Partitioning Posets

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Abstract

Given a poset $P = (X, \prec)$, a partition $X_1, \ldots, X_k$ of X is called an ordered partition of P if, whenever $x \in X_i$ and $y \in X_j$ with $x \prec y$, then $i \leq j$. In this paper, we show that for every poset $P = (X, \prec)$ and every integer $k \geq 2$, there exists an ordered partition of P into k parts such that the total number of comparable pairs within the parts is at most $(m-1)/k$, where $m \geq 1$ is the total number of edges in the comparability graph of P. We show that this bound is best possible for $k = 2$, but we give an improved bound, $m/k - c(k)\sqrt{m}$, for $k \geq 3$, where $c(k)$ is a constant depending only on k. We also show that, given a poset $P = (X, \prec)$, we can find an ordered partition of P that minimises the total number of comparable pairs within parts in polynomial time. We prove more general, weighted versions of these results.

1 Introduction

In this paper, we consider an analogue of the graph theoretic max-cut problem, for posets. Given a finite poset, we look for partitions of the ground set respecting the order and maximising the number of comparable pairs between parts. The extremal problem has some similarities to and some differences

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from the graph case, while the algorithmic problem is quite different as there is a polynomial time algorithm to find the best partition.

We shall be concerned only with finite partially ordered sets. Let $P = (X, \prec)$ be a poset. The *comparability graph* of P, denoted by $C(P)$, is an undirected graph on the vertex set X, such that for $a, b \in X$, ab is an edge of $C(P)$ if and only if a and b are comparable in P.

For $A \subseteq X$, let $e^P(A)$ be the number of edges in the graph induced by $C(P)$ on A. For A, B a partition of X, let $e^P(A, B)$ be the number of edges in $C(P)$ that have one end in A and the other in B. We drop the superscript when it is obvious which poset is being referred to.

For $P = (X, \prec)$, a partition of X into k disjoint parts, $X_1, \dots, X_k$, is called an *ordered partition* of P if, whenever $x \prec y$ with $x \in X_i$ and $y \in X_j$, we have $i \leq j$. If $k = 2$, this means that X_1 is a down-set and X_2 is an upset.

Given a poset $P = (X, \prec)$ and positive real numbers $a_1, \dots, a_k$, define

$$f(P; a_1, \dots, a_k) = \min\Big(\sum_{i=1}^{k} a_i e^P(X_i)\Big),$$

where the minimum is taken over all ordered partitions $X_1, \dots, X_k$ of P. Define

$$f(m; a_1, \dots, a_k) = \max(f(P; a_1, \dots, a_k)),$$

where the maximum is taken over all posets $P = (X, \prec)$, for which $e^P(X) = m$.

The case where $a_1 = \cdots = a_k = 1$ is the most interesting, and arises naturally from a well studied problem in graph theory. We study the more general weighted case, which turns out to be crucial to our proof techniques.

In Sections 2 and 3, we consider the problem of bounding $f(m)$. In Section 2, we prove the following theorem.

Theorem 1.1 *Let k be a positive integer. For positive real numbers $a_1, \dots, a_k$, and a positive integer m, we have that*

$$f(m; a_1, \dots, a_k) \leq \Big(\sum_{i=1}^{k} a_i^{-1}\Big)^{-1} m.$$

Let us compare a few special cases of this result with analogous ones for graphs. Consider first the case when $a_1 = \cdots = a_k = 1$. Then $(\sum_{i=1}^{k} a_i^{-1})^{-1} = \frac{1}{k}$, and Theorem 1.1 tells us that every poset $P = (X, \prec)$ has an ordered partition $X_1, \dots, X_k$ such that

$$e^P(X_1) + \cdots + e^P(X_k) \leq \frac{1}{k} e^P(X).$$

Looking more specifically at the case when $k = 2$ and $a_1 = a_2 = 1$, Theorem 1.1 tells us that every poset $P = (X, \prec)$ has an ordered partition X_1, X_2, such that $e^P(X_1) + e^P(X_2) \le \frac{1}{2}e^P(X)$, or equivalently, that $e^P(X_1, X_2) \ge \frac{1}{2}e^P(X)$. If we drop the condition that the partition should be ordered, then this result is easy to prove; indeed, we are asking for a cut in $C(P)$ containing at least half the edges of $C(P)$. It is well known and easy to show that such a cut exists for every graph (see [10] for more on graph cuts). An easy extension of this is the following result, which has presumably been proven before, but we give a proof here for convenience.

Given a graph $G = (V, E)$, for $U \subseteq V$, we define $E^G(U) = \{ab \in E : a, b \in U\}$, and $e^G(U) = |E^G(U)|$.

Theorem 1.2 *Given positive real numbers, $a_1, \dots, a_k$, and a graph $G = (V, E)$, there exists a partition of V into sets $V_1, \dots, V_k$ such that*

$$\sum_{i=1}^{k} a_i e^G(V_i) \le \Big(\sum_{i=1}^{k} a_i^{-1}\Big)^{-1} |E|.$$

Thus Theorem 1.1 is an analogue of Theorem 1.2 for posets. Of course, for graphs, there is no restriction on the way we partition the vertices. Although Theorem 1.2 is a natural bound, it can be sharpened, something which we discuss later for the case $k = 2$. Let us see first why Theorem 1.2 is true.

Proof We partition V into sets $V_1, \dots, V_k$ by assigning each vertex independently at random to one of $V_1, \dots, V_k$, where

$$\Pr(v \in V_j) = \frac{a_j^{-1}}{\sum_{i=1}^{k} a_i^{-1}}.$$

These probabilities are chosen optimally for the argument that follows. Given an edge $e = ab \in E$, we have that $\Pr(e \in E^G(V_j)) = a_j^{-2}/(\sum_{i=1}^{k} a_i^{-1})^2$, and so

$$\mathbb{E}(\sum_{j=1}^{k} a_j e^G(V_j)) = \sum_{j=1}^{k} a_j \sum_{e \in E} \Pr(e \in E^G(V_j)) = \sum_{j=1}^{k} a_j \sum_{e \in E} \frac{a_j^{-2}}{(\sum_{i=1}^{k} a_i^{-1})^2}$$

$$= \sum_{e \in E} \frac{1}{\sum_{i=1}^{k} a_i^{-1}} = \Big(\sum_{i=1}^{k} a_i^{-1}\Big)^{-1} |E|.$$

There must be some partition $V_1', \dots, V_k'$ of V for which $\sum_{i=1}^{k} a_i e^G(V_i)$ is at most its expected value, proving the theorem. $\square$

We cannot use a probabilistic argument like the one above to prove Theorem 1.1 because we know of no easy way of randomly picking ordered partitions of a poset in a way that would allow us to compute an expectation.

Let us return to the case $k = 2$ with $a_1 = a_2 = 1$. In the case of graphs, Edwards [1, 2] showed that for every graph $G = (V, E)$, there exists a partition of V into sets V_1 and V_2 such that

$$e^G(V_1) + e^G(V_2) \leq \frac{1}{2}m - \left(\sqrt{\frac{m}{8} + \frac{1}{64}} - \frac{1}{8}\right) = \frac{1}{2}m - \Theta(\sqrt{m}),$$

where $m = |E|$. This bound is achieved by complete graphs of odd order. However, this bound does not carry over to ordered partitions of posets.

We show in Section 3, that for every positive integer m, we have

$$f(m; 1, 1) = \left\lfloor \frac{m-1}{2} \right\rfloor.$$

We show further that for m a fixed odd positive integer, there is a unique poset $P_r = (X_r, \prec)$ (Figure 1 below), for which $f(P_r; 1, 1) = f(m; 1, 1)$ (here $r = (m-1)/2$). Describing P_r in words, we have the ground set $X_r = \{y_1, y_2, x_1, \ldots, x_r\}$, with $\{x_1, \ldots, x_r\}$ an antichain, and $y_1 \prec x_i \prec y_2$ for $i = 1, \ldots, r$ (and the transitive relation $y_1 \prec y_2$).

