STABLE RANKS OF BANACH ALGEBRAS OF OPERATOR-VALUED H^∞ FUNCTIONS

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ABSTRACT. Let E be an infinite-dimensional Hilbert space, and let $H^\infty_{\mathcal{L}(E)}$ denote the Banach algebra of all functions $f : \mathbb{D} \to \mathcal{L}(E)$ that are holomorphic and bounded, equipped with the supremum norm $\|f\|_\infty := \sup_{z\in\mathbb{D}} \|f(z)\|_{\mathcal{L}(E)}$, $f \in H^\infty_{\mathcal{L}(E)}$. We show that the Bass and topological stable ranks of $H^\infty_{\mathcal{L}(E)}$ are infinite. If S is an open subset of $\mathbb{T}$, then let $A^S_{\mathcal{L}(E)}$ denote the subalgebra of $H^\infty_{\mathcal{L}(E)}$ of all functions that have a continuous extension to S. We also prove that $A^S_{\mathcal{L}(E)}$ has infinite Bass and topological stable ranks.

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1. INTRODUCTION

In this paper, we prove that the Bass and topological stable ranks of the Banach algebras $H^\infty_{\mathcal{L}(E)}$ and $A^S_{\mathcal{L}(E)}$ (Definition 1.3) are all infinite when E is infinite-dimensional.

The notions of Bass/topological stable ranks play important roles in algebraic/topological K-theory (see [1] and [10]), but they also have applications in the control-theoretic problem of stabilization via a factorization approach (see [18], [5], [9]). We recall the definition of Bass stable rank and topological stable rank below.

Definition 1.1. Let $\mathcal{A}$ be a ring with identity e and $n \in \mathbb{N}$. An element $a = (a_1, \dots, a_n) \in \mathcal{A}^n$ is called *(left) unimodular* if there exists $b = (b_1, \dots, b_n) \in \mathcal{A}^n$ such that

$$\sum_{k=1}^{n} b_k a_k = e. \tag{1.1}$$

We denote the set of unimodular elements of $\mathcal{A}^n$ by $U_n(\mathcal{A})$.

An element $a \in U_{n+1}(\mathcal{A})$ is called *(left) reducible* if there exists $x = (x_1, \dots, x_n) \in \mathcal{A}^n$ such that

$$(a_1 + x_1 a_{n+1}, \dots, a_n + x_n a_{n+1}) \in U_n(\mathcal{A}). \tag{1.2}$$

The *(left) Bass stable rank of* $\mathcal{A}$, denoted by bsr $\mathcal{A}$, is the least integer $n \geq 1$ such that every $a \in U_{n+1}(\mathcal{A})$ is reducible, and it is infinite if no such integer n exists.

Now let $\mathcal{A}$ denote a Banach algebra[1]. The *(left) topological stable rank of* $\mathcal{A}$, denoted by tsr $\mathcal{A}$, is the least integer $n \geq 1$ such that $U_n(\mathcal{A})$ is dense in $\mathcal{A}^n$, and it is infinite if no such integer exists.

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[1]By a Banach algebra we mean a complex Banach algebra with a unit element e; we do not assume commutativity.

Remark 1.2. Analogously one can define a *right* Bass/topological stable rank, by changing the multiplication order in (1.1).

It turns out that for any ring $\mathcal{A}$ the left Bass stable rank is always equal to the right stable rank (see [19]).

Moreover, it is known (see [10, Proposition 1.6]) that the left and right tsr's are equal whenever one has a Banach algebra with a continuous involution $\star$, and so in this case one can unambiguously talk about the tsr. In our case of $H^\infty_{\mathcal{L}(E)}$ and $A_{\mathcal{L}(E)}$, we use the involution $\cdot^\star$ defined as follows: $f^\star = (f(\overline{\cdot}))^*$, and $\cdot^*$ denotes the adjoint.

In this article we will study the Bass and topological stable ranks of the Banach algebras $A^S_{\mathcal{L}(E)}$, which we define below. We will denote the unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$ by $\mathbb{D}$, the closed unit disk $\{z \in \mathbb{C} \mid |z| \leq 1\}$ by $\overline{\mathbb{D}}$, and the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ by $\mathbb{T}$. Throughout this article, unless otherwise stated, E is an infinite-dimensional complex Hilbert space. We use the notation $\mathcal{L}(X, Y)$ for the complex Banach space of bounded linear operators between Hilbert spaces X to Y, equipped with the operator norm.

