\omega} \frac{1}{m+1} G$ is a vector subspace which is also Borel and, being null, is proper). It is not, however, σ-compact, as otherwise the representation $\bigcup_{n\in\omega} K_n$ would force, for some n, the sum $K_n + K_n$ to contain an interval, thus contradicting the fact that the subspace is null. It would be interesting to know whether there exists a $\mathcal{G}_\delta$ proper vector subspace of $\mathbb{R}$ which is not σ-compact. (Lavrentieff's theorem on the topological invariance of $\mathcal{G}_\delta$ sets – see [Eng, p. 276] – would at best permit a direct modification to Laczkovich's construction to yield a $\mathcal{G}_{\delta\sigma}$ proper vector subspace which is not σ-compact.)

Secondly, there is *Solecki's analytic dichotomy theorem* (reformulating and generalizing a specific instance discovered by Petruska, [Pet]) as follows. For $\mathcal{I}$ a family of closed sets (in any Polish space), let $\mathcal{I}_{\text{ext}}$ denote the sets covered by a countable union of sets in I. Then, for A an analytic set, either $A \in \mathcal{I}_{\text{ext}}$, or A contains a $\mathcal{G}_\delta$ set not in $\mathcal{I}_{\text{ext}}$. See [Sol1], where a number of classical theorems, asserting that a 'large' analytic set contains a 'large' compact subset, are deduced, and also [Sol2] for further applications of dichotomy.

We note that, by an appeal directly to Petruska's theorem in [Pet], Laczkovich [Lacz] shows that any proper analytic subgroup of the reals is covered by a null σ-compact set.

4 Proof of the Main Theorem

Suppose that T is analytic and, for simplicity, contains a Hamel basis. Let $h \in \mathcal{F}$ be such that for every $\{\mathbf{x}_n\} \to \mathbf{x}_0$ with $\{\mathbf{x}_n\} \subseteq T$ (i.e. for every convergent sequence which together with its limit lies in T), we have $h|\{\mathbf{x}_n\} \in \mathcal{Q}(\{\mathbf{x}_n\})$, but that $h \notin \mathcal{P}$. Then $h \notin \mathcal{Q}$ (by the hypothesis that $\mathcal{Q} \implies \mathcal{P}$). Since $\mathcal{Q}$ is completely sequential, there is a convergent sequence $\{\mathbf{u}_m : m \in \omega\}$

such that $h|\{\mathbf{u}_m : m \in \omega\} \notin \mathcal{Q}$.

Put $T_k = T \cap (-k, k)$. Since T contains a Hamel basis, we have by the Expansion Lemma of Section 2 that, for some $k, n \in \mathbb{N}$ and $\mathbf{r} \in \mathbb{Z}^n$, the set

$$r_1 T_k + ... + r_n T_k$$

has positive measure. By the Compact Contraction Lemma of Section 3 there is a compact subset S of T_k such that $r_1 S + ... + r_n S$ has positive measure.

By the Kestelman-Borwein-Ditor Theorem, for some $t \in S \subseteq T$ and for $i = 1, ..., n$, there are sequences $\{\mathbf{v}_i^m : m \in \mathbb{M}\} \subseteq S \subseteq T$ such that

$$t + \mathbf{u}_m = r_1 \mathbf{v}_1^m + ... + r_n \mathbf{v}_n^m.$$

By the local compactness of $\mathbb{R}^m$, the compactness of S and passage to an infinite subset $\mathbb{M}' \subseteq \mathbb{M}$, we may assume that each sequence $\mathbf{v}_i = \{\mathbf{v}_i^m : m \in \mathbb{M}\}$ is convergent to a point of S. As each $\mathbf{v}_i$ is in T, it follows that $h|\{\mathbf{v}_n^i\} \in \mathcal{Q}(\{\mathbf{v}_n^i\})$. Hence, since $\mathcal{F}$ preserves $\mathcal{Q}$ under shift and under vector addition and subtraction on the sets $\{\mathbf{v}_i^m : m \in \mathbb{M}\}$, currently assumed compact, $h|\{\mathbf{u}_m\}_{m \in \mathbb{M}} \in \mathcal{Q}(\{\mathbf{u}_m\}_{m \in \mathbb{M}})$. But this contradicts $h \notin \mathcal{Q}$, since $h|\{\mathbf{u}_m : m \in \omega\} \in \mathcal{Q}$ iff $h|\{\mathbf{u}_m : m \in \mathbb{M}\} \in \mathcal{Q}$ for every infinite $\mathbb{M}$. Hence after all $h \in \mathcal{P}$. $\square$