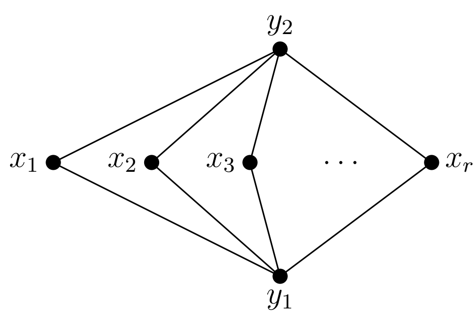

Figure 1: Hasse diagram of P_r

We can see immediately that to minimise $e^{P_r}(X_1) + e^{P_r}(X_2)$ (over all ordered partitions X_1, X_2 of P), we must have $y_1 \in X_1$ and $y_2 \in X_2$. Then, no matter how we place each x_i, exactly one of the edges y_1x_i and x_iy_2 lies inside a part. This gives us that $f(P_r; 1, 1) = r = (m-1)/2$, where m is the number of comparable pairs in P_r. (If we drop the condition that X_1, X_2 should be ordered, then the optimal partition of $C(P_r)$ is one with only a single edge inside parts.)

Thus we see that when $k = 2$, with $a_1 = a_2 = 1$, we cannot improve the bound given in Theorem 1.1 by more than a constant (independent of m), and we show in Section 3.1 that this is the case for general rational

a_1, a_2, using examples similar to P_r. However, for $k \geq 3$, we find that we can improve the bound in Theorem 1.1 by at least $c\sqrt{m}$, where c is a constant independent of m. These results are summarised in the next theorem.

Theorem 1.3 *(a) For fixed positive real numbers, a_1 and a_2, where a_2/a_1 is rational, we have*

$$f(m; a_1, a_2) = \left(\frac{1}{a_1} + \frac{1}{a_2}\right)^{-1} m - \Theta(1).$$

(b) For a fixed integer $k \geq 3$, and fixed positive real numbers $a_1, \ldots, a_k$, we have

$$f(m; a_1, \ldots, a_k) = \left(\sum_{i=1}^{k} a_i^{-1}\right)^{-1} m - \Theta(\sqrt{m}).$$

Note that we do not allow arbitrary real values for a_1 and a_2 in the statement of Theorem 1.3(a). In Section 3.2, we show by giving an explicit example, that Theorem 1.3(a) does not hold in general for real values of a_1 and a_2. We show that when $a_1 = 1$ and $a_2 = (1 + \sqrt{5})/2$, we have

$$f(m; a_1, a_2) = \left(\frac{1}{a_1} + \frac{1}{a_2}\right)^{-1} m - \Omega(\log m).$$

For general real values of a_1 and a_2, we know only that

$$f(m; a_1, a_2) = \left(\frac{1}{a_1} + \frac{1}{a_2}\right)^{-1} m - O(\sqrt{m}).$$

The example of the chain, C_n, on n elements gives the bound above and shows that the error term in Theorem 1.3(b) is of the correct order of magnitude. For $\binom{n}{2} \leq m \leq \binom{n+1}{2}$, an easy calculation, which we give at the end of Section 3.3, shows that

$$f(m; a_1, \ldots, a_k) \geq f(C_n; a_1, \ldots, a_k) = \left(\sum_{i=1}^{k} a_i^{-1}\right)^{-1} m - \Theta(\sqrt{m}),$$

where $k \geq 2$ and $a_1, \ldots, a_k$ are real numbers.

Finally, in Section 4, we show that, given positive rational numbers $a_1, \ldots, a_k$, and a poset P, we can find $f(P; a_1, \ldots, a_k)$ and a corresponding ordered partition in strongly polynomial time. Note the contrast with the situation for graphs, where finding a cut of maximum size in a graph is known to be NP-complete (see [3]).

2 Good Partitions

In this section, we prove Theorem 1.1. The key step is to prove the result for the case $k = 2$; the full result will then follow via a straightforward induction argument. We begin with some notation.

Let $P = (X, \prec)$ be a poset. For $A \subseteq X$, we write $\max^P(A)$ (resp. $\min^P(A)$) for the maximal (resp. minimal) elements of the poset induced by P on A.

For $x \in X$, let

$$\begin{aligned} U^P(x) &= \{y \in X : y \succ x\} \quad \text{with} \quad u^P(x) = |U^P(x)|, \\ \text{and } D^P(x) &= \{y \in X : y \prec x\} \quad \text{with} \quad d^P(x) = |D^P(x)|. \end{aligned}$$

Given positive real numbers, a_1 and a_2, define the function $h^P_{a_1,a_2} : X \to \mathbb{R}$ by

$$h^P_{a_1,a_2}(x) = a_2 u^P(x) - a_1 d^P(x).$$

Again, we drop subscripts and/or superscripts when it is clear what these are. Observe that $h^P_{a_1,a_2}$ is a strictly decreasing function, that is, for $x, y \in X$ with $x \prec y$, we have $h^P_{a_1,a_2}(y) < h^P_{a_1,a_2}(x)$.

A partition of X into parts X_1 and X_2 is called an (a_1, a_2)-*good* partition of P if $h^P_{a_1,a_2}(x) \geq 0$ for all $x \in X_1$ and $h^P_{a_1,a_2}(x) \leq 0$ for all $x \in X_2$ (thus an (a_1, a_2)-good partition of P is uniquely defined except that any element x with $h(x) = 0$ can be in either X_1 or X_2). It is clear that every (a_1, a_2)-good partition of P is an ordered partition of P, since h respects the order of P.

We have the following lemma, which is the case $k = 2$ of Theorem 1.1.

Lemma 2.1 *Fix positive real numbers a_1 and a_2. For $P = (X, \prec)$ a poset, let X_1, X_2 be an (a_1, a_2)-good partition of P. Then*

$$a_1 e(X_1) + a_2 e(X_2) \leq \frac{a_1 a_2}{a_1 + a_2} e(X) = \left(\frac{1}{a_1} + \frac{1}{a_2}\right)^{-1} e(X).$$

Proof The proof is by induction on $|X|$. The lemma is trivially true when $|X| = 1$.

Define $r : X \to \mathbb{R}$ by

$$r(x) = \begin{cases} a_1 d^P(x) & \text{if} \quad x \in X_1; \\ a_2 u^P(x) & \text{if} \quad x \in X_2. \end{cases}$$

Choose x^* to be any element of X that maximises r. Let $B = \max^P(X_1) \cup \min^P(X_2)$. It is clear that $x^* \in B$. We assume that $x^* \in \max^P(X_1)$; the case $x^* \in \min^P(X_2)$ follows in a similar way.

Let $X' = X \backslash \{x^*\}$, $P' = (X', \prec)$, $X'_1 = X_1 \backslash \{x^*\}$, and $X'_2 = X_2 \backslash \{x^*\}$. We claim that X'_1, X'_2 is an (a_1, a_2)-good partition of P'. Let us assume that the claim is true and continue with the proof. We have

$$\begin{aligned} & e^P(X_1) = e^{P'}(X'_1) + d^P(x^*), \\ & e^P(X_2) = e^{P'}(X'_2), \\ \text{and} \quad & e^P(X) = e^{P'}(X') + u^P(x^*) + d^P(x^*). \end{aligned}$$

Putting this together, we have

$$\begin{aligned} \frac{a_1 a_2}{a_1 + a_2} e^P(X) - a_1 e^P(X_1) - a_2 e^P(X_2) &= \frac{a_1 a_2}{a_1 + a_2} e^{P'}(X') - a_1 e^{P'}(X'_1) \\ & -a_2 e^{P'}(X'_2) + u^P(x^*) \frac{a_1 a_2}{a_1 + a_2} + d^P(x^*) \Big(\frac{a_1 a_2}{a_1 + a_2} - a_1 \Big) \\ = \frac{a_1 a_2}{a_1 + a_2} e^{P'}(X') - a_1 e^{P'}(X'_1) - a_2 e^{P'}(X'_2) &+ \frac{a_1}{a_1 + a_2} h^P(x^*) \geq 0, \end{aligned}$$

where the last inequality follows by induction and the fact that $h^P(x^*) \geq 0$ (since $x^* \in X_1$).