Definition 1.3. Let $H^\infty_{\mathcal{L}(E)}$ denote the Banach algebra of functions $f : \mathbb{D} \to \mathcal{L}(E)$ that are holomorphic and bounded, equipped with the supremum norm $\|f\|_\infty := \sup_{z\in\mathbb{D}} \|f(z)\|_{\mathcal{L}(E)}$.

The Banach algebra $A_{\mathcal{L}(E)}$ is defined as the subalgebra of $H^\infty_{\mathcal{L}(E)}$ of all functions $f \in H^\infty_{\mathcal{L}(E)}$ that have a continuous extension to $\mathbb{T}$.

More generally, let S be an open subset of $\mathbb{T}$. The Banach algebra $A^S_{\mathcal{L}(E)}$ is defined as the subalgebra of $H^\infty_{\mathcal{L}(E)}$ of all functions $f \in H^\infty_{\mathcal{L}(E)}$ that have a continuous extension to S.

If $E = \mathbb{C}$, then we denote $H^\infty_{\mathcal{L}(E)}$, $A_{\mathcal{L}(E)}$, simply by H^∞, A, respectively.

Sergei Treil proved that bsr $H^\infty = 1$ [16], and Daniel Suárez showed that tsr $H^\infty = 2$ [15]. The Bass stable rank of the ring of all finite square matrices of size n with entries from the ring $\mathcal{A}$ is related to the Bass stable rank of $\mathcal{A}$ [17, Theorem 3]:

$$\text{bsr } \mathcal{A}^{n\times n} = \lfloor -(\text{bsr } \mathcal{A} - 1)/n \rfloor + 1. \tag{1.3}$$

Here for $r \in \mathbb{R}$, $\lfloor r \rfloor$ denotes the largest integer less than or equal to r. So when E is finite-dimensional, bsr $H^\infty_{\mathcal{L}(E)} = 1$. There is also a similar relation relating the tsr 's when $\mathcal{A}$ is a Banach algebra with a continuous involution (see the proof of [10, Theorem 6.1]):

$$\text{tsr } \mathcal{A}^{n\times n} = \lceil (\text{tsr } \mathcal{A} - 1)/n \rceil + 1. \tag{1.4}$$

Here $\lceil r \rceil$ denotes the least integer greater than r. Hence tsr $H^\infty_{\mathcal{L}(E)} = 2$ when E is finite-dimensional.

The above results are also known for the disk algebra A: bsr $A = 1$ [7], tsr $A = 2$ [10]. Using the relations (1.3) and (1.4) above, we also have bsr $A_{\mathcal{L}(E)} = 1$ and tsr $A_{\mathcal{L}(E)} = 2$ when E is finite-dimensional.

In this article our main result is the following, which is proved in Section 2:

Theorem 1.4. *If* $\dim E = \infty$*, then* bsr $H^\infty_{\mathcal{L}(E)} = \infty$.

It is known that bsr $\mathcal{A}$ is less than or equal to the minimum of the left and right tsr's of $\mathcal{A}$ [10, Corollary 2.4]. So we have the following:

Corollary 1.5. *If* $\dim E = \infty$*, then* tsr $H^\infty_{\mathcal{L}(E)} = \infty$.

Analogously, we also show that bsr $A^S_{\mathcal{L}(E)} =$ tsr $A^S_{\mathcal{L}(E)} = \infty$ in Section 3 (disk algebra case; $S = \mathbb{T}$) and Section 4 (general S).

2. Proof of Theorem 1.4

In this section we will prove our main result (Theorem 1.4) that bsr $H^\infty_{\mathcal{L}(E)} = \infty$ when $\dim E = \infty$. The proof is similar to that of [2, Corollary, p. 292] that bsr $\mathcal{L}(E) = \infty$ if E is an infinite-dimensional Hilbert space. We will use the following result [2, Theorem 2]:

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra with Bass stable rank not greater than n. Then for any $m \geq n+1$, $U_m(\mathcal{A})$ is connected by arcs in $\mathcal{A}^m$.*

If $a \in U_m(\mathcal{A})$, let t_a be the map $t_a : GL_m(\mathcal{A}) \to U_m(\mathcal{A})$ given by $t_a(\sigma) = \sigma(a)$, $\sigma \in GL_m(\mathcal{A})$. Then the map t_a is surjective.

In the above, $GL_m(\mathcal{A})$ denotes the group of all invertible elements in the set of all matrices of size $m \times m$ in the ring $\mathcal{A}$. If $m = 1$, then we will denote the set $GL_m(\mathcal{A})$ simply by $GL(\mathcal{A})$.

We begin by showing that $GL(H^\infty_{\mathcal{L}(E)})$ is path connected.