5 Variants, ideals under $\mathbb{R}^d$-shifts

Here we point to two generalizations. The first such, Theorem 1, makes explicit reference to the sequential combinatorics used in the proof of Section 4. The second, Theorem 2, restricts attention to analytic, spanning sets T which are 'shifted-symmetric'.

Definitions. Let $c(\mathbb{R}^d)$ denote the *the sequence space* of $\mathbb{R}^d$, i.e. the additive group of convergent sequences $\mathbf{u} = \{u_n\}$ of vectors in $\mathbb{R}^d$ (with term-wise addition). A subgroup $\mathcal{G}$, invariant under the action of (termwise) shifts by elements of $\mathbb{R}^d$, will be called an $\mathbb{R}^d$-*shift ideal (in the sequence space* of $\mathbb{R}^d$); it will be called a *complete* $\mathbb{R}^d$-*shift ideal* if it is also closed under subsequence formation, that is:

(i) $\mathbf{u}, \mathbf{v} \in \mathcal{G}$ implies that $\mathbf{u} \pm \mathbf{v} \in \mathcal{G}$ (subgroup property),

(ii) $\mathbf{u} \in \mathcal{G}$ implies $t + \mathbf{u} = \{t + u_n\} \in \mathcal{G}$, for each t in $\mathbb{R}^d$ ($\mathbb{R}^d$-shift invariance),

(iii) $\mathbf{u} \in \mathcal{G}$ implies that $\mathbf{u}_{\mathbb{M}} = \{u_m : m \in \mathbb{M}\} \in \mathcal{G}$, for every infinite $\mathbb{M}$ (completeness).

Definition. Say that a sequence $\mathbf{u} = \{u_n\}$ is $\mathcal{Q}$-*good* for h if

$$h|\{u_n\} \in \mathcal{Q}|\{u_n\},$$

and put

$$\mathcal{G}_{h\mathcal{Q}} = \{\mathbf{u} : h|\{u_n\} \in \mathcal{Q}|\{u_n\}\}.$$

If $\mathcal{Q}$ is completely sequential, then $\mathbf{u}$ is $\mathcal{Q}$-good for h iff every subsequence of $\mathbf{u}$ is $\mathcal{Q}$-good for h. One then has:

Lemma. *If $\mathcal{Q}$ is completely sequential and $\mathcal{F}$ preserves $\mathcal{Q}$ under shift and under vector addition and subtraction on compacts, then $\mathcal{G}_{h\mathcal{Q}}$ for $h \in \mathcal{F}$ is a complete $\mathbb{R}^d$-shift ideal.*

Example 1. The convergent sequences form a complete $\mathbb{R}^d$-shift ideal, and so do the eventually constant sequences. We now refer to the examples of Section 1.

Example 2. For $\mathcal{Q} = \mathcal{B}_{\text{loc}}$, and $\mathcal{F} = \mathcal{A}dd$, then $\mathcal{Q}$ is sequential, $\mathcal{F}$ preserves $\mathcal{Q}$ under vector addition and subtraction on bounded sets and also under shift and , so that $\mathcal{G}_{h\mathcal{Q}}$ is a complete $\mathbb{R}^d$-shift ideal when $h \in \mathcal{F}$.

Example 3. For $\mathcal{Q} = \bar{\mathcal{C}}$, and $\mathcal{F} = \mathcal{A}dd$, then $\mathcal{Q}$ is sequential, $\mathcal{F}$ preserves $\mathcal{Q}$ under vector addition and subtraction on bounded sets and also under shift, so that $\mathcal{G}_{\mathcal{Q}}$ is a complete $\mathbb{R}^d$-shift ideal.