It remains only to show that X'_1, X'_2 is an (a_1, a_2)-good partition of P', that is, we must show that

$$h^{P'}(x) \begin{cases} \geq 0 & \forall x \in X'_1; \\ \leq 0 & \forall x \in X'_2. \end{cases}$$

Observe that $h^{P'}(x) = h^P(x)$ if x and x^* are incomparable in P and so the above holds for such elements x.

If $x \prec x^*$ in P, then $x \in X_1$ and

$$\begin{aligned} h^{P'}(x) &= a_2 u^{P'}(x) - a_1 d^{P'}(x) \\ &\geq a_2 u^P(x^*) - a_1 (d^P(x^*) - 1) \\ &= h^P(x^*) + a_1 \\ &\geq 0. \end{aligned}$$

If $x \succ x^*$ then $x \in X_2$ and

$$\begin{aligned} h^{P'}(x) &= a_2 u^{P'}(x) - a_1 d^{P'}(x) \\ &\leq a_2 u^P(x) - a_1 d^P(x^*) \\ &\leq 0 \quad \text{(by our choice of } x^*\text{).} \end{aligned}$$

This completes the proof. $\square$

We make a couple of remarks, which we shall make use of later.

Lemma 2.1 says that $f(P; a_1, a_2) \geq (a_1^{-1} + a_2^{-1})^{-1} e^P(X)$ for all posets P. Analysing the proof of Lemma 2.1, we see that we make a gain on this bound every time we remove a vertex x^* (in the induction) for which $|h(x^*)| > 0$. Hence, one way to construct a poset P for which $f(P; a_1, a_2)$ is close to our bound, would be to include many vertices x for which $h(x) = 0$. In fact, this is necessary in light of Lemma 3.2, which we prove in the next section.

We note also that we have strict inequality in Lemma 2.1 if $e^P(X) \geq 1$. This is because, as we inductively remove vertices from our poset P, we will eventually be left with a poset of height 2. For such a poset, Lemma 2.1 holds with strict inequality. Then, working backwards through the induction, we find that Lemma 2.1 holds with strict inequality for P.

We now prove Theorem 1.1 via an easy induction argument, using Lemma 2.1 as the induction step.

Proof (of Theorem 1.1) Given $a_1, \ldots, a_k$, it is sufficient to show that for every poset $P = (X, \prec)$, there exists an ordered partition of P into sets $X_1, \ldots, X_k$ such that

$$\sum_{i=1}^{k} a_i e(X_i) \leq \Big(\sum_{i=1}^{k} a_i^{-1} \Big)^{-1} e(X).$$

We use induction on k. The above is trivially true for $k = 1$. Assume it is true for $k - 1$.

Let $b_1 = (\sum_{i=1}^{k-1} a_i^{-1})^{-1}$ and $b_2 = a_k$. By Lemma 2.1, there exists an ordered partition of P into parts Y_1 and Y_2 such that

$$b_1 e(Y_1) + b_2 e(Y_2) \leq \Big(\frac{1}{b_1} + \frac{1}{b_2} \Big)^{-1} e(X) = \Big(\sum_{i=1}^{k} a_i^{-1} \Big)^{-1} e(X).$$

By the induction hypothesis, there exists an ordered partition of Y_1 into parts $X_1, \ldots, X_{k-1}$ such that

$$\sum_{i=1}^{k-1} a_i e(X_i) \leq \Big(\sum_{i=1}^{k-1} a_i^{-1} \Big)^{-1} e(Y_1) = b_1 e(Y_1).$$

Setting $X_k = Y_2$ gives the desired ordered partition of P. $\square$

3 Better Partitions

3.1 Rational Weights in Bipartitions

Our first task is to prove Theorem 1.3(a), which says that, for the case $k = 2$, Theorem 1.1 is close to best possible. We do this, as the remark after

Lemma 2.1 suggests, by constructing posets that include a large number of vertices x for which $h^P_{a_1,a_2}(x) = 0$.

Proof (of Theorem 1.3(a)) In light of Lemma 2.1, it is sufficient to prove the lower bound

$$f(m; a_1, a_2) \geq \frac{a_1 a_2}{a_1 + a_2} m - \Theta(1).$$

Let us assume for the moment that a_1 and a_2 are integers. For an integer $t \geq 0$, let $P(t) = P(a_1, a_2, t)$ be the complete three-layer poset with a_1 elements in the top layer A_1, a_2 elements in the bottom layer A_2, and t elements in the middle layer T (so, A_1, T, and A_2 are antichains and every element in T is below every element in A_1 and above every element in A_2).

Let X_1', X_2' be an ordered partition that minimises $a_1 e^{P(t)}(X_1) + a_2 e^{P(t)}(X_2)$ over all ordered partitions X_1, X_2 of $P(t)$. A little thought should convince the reader that it is necessary to have $A_2 \subseteq X_1'$ and $A_1 \subseteq X_2'$. A little further thought should convince the reader that this is in fact, sufficient. Therefore, we can assume that $X_1' = A_2 \cup T$ and $X_2' = A_1$. Noting also that $P(t)$ has $m(t) = (a_1 + a_2)t + a_1 a_2$ comparable pairs, we have that

$$\begin{aligned}
\min(a_1 e^{P(t)}(X_1) + a_2 e^{P(t)}(X_2)) &= a_1 e^{P(t)}(X_1') + a_2 e^{P(t)}(X_2') \\
&= a_1 a_2 t \\
&= \frac{a_1 a_2}{a_1 + a_2}(m(t) - a_1 a_2) \\
&= \frac{a_1 a_2}{a_1 + a_2} m(t) - d(a_1, a_2)
\end{aligned}$$

for all t, where $d(a_1, a_2)$ is a constant independent of $m(t)$. Given m, we choose t so that $m(t) \leq m < m(t+1)$, so that $m - m(t) < a_1 + a_2$. Now we have

$$\begin{aligned}
f(m; a_1, a_2) \geq f(m(t); a_1, a_2) &\geq f(P(t); a_1, a_2) \\
&= \frac{a_1 a_2}{a_1 + a_2} m(t) - d(a_1, a_2) \\
&= \frac{a_1 a_2}{a_1 + a_2} m - \Theta(1).
\end{aligned}$$

The above is also true for rational values of a_1 and a_2, and more generally when a_1/a_2 is rational, since

$$f(m; \frac{a_1}{r}, \frac{a_2}{r}) = \frac{1}{r} f(m; a_1, a_2),$$

for any real $r > 0$. $\square$

3.2 Irrational Weights in Bipartitions

Next, we give an example to show that Theorem 1.3(a) does not hold in general for real values of a_1 and a_2. We shall make use of some elementary results in the theory of continued fractions and Diophantine approximation, all of which can be found in, for example, [6].

Theorem 3.1 *For $a_1 = 1$ and $a_2 = (1+\sqrt{5})/2$ we have that*

$$f(m; a_1, a_2) = \left(\frac{1}{a_1} + \frac{1}{a_2}\right)^{-1} m - \Omega(\log m).$$

Proof We start with some preliminaries. We make use of the result that $\phi = (1+\sqrt{5})/2$, the *golden ratio*, has best rational approximation given by ratios of successive *Fibonacci numbers*. Let us go into more detail. We define the Fibonacci sequence F_n by the recursive relation $F_n = F_{n-1} + F_{n-2}$, with the initial conditions that $F_0 = 0$ and $F_1 = 1$. We have

$$F_n = \frac{1}{\sqrt{5}}(\phi^{n+1} + \hat{\phi}^{n+1}),$$

where $\hat{\phi} = (1-\sqrt{5})/2$. Note that $\phi + \hat{\phi} = 1$, $\phi\hat{\phi} = -1$, and $|\hat{\phi}| < 1$; we shall use these in later calculations.