Theorem 2.2. *$GL(H^\infty_{\mathcal{L}(E)})$ is path connected.*

In order to prove this theorem, we will use the characterization of $H^\infty_{\mathcal{L}(E)}$ functions as being those bounded linear operators on H^2_E that commute with the shift operator [11]. We quote this result below, but first we recall the definition of H^2_E and the shift operator:

Definition 2.3. We denote by H^2_E the Hilbert space of functions $f : \mathbb{D} \to E$ that are holomorphic in $\mathbb{D}$ such that

$$\|f\|_2 := \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_E^2 d\theta \right)^{1/2} < \infty.$$

The *shift operator* S is the bounded linear transformation on H^2_E of multiplication by z.

We recall the following result [11, Theorem B, Section 1.15]:

Theorem 2.4. *A bounded linear operator T on H^2_E commutes with S iff T is the multiplication map Λ_g (that is, $f \mapsto \Lambda_g(h) = gh$, $h \in H^2_E$) for some $g \in H^\infty_{\mathcal{L}(E)}$. Moreover, in this case $\|T\|_{\mathcal{L}(H^2_E)} = \|g\|_\infty$.*

Proof of Theorem 2.2. We prove that given $f \in GL(H^\infty_{\mathcal{L}(E)})$, there exists a continuous $\varphi : [0,1] \to GL(H^\infty_{\mathcal{L}(E)})$ such that $\varphi(0) = I$ and $\varphi(1) = f$.

Given $g \in H^\infty_{\mathcal{L}(E)}$, let $\Lambda_g \in \mathcal{L}(H^2_E)$ denote the multiplication map by g. Then Λ_g commutes with the shift operator S on H^2_E.

By [12, Theorem 12.35.(a)], it follows that the invertible operator $\Lambda_f \in \mathcal{L}(H^2_E)$ has a unique polar decomposition $T = UP$, where P is the invertible positive square root of $\Lambda_f^*\Lambda_f$, and $U = \Lambda_f P^{-1}$ is an invertible unitary operator. Also P commutes with every operator that Λ_f commutes with (see for example [12, Section 12.24]), and in particular, P commutes with the shift operator S. Hence U also commutes with S.

Since $\sigma(P) \subset (0,\infty)$, log is a continuous real function on $\sigma(P)$. It follows from the symbolic calculus [12, Section 12.24] that there is a self-adjoint $X \in \mathcal{L}(H^2_E)$ such that $P = e^X$. Moreover, by [12, Section 12.24] it follows that X also commutes with S. Since U is unitary, $\sigma(U)$ lies on the unit circle, so that there is a real bounded Borel function ψ on $\sigma(U)$ that satisfies $e^{i\psi(\lambda)} = \lambda$, $\lambda \in \sigma(U)$. Put $Y = \psi(U)$. Then $Y \in \mathcal{L}(H^2_E)$ is self-adjoint, $U = e^{iY}$, and Y commutes with S. Thus $\Lambda_f = UP = e^{iY}e^X$. Now define $T : [0,1] \to \mathcal{L}(H^2_E)$ by $T(t) = e^{itY}e^{tX}$. Since X and Y commute with S, so does $T(t)$. Moreover, $T(t)$ is invertible in $\mathcal{L}(H^2_E)$ for each t: $T(t)^{-1} = e^{-tX}e^{-itY}$. By Theorem 2.4, it follows that for each $t \in [0,1]$, there exists a $\varphi(t) \in H^\infty_{\mathcal{L}(E)}$ such that $T(t) = \Lambda_{\varphi(t)}$, and $\|T(t)\|_{\mathcal{L}(H^2_E)} = \|\varphi(t)\|_\infty$. Since $T(t)$ is invertible, it can be seen that $\varphi(t) \in GL(H^\infty_{\mathcal{L}(E)})$. Since $\varphi(0) = I$ and $\varphi(1) = f$, we conclude that φ can be taken as the desired path. $\square$

Theorem 2.5. *Let $n \in \mathbb{N}$. Then $GL_n(H^\infty_{\mathcal{L}(E)})$ is connected.*

Proof. The set of matrices of size $n \times n$ with entries in $H^\infty_{\mathcal{L}(E)}$ is isomorphic to $H^\infty(\mathcal{L}(E^n))$ as a Banach algebra, and so $GL_n(H^\infty_{\mathcal{L}(E)})$, being homeomorphic to $GL(H^\infty(\mathcal{L}(E^n)))$, is connected by Theorem 2.2 above. $\square$