Example 4. In the case of the slowly varying functions, i.e. for

$$\mathcal{F} = \mathcal{S}v = \{h : (\forall\{\mathbf{x}_n \to \infty\})(\forall u) \lim_{n\to\infty} |h(u + x_n) - h(x_n)| = 0\},$$

and for h slowly varying ($h \in \mathcal{S}v$), let $\mathcal{G}_h$ be the set of convergent sequences $\mathbf{u} = \{u_n\}$ good for h, i.e. those satisfying: for all $\{x_n\} \to \infty$,

$$\lim_{n\to\infty} |h(u_n + x_n) - h(x_n)| = 0.$$

In [BOst6] we show that $\mathcal{G}_h$ for $h \in \mathcal{F}$ is a complete $\mathbb{R}^d$-shift ideal.

Theorem 1. (Analytic Automaticity Theorem - combinatorial form).

Suppose that

(a) *functions of $\mathcal{F}$ having $\mathcal{Q}$ on $\mathbb{R}^d$ have $\mathcal{P}$ on $\mathbb{R}^d$, where $\mathcal{Q}$ is a property of functions from $\mathbb{R}^d$ to $\mathbb{R}$ that is completely sequential on $\mathbb{R}$;*

(b) *for all $h \in \mathcal{F}$, the family $\mathcal{G}_{h\mathcal{Q}}$ of $\mathcal{Q}$-good sequences is a complete $\mathbb{R}^d$-shift ideal (closed under vector addition and subtraction, invariant under shift, closed under subsequence formation).*

Then, for any analytic set T spanning $\mathbb{R}$ as a vector space over $\mathbb{Q}$ (e.g. containing a Hamel basis), functions of $\mathcal{F}$ having $\mathcal{Q}$ on T have $\mathcal{P}$ on $\mathbb{R}^d$.

Proof of Theorem 1. This is a rephrasing of the main theorem in the language of $\mathbb{R}^d$-shift ideals. $\square$

Theorem 2. (Symmetric Analytic Automaticity Theorem).
Suppose that

(a) *functions of $\mathcal{F}$ having the property $\mathcal{Q}$ on $\mathbb{R}^d$ have a property $\mathcal{P}$ on $\mathbb{R}^d$, where $\mathcal{Q}$ is a property of functions from $\mathbb{R}^d$ to $\mathbb{R}$ that is completely sequential on $\mathbb{R}^d$,*

(b) *$\mathcal{F}$ preserves $\mathcal{Q}$ under vector addition on compact sets and also under shift, that is:*

(i), *for compact sets S and T, functions of $\mathcal{F}$ having $\mathcal{Q}$ on S and T have $\mathcal{Q}$ on $S+T$;*

(ii) *functions of $\mathcal{F}$ having $\mathcal{Q}$ on any $T \subseteq \mathbb{R}^d$ have $\mathcal{Q}$ on $\tau+T := \{\tau+t : t \in T\}$, for any $\tau \in \mathbb{R}^d$;*

Then, for any analytic set T spanning $\mathbb{R}^d$ as a vector space over $\mathbb{Q}$ (e.g. containing a Hamel basis) such that, for some τ, $S := \tau + T$ is symmetric (i.e. $S = -S$), functions of $\mathcal{F}$ having $\mathcal{Q}$ on T have $\mathcal{P}$ on $\mathbb{R}^d$.

Proof of Theorem 2. Since $S = \tau + T$ is an analytic spanning set if T is, we may as well assume by (ii) that T is in fact symmetric. It follows that in the Expansion Lemma all the factors r_i may be taken positive. In this case the proof of the Main Theorem requires only that $\mathcal{F}$ preserves $\mathcal{Q}$ under vector addition on compact sets and also under shift. $\square$

Note. 1. In [BOst6] we show that Theorem 2 applies also to subadditive functions and to convex functions.

2. Much of the work here carries over from the present Euclidean setting to topological groups. We will develop this programme elsewhere.

3. We thank Anatole Beck for the eventual constancy part of Example 1 of this section.

Postcript. Dellacherie ([Del1, p.43]) points out the important yet neglected work on capacities of R. O. Davies [ROD] in 1952, at the same time as Choquet's three notes ([Ch1]–[Ch3]) in the *Comptes Rendus*, and well before his seminal paper of 1955 [Ch4]. Roy Davies has been a long-standing friend, mentor and collaborator of the second author. It is a pleasure for both authors to dedicate this paper to him, on the occasion of his eightieth birthday.

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