A consequence of what is sometimes referred to as the *law of best approximation* (Theorem 182 in [6]) is the following. For any natural numbers r, s where $s \leq F_n$, we have that

$$|s\phi - r| \geq |F_n\phi - F_{n+1}|.$$

We prove that for every poset $P = (X, \prec)$, where $|X| = n$, there exists an ordered partition X_1, X_2, such that

$$a_1 e^P(X_1) + a_2 e^P(X_2) = (a_1^{-1} + a_2^{-1})^{-1} e^P(X) - \Omega(\log n).$$

This then proves the theorem, since $\log n > \log \sqrt{m} = \frac{1}{2}\log m$.

Note that every poset $P = (X, \prec)$ without isolated elements has a unique (a_1, a_2)-good partition for our choice of a_1 and a_2, since $h^P_{a_1,a_2}(x)$ is an integral linear function of ϕ, and so is non-zero for all $x \in X$.

Now fix $P = (X, \prec)$ with $|X| = n$. Define the sequence of posets $P_i = (X_i, \prec)$, $i = 0, \ldots, n-1$ as follows. Let $P_0 = P$. Given $P_i = (X_i, \prec)$, let X_1^i, X_2^i be the (a_1, a_2)-good partition of P_i. We know from the proof of Lemma 2.1 that there exists some $x_i^* \in X_i$ such that, defining $X_{i+1} =$

$X_i \backslash \{x_i^*\}$, $P_{i+1} = (X_{i+1}, \prec)$, $X_1^{i+1} = X_1^i \backslash \{x_i^*\}$, and $X_2^{i+1} = X_2^i \backslash \{x_i^*\}$, we have that X_1^{i+1}, X_2^{i+1} is the (a_1, a_2)-good partition of P_{i+1}. Furthermore

$$(a_1^{-1} + a_2^{-1})^{-1} e^{P_i}(X_i) - a_1 e^{P_i}(X_1^i) - a_2 e^{P_i}(X_2^i) = (a_1^{-1} + a_2^{-1})^{-1} e^{P_{i+1}}(X_{i+1}) - a_1 e^{P_{i+1}}(X_1^{i+1}) - a_2 e^{P_{i+1}}(X_2^{i+1}) + c|h^{P_i}(x_i^*)|,$$

where c is either $a_1(a_1^{-1} + a_2^{-1})^{-1}$ or $a_2(a_1^{-1} + a_2^{-1})^{-1}$ depending on whether x_i^* is in X_1^i or X_2^i. Thus the above gives us that

$$(a_1^{-1} + a_2^{-1})^{-1} e^P(X_i) - a_1 e^P(X_1) - a_2 e^P(X_2) = \Theta\Big(\sum_{i=0}^{n-1} |h^{P_i}(x_i^*)| \Big),$$

and it remains for us to show that $\sum_{i=0}^{n-1} |h^{P_i}(x_i^*)| = \Omega(\log n)$.

Note that P_{n-i} has i elements. Let $k(i)$ be the smallest integer so that $F_{k(i)} \geq i$. Observe that $k(i) = \Theta(\log i)$. Then we have that

$$\begin{aligned} |h^{P_{n-i}}(x_{n-i}^*)| &= |\phi u^{P_{n-i}}(x_{n-i}^*) - d^{P_{n-i}}(x_{n-i}^*)| \geq |F_{k(i)}\phi - F_{k(i)+1}| \\ &= \frac{1}{\sqrt{5}} \Big| (\phi^{k(i)+1} + \hat{\phi}^{k(i)+1})\phi - (\phi^{k(i)+2} + \hat{\phi}^{k(i)+2}) \Big| = |\hat{\phi}^{k(i)+1}|. \end{aligned}$$

Now we have that

$$\sum_{i=0}^{n-1} |h^{P_i}(x_i^*)| = \sum_{i=1}^{n} |h^{P_{n-i}}(x_{n-i}^*)| \geq \sum_{i=1}^{n} |\hat{\phi}^{k(i)+1}| \geq \sum_{j=1}^{k(n)-1} |\hat{\phi}^{j+1}|(F_j - F_{j-1}).$$

Also

$$\begin{aligned} |\hat{\phi}^{j+1}|(F_j - F_{j-1}) &= |\hat{\phi}^{j+1}| F_{j-2} = \frac{1}{\sqrt{5}} |\hat{\phi}^{j+1}(\phi^{j-2} + \hat{\phi}^{j-2})| \\ &= \frac{1}{\sqrt{5}} |\hat{\phi}^3(-1)^{j-2} + \hat{\phi}^{2j-1}| \geq \frac{1}{\sqrt{5}} (|\hat{\phi}^3| - |\hat{\phi}^{2j-1}|), \end{aligned}$$

where the last inequality holds for $j \geq 2$. Finally, we have that

$$\begin{aligned} \sum_{i=0}^{n-1} |h^{P_i}(x_i^*)| &\geq \frac{1}{\sqrt{5}} \Big(\sum_{j=3}^{k(n)-1} |\hat{\phi}^3| - \sum_{j=3}^{k(n)-1} |\hat{\phi}^{2j-1}| \Big) \\ &= \Theta(k(n)) = \Theta(\log n), \end{aligned}$$

as required. □

It is unclear if the bound in Theorem 3.1 gives the correct asymptotic value for $f(m; 1, (1+\sqrt{5})/2)$. More generally, it seems that the growth of $(a_1^{-1} + a_2^{-1})^{-1} m - f(m; a_1, a_2)$ depends on how well we can approximate a_2/a_1 by rationals.

3.3 Weighted k-partitions

Our next lemma will be the key step in proving the upper bound for Theorem 1.3(b). It will also enable us to prove the uniqueness of certain extremal posets in the next section. First we introduce the notion of a *balanced* element.

Fix positive real numbers a_1 and a_2 and let $P = (X, \prec)$ be a poset. For t a positive real number, define

$$\mathrm{Bal}^P_{a_1,a_2}(t) = \{x \in X : |h^P_{a_1,a_2}(x)| \le t\}$$

and let $\mathrm{bal}^P_{a_1,a_2}(t) = |\mathrm{Bal}^P_{a_1,a_2}(t)|$. We refer to elements in $\mathrm{Bal}^P_{a_1,a_2}(t)$ as *balanced* elements. Once again, subscripts and superscripts may be dropped.

Here is the aforementioned lemma, which gives us an upper bound on $f(P; a_1, a_2)$ that takes into account the number of balanced elements in P.

Lemma 3.2 *Fix positive real numbers a_1 and a_2, and let $P = (X, \prec)$ be a poset, with X_1, X_2 an (a_1, a_2)-good partition of P. For $0 \le t < \frac{1}{2}\min(a_1, a_2)$, we have*

$$a_1 e^P(X_1) + a_2 e^P(X_2) \le \frac{a_1 a_2}{a_1 + a_2} e^P(X) - t\frac{\min(a_1, a_2)}{2(a_1 + a_2)}(|X| - \mathrm{bal}^P_{a_1,a_2}(t)).$$

Furthermore, if $a_1 = a_2$, then the above inequality holds under the weaker condition, $0 \le t < \min(a_1, a_2)$.

We concentrate on the proof of the first part of the lemma. The second part has almost the same proof; we make remarks where the proofs differ.

Proof The proof is again by induction on $|X|$. The lemma is true for $|X| = 1$, since an isolated element is balanced. Assume it is true for all posets with fewer than $|X|$ elements.

Let x^* be as in the proof of Lemma 2.1, and as before, let $X' = X\backslash\{x^*\}$, $P' = (X', \prec)$, $X_1' = X_1\backslash\{x^*\}$, and $X_2' = X_2\backslash\{x^*\}$. We know from the proof of Lemma 2.1 that X_1', X_2' is an (a_1, a_2)-good partition of P'. We shall assume that $x^* \in \max^P(X_1)$ so that $h^P(x^*) \ge 0$; a similar argument holds if $x^* \in \min^P(X_2)$. We must consider the two cases $|h^P(x^*)| \le t$ and $|h^P(x^*)| > t$ separately.