Theorem 2.6. *Let $\mathcal{A}$ be a Banach algebra with identity e.*

(1) *$GL(\mathcal{A})$ is a closed subset of $U_1(\mathcal{A})$.*
(2) *$U_1(\mathcal{A}^n)$ is homeomorphic to $(U_1(\mathcal{A}))^n$.*

Proof. (1) Suppose $a_n \in GL(\mathcal{A})$ and $a_n \to a \in U_1(\mathcal{A})$. If $ba = e$, then $ba_n \to (ba =)e$. Since $GL(\mathcal{A})$ is open, there exists a N such that for all $n \geq N$, $ba_n \in GL(\mathcal{A})$. Thus $b = (ba_n)a_n^{-1} \in GL(\mathcal{A})$ and $a = b^{-1}(ba) \in GL(\mathcal{A})$.

(2) It is known that if $\mathcal{A}$ is a Banach algebra with unit and X is a compact Hausdorff space, then $U_1(C(X, \mathcal{A}))$ is homeomorphic to $C(X, U_1(\mathcal{A}))$ [4, Theorem 2.4]. Here the notation $C(X, Y)$ means the space of all continuous maps from X to the Banach space Y, equipped with the supremum norm: $\|f\|_{C(X,Y)} = \max_{x \in X} \|f(x)\|_Y$. Taking $X = \{1, \dots, n\}$ with the discrete topology, the result follows. $\square$

Proof of Theorem 1.4. Since $GL(H^\infty_{\mathcal{L}(E)})$ is

(1) a connected subset of $U_1(H^\infty_{\mathcal{L}(E)})$ (Theorem 2.2),
(2) an open subset of $U_1(H^\infty_{\mathcal{L}(E)})$ ([12, Theorem 10.12]), and
(3) a closed subset of $U_1(H^\infty_{\mathcal{L}(E)})$ (Theorem 2.6),

we conclude that $GL(H^\infty_{\mathcal{L}(E)})$ is a connected component of $U_1(H^\infty_{\mathcal{L}(E)})$.

We observe that $GL(H^\infty_{\mathcal{L}(E)}) \neq U_1(H^\infty_{\mathcal{L}(E)})$ because E is infinite-dimensional. Since we have proved above that $GL(H^\infty_{\mathcal{L}(E)})$ is a connected component of $U_1(H^\infty_{\mathcal{L}(E)})$, it follows that $U_1(H^\infty_{\mathcal{L}(E)})$ is disconnected. For topological spaces X_α, if $\prod X_\alpha$ is connected and nonempty, then each X_α is connected [8, Exercise 2, p. 151]. Consequently, $(U_1(H^\infty_{\mathcal{L}(E)}))^n$ is disconnected for all $n \in \mathbb{N}$.

From Theorem 2.6, we know that $(U_1(H^\infty_{\mathcal{L}(E)}))^n$ is homeomorphic to $U_n(H^\infty_{\mathcal{L}(E)})$. So from the above, $U_n(H^\infty_{\mathcal{L}(E)})$ is disconnected as well for all $n \in \mathbb{N}$. Now let $m \in \mathbb{N}$ and $f \in U_m(H^\infty_{\mathcal{L}(E)})$. Consider the map $t_f : GL_m(H^\infty_{\mathcal{L}(E)}) \to U_m(H^\infty_{\mathcal{L}(E)})$ given by $t_f(A) = Af$. Then t_f is continuous [3, Lemma 1.2.(ii)]. From Theorem 2.5, we know that $GL_m(H^\infty_{\mathcal{L}(E)})$ is connected. But the image of a connected space under a continuous map is connected [8, Theorem 1.5, Section 3-1], and so t_f can never be surjective. From Theorem 2.1, it follows that bsr $H^\infty_{\mathcal{L}(E)} = \infty$. $\square$

3. Disk algebra

In this section we prove that the Bass and topological stable ranks of the disc algebra $A_{\mathcal{L}(E)}$ are infinite. (Although this is a special case of the more general result in the next section, we include a separate proof here since it is simpler than the most general case.)

Theorem 3.1. *If $\dim E = \infty$, then* $\text{bsr} A_{\mathcal{L}(E)} = \infty$.