Suppose $h^P(x^*) \le t$. (Regarding the case $a_1 = a_2$, if $h^P(x^*) < t \le \min(a_1, a_2)$, then $h^P(x^*) = 0$.) We claim that $\mathrm{bal}^{P'}(t) \le \mathrm{bal}^P(t) - 1$. We prove this by showing that the removal of x^* from P does not create any new balanced elements in P'. Fix $x \in X$ with $x \notin \mathrm{Bal}^P(t)$.

If x is incomparable to x^*, then $h^{P'}(x) = h^P(x)$, hence $|h^{P'}(x)| > t$ and $x \notin \mathrm{Bal}^{P'}(t)$.

If $x \prec x^*$ then

$$\begin{aligned} h^{P'}(x) &= a_2 u^{P'}(x) - a_1 d^{P'}(x) \\ &\geq a_2 u^P(x^*) - a_1(d^P(x^*) - 1) \\ &= h^P(x^*) + a_1 > t, \end{aligned}$$

so $x \notin \mathrm{Bal}^{P'}(t)$.

If $x \succ x^*$ then

$$\begin{aligned} h^{P'}(x) &= a_2 u^{P'}(x) - a_1 d^{P'}(x) \\ &\leq a_2(u^P(x^*) - 1) - a_1(d^P(x^*)) \\ &= h^P(x^*) - a_2 < -t, \end{aligned}$$

so $x \notin \mathrm{Bal}^{P'}(t)$.

We have proved our claim and we have, as in Lemma 2.1, that

$$\begin{aligned} &\frac{a_1 a_2}{a_1 + a_2} e^P(X) - a_1 e^P(X_1) - a_2 e^P(X_2) \\ &\qquad = \frac{a_1 a_2}{a_1 + a_2} e^{P'}(X') - a_1 e^{P'}(X_1') - a_2 e^{P'}(X_2') + \frac{a_1}{a_1 + a_2} h^P(x^*) \\ &\qquad \geq t \frac{\min(a_1, a_2)}{2(a_1 + a_2)} (|X'| - \mathrm{bal}^{P'}(t)) \text{ (induction)} \\ &\qquad \geq t \frac{\min(a_1, a_2))}{2(a_1 + a_2)} (|X| - \mathrm{bal}^P(t)), \end{aligned}$$

where the last inequality follows because $|X'| = |X| - 1$ and $\mathrm{bal}^{P'}(t) \leq \mathrm{bal}^P(t) - 1$.

Next, suppose $h^P(x^*) > t$. We claim that

$$h^P(x^*) \geq \begin{cases} t(\mathrm{bal}^{P'}(t) - \mathrm{bal}^P(t)) & \text{if } \mathrm{bal}^{P'}(t) > \mathrm{bal}^P(t); \\ t & \text{if } \mathrm{bal}^{P'}(t) \leq \mathrm{bal}^P(t), \end{cases}$$

the second case being trivial. Again, we consider which elements in X change from being unbalanced to balanced when x^* is removed from X. Fix $x \in X'$ with $x \notin \mathrm{Bal}^P(t)$.

If x and x^* are incomparable, then as before $h^P(x) = h^{P'}(x)$, and $x \notin \mathrm{Bal}^{P'}(t)$.

If $x \prec x^*$, then as before

$$h^{P'}(x) \geq h^P(x^*) + a_1 > t,$$

and $x \notin \mathrm{Bal}^{P'}(t)$.

If $x \succ x^*$, then x may become balanced upon removal of x^*, so we must consider all such elements. Let $x_1, \dots, x_r$ be the elements of X such that for each i, $x_i \succ x^*$ and x_i is balanced in P' but not in P. We first show that $x_1, \dots, x_r$ is an antichain in P'. If $x_j \succ x_i$ for some $1 \le i, j \le r$, then we have $-t \le h^{P'}(x_i) \le 0$ and $h^{P'}(x_j) \le h^{P'}(x_i) - (a_1 + a_2)$, giving that $h^{P'}(x_j) < -t$ (by our choice of t), contradicting that x_j is balanced in P'. Thus, $x_1, \dots, x_r$ must be an antichain.

Now, we have

$$\begin{aligned} h^P(x^*) &= a_2 u^P(x^*) - a_1 d^P(x^*) \\ &= a_2\left(r + \Big|\bigcup_{i=1}^{r} U^{P'}(x_i)\Big|\right) - a_1(d^{P'}(x_r)) \\ &\ge a_2 r + h^{P'}(x_r) \\ &\ge tr, \end{aligned}$$

where the last inequality follows since $h^{P'}(x_r) > -t$ and $a_2 > 2t$ (by our choice of t). (Regarding the case $a_1 = a_2$, $h^{P'}(x_r) = 0$ and $a_2 > t$ (by our choice of t).) This proves the claim since $r = \mathrm{bal}^{P'}(t) - \mathrm{bal}^P(t)$. Combining the two cases of the claim, we have the following (weaker) inequality,

$$h^P(x^*) \ge \frac{1}{2} t\big(1 + \mathrm{bal}^{P'}(t) - \mathrm{bal}^P(t)\big).$$

Finally, to complete the induction, we have

$$\begin{aligned} &\frac{a_1 a_2}{a_1 + a_2} e^P(X) - a_1 e^P(X_1) - a_2 e^P(X_2) \\ &\qquad = \frac{a_1 a_2}{a_1 + a_2} e^{P'}(X') - a_1 e^{P'}(X_1') - a_2 e^{P'}(X_2') + \frac{a_1}{a_1 + a_2} h^P(x^*) \\ &\qquad \ge t \frac{\min(a_1, a_2)}{2(a_1 + a_2)}(|X'| - \mathrm{bal}^{P'}(t)) + \frac{\min(a_1, a_2)}{a_1 + a_2} h^P(x^*) \\ &\qquad \ge t \frac{\min(a_1, a_2)}{2(a_1 + a_2)}(|X| - \mathrm{bal}^P(t)), \end{aligned}$$

where the first inequality follows by induction and the last inequality follows from our bound on $h^P(x^*)$. $\square$

Next, we use Lemma 3.2 to prove the following theorem, which is the upper bound in Theorem 1.3(b).

Theorem 3.3 *Given an integer $k \ge 3$, and positive real numbers, $a_1, \dots, a_k$, there exists a constant $c = c(k, a_1, \dots, a_k)$ such that*

$$f(m; a_1, \dots, a_k) \le \Big(\sum_{i=1}^{k} a_i^{-1}\Big)^{-1} m - c\sqrt{m}.$$

Proof For $r = 1, \ldots, k-1$, let $b_r = (\sum_{i=1}^{r} a_i^{-1})^{-1}$ and let $b'_r = (\sum_{i=r+1}^{k} a_i^{-1})^{-1}$. For $r < s$,

$$v_{r,s} = \left| \frac{b'_r b_r^{-1} - b'_s b_s^{-1}}{b_r^{-1} + b_s^{-1}} \right| \quad \text{and} \quad w_{r,s} = \left| \frac{b_r b_r'^{-1} - b_s b_s'^{-1}}{b_r'^{-1} + b_s'^{-1}} \right|.$$

Let $t_{r,s} = \frac{1}{2} \min(v_{r,s}, w_{r,s}, b_r, b'_r b_s, b'_s)$ and let $t = \min_{r<s} t_{r,s}$. Finally, let

$$c = c(k, a_1, \ldots, a_k) = \Big(\min_{i=1,\ldots,k-1} \frac{\min(b_i, b'_i)}{2(b_i + b'_i)} \Big) t \Big(1 - \frac{1}{k-1} \Big).$$

Note that $c > 0$ since $k \geq 3$.

Let $P = (X, \prec)$ be a poset, which, we may assume, has no isolated elements. It is sufficient to show that there exists an ordered partition $X_1, \ldots, X_k$ of P such that

$$\sum_{i=1}^{k} a_i e^P(X_i) \leq \Big(\sum_{i=1}^{k} a_i^{-1} \Big)^{-1} e^P(X) - c\sqrt{m}.$$

We show first that our choice of t ensures that for $x \in X$, there can be at most one value of r for which $x \in \mathrm{Bal}^P_{b_r,b'_r}(t)$. Suppose not. Then there exists $1 \leq r < s \leq k-1$ such that

$$\begin{aligned} &|h_{b_r,b'_r}(x)| = |b_r u(x) - b'_r d(x)| \leq t \\ \text{and} \quad &|h_{b_s,b'_s}(x)| = |b_s u(x) - b'_s d(x)| \leq t. \end{aligned}$$

Dividing the first equation by b_r and the second by b_s and subtracting the resulting equations, we obtain

$$|b'_r b_r^{-1} - b'_s b_s^{-1}| d(x) \leq \Big| |u(x) - b'_r b_r^{-1} d(x)| - |u(x) - b'_s b_s^{-1} d(x)| \Big| \leq (b_r^{-1} + b_s^{-1}) t.$$

Thus we have that $v_{r,s} d(x) \leq t$, and by a similar argument, we have $w_{r,s} u(x) \leq t$. Since x is not isolated, either $d(x) \geq 1$ or $u(x) \geq 1$, whence we have that $t \geq \min(v_{r,s}, w_{r,s})$. But this contradicts our choice of t.

Hence, $x \in \mathrm{Bal}^P_{b_r,b'_r}(t)$ for at most one value of $r = 1, \ldots, k-1$. Let R be the value of r that minimises $\mathrm{bal}^P_{b_r,b'_r}(t)$. Then we have that

$$\mathrm{bal}^P_{b_R,b'_R}(t) \leq \frac{|X|}{k-1}.$$

By Lemma 3.2 there exists an ordered partition Y_1, Y_2 of P, such that

$$\begin{aligned}
& b_R e^P(Y_1) + b'_R e^P(Y_2) \\
& \qquad \le \frac{b_R b'_R}{b_R + b'_R} e^P(X) - t\frac{\min(b_R, b'_R)}{2(b_R + b'_R)}(|X| - \mathrm{bal}^P_{b_R, b'_R}(t)) \\
& \qquad \le \Big(\sum_{i=1}^{k} a_i^{-1}\Big)^{-1} e^P(X) - t\frac{\min(b_R, b'_R)}{2(b_R + b'_R)}\Big(1 - \frac{1}{k-1}\Big)|X| \\
& \qquad \le \Big(\sum_{i=1}^{k} a_i^{-1}\Big)^{-1} e^P(X) - c\sqrt{m}.
\end{aligned}$$

(Note that our choice of t is consistent with the condition in Lemma 3.2.) By Lemma 2.1, we can find an ordered partition of Y_1 into sets $X_1, \dots, X_R$ and of Y_2 into sets $X_{R+1}, \dots, X_k$ such that

$$\sum_{i=1}^{R} a_i e^P(X_i) \le b_R e^P(Y_1)$$

$$\text{and} \quad \sum_{i=R+1}^{k} a_i e^P(X_i) \le b'_R e^P(Y_2).$$

Then $X_1, \dots, X_k$ is the desired ordered partition of P. □

We end this subsection by showing that the error term in the bound given by Theorem 3.3 is of the correct order of magnitude, that is, we complete the proof of Theorem 1.3(b).

Proof (of Theorem 1.3(b)) We have shown the upper bound in Theorem 3.3. For the lower bound, assuming that $\binom{n}{2} \le m < \binom{n+1}{2}$ and recalling that C_n is the chain on n elements, we have

$$f(m; a_1, \dots, a_k) \ge f\Big(\binom{n}{2}, a_1, \dots, a_k\Big) \ge f(C_n; a_1, \dots, a_k)$$

Let $X_1, \dots, X_k$ be the ordered partition of C_n, where $|X_i| = x_i$ for each i and $x_1 + \dots + x_k = n$. Then we have that

$$\sum_{i=1}^{k} a_i e^{C_n}(X_i) = \sum_{i=1}^{k} a_i \binom{x_i}{2} \ge \sum_{i=1}^{k} \frac{1}{2} a_i (x_i - 1)^2.$$

For $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$, define $g(\mathbf{x}) = \sum_{i=1}^{k} \frac{1}{2} a_i (x_i - 1)^2$, so that

$$f(C_n; a_1, \dots, a_k) \ge \min_{x_1 + \dots + x_k = n} g(\mathbf{x}).$$

We find that the minimum occurs at $\mathbf{y} = (y_1, \ldots, y_k)$ where

$$y_i - 1 = \Big(a_i \sum_{j=1}^{k} a_j^{-1}\Big)^{-1}(n-k).$$

This is because g is a convex function and $g'(\mathbf{y})$ is parallel to $(1, 1, \ldots, 1)$, which is normal to the plane $x_1 + \cdots + x_k = 0$. Therefore

$$\begin{aligned} f(m; a_1, \ldots, a_k) \geq f(C_n; a_1, \ldots, a_k) \geq g(\mathbf{y}) &= \Big(\sum_{i=1}^{k} a_i^{-1}\Big)^{-1}\frac{1}{2}(n-k)^2 \\ &= \Big(\sum_{i=1}^{k} a_i^{-1}\Big)^{-1} m - \Theta(\sqrt{m}). \end{aligned}$$

□

We remark that the calculation above when $k = 2$ shows that for arbitrary positive real values of a_1 and a_2, we have that

$$f(m; a_1, a_2) = \Big(\frac{1}{a_1} + \frac{1}{a_2}\Big)^{-1} m - O(\sqrt{m}).$$

3.4 Extremal Results

For the special case when $k = 2$ and $a_1 = a_2 = 1$, we can give the exact values of $f(m; 1, 1)$. We shall make use of the remarks after Lemma 2.1.

Theorem 3.4 *For m a positive integer, we have that*

$$f(m; 1, 1) = \Big\lfloor \frac{m-1}{2} \Big\rfloor.$$

Proof By Lemma 2.1, we have, for every poset $P = (X, \prec)$, that $f(P; 1, 1) \leq \frac{1}{2}e^P(X)$ and that the inequality is strict if $e^P(X) > 0$. Thus for $m > 0$, we have $f(m; 1, 1) < \frac{m}{2}$ or equivalently $f(m; 1, 1) \leq \lfloor (m-1)/2 \rfloor$.

Recall the poset $P_r = P(1, 1, r)$ defined in the introduction. We saw that for m odd, we have $f(m; 1, 1) \geq f(P_{(m-1)/2}; 1, 1) = (m-1)/2$. For m even, taking disjoint copies of P_0 and $P_{(m-2)/2}$, which we denote by $P_0 \sqcup P_{(m-2)/2}$, we have that $f(m; 1, 1) \geq f(P_0 \sqcup P_{(m-2)/2}; 1, 1) = \lfloor (m-1)/2 \rfloor$. This proves that the values of $f(m; 1, 1)$ are as stated. □

We conclude this section by showing how we can use Lemma 3.2 in proving the uniqueness of $P_{(m-1)/2}$ as the extremal poset corresponding to $f(m; 1, 1)$ when m is odd.

Theorem 3.5 *Fix m an odd positive integer. If P is a poset with m comparable pairs and no isolated elements, and*

$$f(P;1,1) = f(m;1,1),$$

then $P = P_{(m-1)/2}$.

Proof Let P be a poset as in the premise of the theorem, so that $f(P;1,1) = f(m;1,1) = (m-1)/2$. We apply Lemma 3.2 to the poset P, where $a_1 = a_2 = 1$, and t is any fixed number in the range $2/3 < t < 1$. Thus, we have

$$\frac{m-1}{2} = f(P;1,1) \leq \frac{1}{2}m - \frac{1}{4}t(|X| - \mathrm{bal}_{1,1}^P(t)).$$

Therefore, we must have that $|X| - \mathrm{bal}_{1,1}^P(t) \leq 2/t$, whence there are at most two elements of P that are not in $\mathrm{Bal}_{1,1}^P(t)$ (since $t > 2/3$). But maximal and minimal elements of P are not in $\mathrm{Bal}_{1,1}^P(t)$ (since $t < 1$), hence there are exactly two elements, which we call y_1 and y_2, that are not in $\mathrm{Bal}_{1,1}^P(t)$.