Proof. The proof is the same, mutatis mutandi, to the proof of Theorem 1.4, and the only key question is that of the connectedness of $GL(A_{\mathcal{L}(E)})$. This can be proved using the corresponding result for $H^{\infty}_{\mathcal{L}(E)}$ as follows. Let $f \in GL(A_{\mathcal{L}(E)})$. We want to find a path in $GL(A_{\mathcal{L}(E)})$ that joins f to I. Choose $\epsilon > 0$ such that the ball $\{g \in A_{\mathcal{L}(E)} \mid \|g - f\|_\infty < \epsilon\}$ is contained in $GL(A_{\mathcal{L}(E)})$ (this can be done since $GL(A_{\mathcal{L}(E)})$ is open in the Banach algebra $A_{\mathcal{L}(E)}$). It is clear that given any such g, f can be connected to g by a path in $GL(A_{\mathcal{L}(E)})$. Now we choose a special g, namely one which is holomorphic across $\mathbb{T}$. To do this, we simply dilate the f. First of all, if $r \in (0, 1)$, then let f_r be the dilated function defined on the dilated disk $\mathbb{D}_r := (1/r)\mathbb{D}$ by $f_r(z) = f(rz)$, $z \in \mathbb{D}_r$. By choosing r close enough to 1, we can ensure that $\|f_r - f\|_\infty < \epsilon$. We take this f_r as our g. Since f_r is a dilation of f, it follows that $f_r \in GL(H^{\infty}_{\mathcal{L}(E)}(\mathbb{D}_r))$, where $H^{\infty}_{\mathcal{L}(E)}(\mathbb{D}_r)$ denotes the the set of all $\mathcal{L}(E)$-valued bounded and holomorphic functions defined on $\mathbb{D}_r$. The path connectedness of $GL(H^{\infty}_{\mathcal{L}(E)}(\mathbb{D}_r))$ follows from the path connectedness of $GL(H^{\infty}_{\mathcal{L}(E)})$. So f_r can be connected to I by a path in $GL(H^{\infty}_{\mathcal{L}(E)}(\mathbb{D}_r))$. But this path is then also a path in $GL(A_{\mathcal{L}(E)})$. $\square$

Corollary 3.2. *If* $\dim E = \infty$*, then* $\operatorname{tsr} A_{\mathcal{L}(E)} = \infty$.

4. General S open in $\mathbb{T}$

We will need the following approximation result can be found in Stray [14] and Gamelin and Garnett [6] for the case of complex valued functions; the operator-valued case was shown in [13, Lemma 2.1].

Lemma 4.1. *Let S be an open subset of $\mathbb{T}$, and $f \in A^S_{\mathcal{L}(E)}$. For all $\epsilon > 0$, there exists an open set O in $\mathbb{C}$ that contains S and a holomorphic function $F : \mathbb{D} \cup O \to \mathcal{L}(E)$ such that $\|F|_{\mathbb{D}} - f\| \leq \epsilon$.*

Theorem 4.2. *Let S be an open subset of $\mathbb{T}$. If* $\dim E = \infty$*, then* $\operatorname{bsr} A^S_{\mathcal{L}(E)} = \infty$.

Proof. The proof is similar to the proof of Theorem 3.1, except that instead of using a dilation for the approximation, we use the approximation result from Lemma 4.1.

Let $f \in GL(A^S_{\mathcal{L}(E)})$. We want to find a path in $GL(A^S_{\mathcal{L}(E)})$ that joins f to I. Choose $\epsilon > 0$ such that the ball $\{g \in A^S_{\mathcal{L}(E)} \mid \|g - f\|_\infty < \epsilon\}$ is contained in $GL(A^S_{\mathcal{L}(E)})$. It is clear that given any such g, f can be connected to g by a path in $GL(A^S_{\mathcal{L}(E)})$. Now by Lemma 4.1, we can choose a $g = F$, where F is holomorphic across S. We can also shrink the set O in Lemma 4.1 so that $\Omega := O \cup \mathbb{D}$ is simply connected, and F belongs to $GL(H^{\infty}_{\mathcal{L}(E)}(\Omega))$. (Here $H^{\infty}_{\mathcal{L}(E)}(\Omega)$ denotes the the set of all $\mathcal{L}(E)$-valued bounded and holomorphic functions defined on Ω.) The path connectedness of $GL(H^{\infty}_{\mathcal{L}(E)}(\Omega))$ follows from the path connectedness of $GL(H^{\infty}_{\mathcal{L}(E)})$ and the Riemann mapping theorem. So g can be connected to I by a path in $GL(H^{\infty}_{\mathcal{L}(E)}(\Omega))$. But this path is then also a path in $GL(A^S_{\mathcal{L}(E)})$. $\square$

Corollary 4.3. *Let S be an open subset of $\mathbb{T}$. If* $\dim E = \infty$*, then the left/right topological stable rank of $A^S_{\mathcal{L}(E)}$ is infinite.*

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