Observe that, since $h_{1,1}$ is an integer function, if $x \in \mathrm{Bal}_{1,1}^P(t)$ for $t < 1$, then $h_{1,1}(x) = 0$. Also, since $h_{1,1}$ is a strictly decreasing function, then $\mathrm{Bal}_{1,1}^P(t)$ must be an antichain. Since the elements in $\mathrm{Bal}_{1,1}^P(t)$ are not isolated, they must each be (without loss of generality) above y_1 and below y_2 (in order to ensure that $h_{1,1}(x) = 0$ for each $x \in \mathrm{Bal}_{1,1}^P(t)$). Thus $P = P_r$ for some r and since P has m comparable pairs, we must have that $r = (m-1)/2$. $\square$

We end this subsection with the following conjecture about the exact value of f when $k \geq 3$.

Conjecture Let $k \geq 3$ be a fixed integer, and $a_1 = \cdots = a_k = 1$. For $m = \binom{n}{2}$, we have

$$f(m; a_1, \ldots, a_k) = f(C_n; a_1, \ldots, a_k).$$

Examples like P_r fail to be extremal when $k \geq 3$ because of the increased freedom we have when partitioning into three or more parts. Informally, it seems that this increased freedom, together with transitivity in posets, allows us to create partitions where a large number of comparable edges go across parts. Thus, in order to construct an extremal example, we also require a large number of comparable edges within parts. Chains seem the most likely candidates to satisfy this.

4 Best Partitions

In this section, we give an algorithm that finds us an optimal ordered partition for any given poset P. More precisely, we have the following theorem.

Theorem 4.1 *There exists a strongly polynomial time algorithm, such that for an input $(P, k, a_1, \ldots, a_k)$, where $P = (X, \prec)$ is a poset, k is a positive integer, and $a_1, \ldots, a_k$ are positive rationals, the algorithm outputs an ordered partition, $X_1, \ldots, X_k$, of P for which*

$$\sum_{i=1}^{k} a_i e(X_i) = f(P; a_1, \ldots, a_k).$$

Before we give the algorithm in general, we give simpler algorithms for two special cases. We start by giving a particularly simple algorithm for the case $k = 2$, where in fact, we are able to find all optimal ordered partitions in polynomial time.

Theorem 4.2 *Let a_1 and a_2 be positive real numbers. For a poset P together with an ordered partition X_1, X_2 of P, we have that*

$$a_1 e(X_1) + a_2 e(X_2) = f(P; a_1, a_2)$$

if and only if X_1, X_2 is an (a_1, a_2)-good partition of P.

We note that if a_1 and a_2 are rationals, then all (a_1, a_2)-good partitions can be found in strongly polynomial time.

Proof Given an ordered partition X_1, X_2 observe that

$$\begin{aligned} a_1 e(X_1) + a_2 e(X_2) &= a_2 \sum_{x \in X_2} u(x) + a_1 \sum_{x \in X_1} d(x) \\ &= a_2 \sum_{x \in X} u(x) + \sum_{x \in X_1} (a_1 d(x) - a_2 u(x)) \\ &= a_2 e(X) - \sum_{x \in X_1} h_{a_1,a_2}(x). \end{aligned}$$

This is minimised if and only if $h_{a_1,a_2}(x) \geq 0$ for all $x \in X_1$ and $h_{a_1,a_2}(x) \leq 0$ for all $x \in X_2$, that is, X_1, X_2 is an (a_1, a_2)-good partition. $\square$

Given that the above theorem tells us exactly which ordered partitions are optimal, one might expect that we can use this information directly to bound $f(P; a_1, a_2)$, rather than using an inductive proof such as Lemma 2.1. Such an argument has eluded us.

Turning now to the case $k \geq 3$, one might expect that we can apply Theorem 4.2 repeatedly to give an optimal ordered partition into k parts. The obstruction to this is that by performing our optimisation sequentially, our choice of partition at one stage affects the poset we are required to partition at subsequent stages, and so sacrificing optimality at an earlier stage can leave us with posets better suited to partitioning at later stages.

Next we consider the case for general k, but where $a_1 = \cdots = a_k = 1$. We thank Omid Amini and Stéphan Thomassé for the argument that follows.

We find that our problem can be reduced to one of finding a maximum sized union of $k-1$ antichains in a poset $P = (X, \prec)$. This is known to be solvable in polynomial time (see [8]). Note that $A \subseteq X$ is a union of $k-1$ antichains if and only if P induces a poset of height at most $k-1$ on A.

We start with some notation and definitions. Let k, a positive integer, and $P = (X, \prec)$, a poset be given. For an ordered partition $X_1, \ldots, X_k$ of P, define

$$E^P(X_1, \ldots, X_k) = \{(a, b) : a \in X_i, b \in X_j, a \prec b, i < j\}.$$

Our problem is equivalent to maximising $|E^P(X_1, \ldots, X_k)|$.

We say that $Y \subseteq X$ is a *maximal* union of $k-1$ antichains, if there is no $Z \supsetneq Y$ that is also a union of $k-1$ antichains.

Next we define the *line poset* $L(P) = (E^P(X), \prec_{L(P)})$ of P, where

$$E^P(X) = \{(a, b) : a \prec b\},$$

and $(a_1, b_1) \prec_{L(P)} (a_2, b_2)$ if and only if $b_1 \preceq a_2$. It is easy to check that $L(P)$ is a well defined poset.

We have the following lemma.

Lemma 4.3 *Given a positive integer k and a poset $P = (X, \prec)$, we have that $Y \subseteq E^P(X)$ is a union of at most $k-1$ antichains in the line poset $L(P)$, if and only if $Y \subseteq E^P(X_1, \ldots, X_k)$ for some ordered partition $X_1, \ldots, X_k$ of P.*

Thus, finding an optimal k-partition is equivalent to finding a maximum sized union of $k-1$ antichains in $L(P)$, where the latter can be found in time polynomial in $|P|$, (since $|L(P)|$ is polynomial in $|P|$).

Proof Let $X_1, \ldots, X_k$ be an ordered partition of P. Then it is clear that if $Y \subseteq E^P(X_1, \ldots, X_k)$, then $L(P)$ induces a poset of height at most $k-1$ on Y.

Conversely, let $Y \subseteq E^P(X)$ be a maximal union of $k-1$ antichains in $L(P)$, that is, $L(P)$ induces a poset of height $k-1$ on Y. We prove, by induction on k, that there exists an ordered partition $X_1, \ldots, X_k$ such that $Y = E^P(X_1, \ldots, X_k)$. This then proves the lemma.

When $k = 1$, Y has height zero, so is empty and corresponds to the ordered partition with one part, namely the whole of X.

For general k, let $Y_1 = \min^{L(P)}(Y)$ and let $X_1 = \{a : \exists (a, b) \in Y_1\}$. Let $Y' = Y \backslash Y_1$ and $X' = X \backslash X_1$, and let $P' = (X', \prec)$. We claim that,

(a) X_1 is a down-set of P, and
(b) Y' is a maximal union of $k-2$ antichains in $L(P')$.

Let us continue with the proof assuming the claim is true. By induction, there exists an ordered partition $X_2, \ldots, X_k$ of P' such that $Y' = E^{P'}(X_2, \ldots, X_k)$. Since X_1 is a down-set, it is clear that $X_1, \ldots, X_k$ forms an ordered partition of P and that $Y = Y_1 \cup Y' \subseteq E^P(X_1, X') \cup E^P(X_2, \ldots, X_k) = E^P(X_1, \ldots, X_k)$. Since Y is maximal, we must have $Y = E^P(X_1, \ldots, X_k)$.

For part (a) of the claim, let $a \in X_1$, with $b \in X$ and $b \prec a$. Since $a \in X_1$, there exists some $a' \in X$ with $a \prec a'$ such that $(a, a') \in Y_1 = \min^{L(P)}(Y)$. We show below that (b, a') is incomparable to every element in $Y_1 = \min^{L(P)}(Y)$, hence $(b, a') \in Y_1$ (by the maximality of Y), thus $b \in X_1$, proving part (a) of the claim.

In order to show that (b, a') is incomparable to every element in Y_1 suppose $(a_1, a_2) \in Y_1$. Then we cannot have $(a_1, a_2) \prec_{L(P)} (b, a')$, otherwise we have $(a_1, a_2) \prec_{L(P)} (a, a')$ contradicting that $(a, a') \in Y_1$. We cannot have $(b, a') \prec_{L(P)} (a_1, a_2)$, otherwise $(a, a') \prec_{L(P)} (a_1, a_2)$, contradicting that $(a_1, a_2) \in Y_1$.

For part (b) of the claim, it is clear that $Y' \subseteq E^{P'}(X')$. It is also clear that $Y' = Y \backslash \min^{L(P)}(Y)$ is a maximal union of $k-2$ antichains, since Y is a maximal union of $k-1$ antichains. This completes the proof. $\square$

We now turn to the proof for the general weighted case, Theorem 4.1, which relies on the fact that a submodular function on a lattice family can be minimised in strongly polynomial time. This result was originally due to Grötschel, Lovász, and Schrijver [4, 5] and was refined most notably by Iwata, Fleischer, and Fujishige [7] and Schrijver [9]. We begin with some preliminaries.

Given a set V, a set L of subsets of V (with the inclusion order) is called a *lattice family* if, whenever $A, B \in L$, we have $A \cap B \in L$ and $A \cup B \in L$. For example, the set of down-sets of a poset P on V, which we denote by $D(P)$, forms a lattice family.

A function $g : L \to \mathbb{R}$ is called *submodular* if

$$g(A) + g(B) \geq g(A \cap B) + g(A \cup B)$$

for all $A, B \in L$.

We have the following special case of a result of Schrijver [9].

Theorem 4.4 *Let $D(P)$ be the set of down-sets of some partial order on $P = (V, \prec)$. Let g be a submodular function on $D(P)$. Given a value-giving oracle for g, a set $U \in D(P)$ that minimises g can be found by an algorithm using a number of calls to the oracle and a number of arithmetic steps that are both polynomial in $|V|$.*

The value-giving oracle is able to access values of g in polynomial time. It is required because we would like an algorithm polynomial in $|V|$, so we do not wish to input all values of g, since this would require $|D(P)|$ operations, where $|D(P)|$ is potentially exponential in $|V|$.

We are now ready to prove Theorem 4.1.

Proof (of Theorem 4.1) A poset $P = (X, \prec)$, a positive integer k, and positive rational numbers $a_1, \ldots, a_k$ are given. Let $|X| = n$. For $X_1, \ldots, X_k$ an ordered partition of P, we define the *partition function* of $X_1, \ldots, X_k$ to be the function $\omega : X \to [k]$ where $\omega(x) = i$ if and only if $x \in X_i$. Let Ω be the set of all partition functions corresponding to the ordered partitions of P. (A partition function ω defines a partition, and so we sometimes refer to ω as a partition, and to Ω as the set of all ordered partitions.[1])

We start by showing that Ω has a natural lattice structure of the form given in Theorem 4.4.

Let $X = \{x_1, \ldots, x_n\}$. Define $P(k-1) = (Y, \prec^*)$, where $Y = Y_1 \cup \cdots \cup Y_{k-1}$, $Y_i = \{y_{i1}, \ldots y_{in}\}$ for $i = 1, \ldots, k-1$, and $y_{ir} \prec^* y_{js}$ if and only if $i \geq j$ and $x_r \preceq x_s$ (assuming y_{ir} and y_{js} are distinct). Recall that $D(P(k-1))$ is the set of down-sets of $P(k-1)$. We give a bijection from Ω to $D(P(k-1))$.

For ω an ordered partition of P (into k parts), define $D(\omega)$ to be the subset of Y such that $y_{ij} \in D(\omega)$ if and only if $i \geq \omega(x_j)$. It is easy to check that $D(\omega)$ is a down-set of $P(k)$ and that $\omega \mapsto D(\omega)$ is a bijection from Ω to $D(P(k-1))$. We note that if $\omega_1 \mapsto D_1$ and $\omega_2 \mapsto D_2$, then $\min(\omega_1, \omega_2) \mapsto D_1 \cup D_2$ and $\max(\omega_1, \omega_2) \mapsto D_1 \cap D_2$, where $\min(\omega_1, \omega_2)$

[1] Ω is often called the set of order-preserving maps from P to $[k]$.

(resp. $\max(\omega_1, \omega_2)$) is the pointwise minimum (resp. maximum) of ω_1 and ω_2.

We define $g : \Omega \to \mathbb{R}$, where $g(\omega) = \sum_{i=1}^{k} a_i e^P(X_i)$ and ω is the partition function of $X_1, \ldots, X_k$. We note that, given any partition ω, we can compute $g(\omega)$ in time polynomial in n and the lengths of the rationals a_i.

Invoking Theorem 4.4, we see that if g is submodular, then we can minimise g over Ω in time polynomial in $|P(k)| = kn \leq n^2$ and the lengths of the rationals a_i. That is, we can minimise g in strongly-polynomial time, thus proving the theorem.

It remains only to show that g is submodular, that is, we wish to show that if ω and ϕ are ordered partitions, then

$$g(\omega) + g(\phi) \geq g(\max(\omega, \phi)) + g(\min(\omega, \phi)).$$

Noting that the sum of submodular functions is submodular, we write g as a sum of indicator functions and show that each indicator function is submodular.

Define $I_{x,y,i} : \Omega \to \{0, 1\}$, where

$$I_{x,y,i}(\omega) = \begin{cases} 1 \text{ if } \omega(x) = \omega(y) = i; \\ 0 \text{ otherwise }. \end{cases}$$

Then we have that

$$g = \sum_{i=1}^{k} a_i \sum_{x \prec y} I_{x,y,i}.$$

We now carry out an easy case analysis to show that if $x \prec y$, then $I_{x,y,i}$ is submodular. We wish to show for every pair of ordered partitions ω and ϕ, that

$$I_{x,y,i}(\omega) + I_{x,y,i}(\phi) \geq I_{x,y,i}(\max(\omega, \phi)) + I_{x,y,i}(\min(\omega, \phi)). \tag{1}$$

Since $x \prec y$, we have that $\omega(x) \leq \omega(y)$ and $\phi(x) \leq \phi(y)$. Henceforth, we drop the subscripts on I.

Suppose $I(\max(\omega, \phi)) + I(\min(\omega, \phi)) = 2$. Then

$$\begin{aligned} & \max(\omega, \phi)(x) = \min(\omega, \phi)(x) = \max(\omega, \phi)(y) = \min(\omega, \phi)(y) = i, \\ \text{so} \quad & \omega(x) = \omega(y) = \phi(x) = \phi(y) = i, \\ \text{hence} \quad & I(\omega) + I(\phi) = 2. \end{aligned}$$

Suppose $I(\max(\omega, \phi)) + I(\min(\omega, \phi)) = 1$. Then without loss of generality, we have $\max(\omega, \phi)(x) = \max(\omega, \phi)(y) = i$. Without loss of generality, $\phi(x) \leq$

$\omega(x) = i$. Now we have one of the following two possibilities:

$$\begin{array}{ll} & \text{(a) } \phi(y) \leq \omega(y) = i, \\ \text{or} & \text{(b) } \omega(y) \leq \phi(y) = i. \end{array}$$

For case (a), we have that $\omega(x) = \omega(y) = i$, so that $I(\omega) + I(\phi) \geq 1$. For case (b) we have $\omega(y) \leq \omega(x) = i$, but we know that $\omega(x) \leq \omega(y)$ (since $x \prec y$). Hence $\omega(x) = \omega(y) = i$ and $I(\omega) + I(\phi) \geq 1$.

If $I(\max(\omega, \phi)) + I(\min(\omega, \phi)) = 0$ then (1) trivially holds. Thus (1) holds in all cases and the proof is complete. $\square$

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