Covering two-edge-coloured complete graphs with two disjoint monochromatic cycles

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Abstract

In 1998 Łuczak, Rödl and Szemerédi [3] proved, by means of the Regularity Lemma, that there exists n_0 such that, for any $n \geq n_0$ and two-edge-colouring of K_n, there exists a pair of vertex disjoint monochromatic cycles of opposite colours covering the vertices of K_n. In this paper we make use of an alternative method of finding useful structure in a graph, leading to a proof of the same result with a much smaller value of n_0. The proof gives a polynomial time algorithm for finding the two cycles.

1 Introduction

Throughout this paper G will be a complete graph on n vertices, whose edges are coloured either red or blue.

We are interested in monochromatic cycles, i.e., sets of vertices of G given a cyclic order such that all edges between successive vertices possess the same colour. Note that in particular we do allow cycles to have length one (a single vertex) or two (an edge)

If the vertices of G can be partitioned into two sets C_r and C_b, with C_r possessing a red Hamilton cycle and C_b a blue Hamilton cycle, we say that G has a *two-cycle partition*.

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Lehel conjectured that, for any n and G, we can find a two-cycle partition of G.

Gyárfás [2] proved the following theorem.

Theorem 1. *For any two-coloured complete graph G we can find within G a red cycle and a blue cycle which together cover the vertices of G and have at most one vertex in common.*

More recently, in 1998 Łuczak, Rödl and Szemerédi [3] showed that there exists n_0 such that for any $n \geq n_0$ there is a two-cycle partition of G. Their proof makes use of the Szemerédi Regularity Lemma [4], and as a result their n_0 is given by a tower of 2s of height 10^{300}.

In this paper we prove, by a different method, that Lehel's conjecture holds for all $n > n_0$; we do not make use of the Regularity Lemma, and so our result gives a much smaller value of n_0.

Theorem 2. *For all $n \geq 2^{18000}$ and all 2-coloured graphs G on n vertices, there exists a two-cycle partition of G.*

The argument of Łuczak, Rödl and Szemerédi and this paper both follow a similar pattern.

First, dispose of a pathological case: when there exists a very large complete bipartite subgraph.

Second, partition the vertices of G into many sets in a way which locates useful structure.

Third, use this partition to find good candidates for the red-cycle and blue-cycle parts of G, together with a relatively small number of problem vertices.

Fourth, find an algorithm to incorporate the problem vertices into the candidate cycle-parts and appeal to the useful structure to construct the actual red and blue cycles.

The argument of Łuczak, Rödl and Szemerédi uses the Regularity Lemma to partition $V(G)$ into a very large but fixed number of parts, finding many ε-regular pairs of parts whose red- or blue-density is significantly greater than zero. In light of the Blow-up Lemma, this structure is essentially as strong as having complete red or blue bipartite graphs between many pairs of parts.

In this paper, we apply instead Ramsey's theorem to find a partition of $V(G)$ into many small cliques of bounded size; we deal with the red and blue cliques separately and eventually find that two red cliques are joined either by a few red edges or by a blue complete bipartite graph. This structure is not so strong as that obtained from the Regularity Lemma (although it does allow us to make our constructions explicitly rather than by appealing to technical results such as the Blow-up Lemma) and as a result the algorithm we use to incorporate our problem vertices into cycles requires a little more care.

2 Large complete bipartite subgraphs

We make use of two theorems from the paper of Łuczak, Rödl and Szemerédi [3].

Theorem 3. *If there exists a partition $V(G) = V_1 \sqcup V_2 \sqcup V_3$, such that $\min(|V_1|, |V_2|) \geq 5 + 2|V_3|$ and V_1, V_2 form the parts of a blue complete bipartite graph, then there is a two-cycle partition of G.*

The second theorem is a variant on 'Fact 4.3' from the same paper, adapted to give us greater control over the relatively small number of paths we will need to claim exist.

Theorem 4. *For every $k \geq 2$ and $n \geq 63k$, the following holds. Either $V(G)$ may be partitioned into three sets satisfying the conditions of Theorem 3, or given disjoint subsets A, B and C of $V(G)$, where $|A|, |B| \geq \frac{n}{2k}$ and $|C| \leq \frac{n}{5k}$, there exists a red path of length at most $100k$ whose initial vertex is in A, whose final vertex is in B, and whose interior vertices are in $V(G) - (A \cup B \cup C)$.*

Proof. Let R be the graph whose edges are the red edges of G on the vertex set $V(G) - C$. Let N_r be the set of vertices at distance exactly r from the set A, and N'_r be the set of vertices at distance exactly r from the set B.

If both $\sum_{r=1}^{50k} |N_r| > \frac{n}{2}$ and $\sum_{r=1}^{50k} |N'_r| > \frac{n}{2}$, then there must be a path of length at most $100k$ from A to B within R as desired.

If there does not exist any such path, then we may assume without loss of generality that $\sum_{r=1}^{50k} |N_r| \leq \frac{n}{2}$, and so there must be r_0, $1 \leq r_0 \leq 50k$, such that $|N_{r_0}| \leq \frac{n}{100k}$.

Now let $V_1 = A \cup \bigcup_{r=1}^{r_0-1} N_r$, $V_3 = N_{r_0} \cup C$ and $V_2 = V(G) - (V_1 \cup V_3)$. We have

$$|V_1| \geq |A| \geq \frac{n}{2k} \geq 5 + 2\left(\frac{n}{5k} + \frac{n}{100k}\right) \geq 5 + 2|V_3| \ ,$$

and similarly $|V_2| \geq 5 + 2|V_3|$. By definition of the N_i and N'_i, all the edges between V_1 and V_2 must be blue, satisfying the conditions of Theorem 3. $\square$

With these theorems in mind, we shall henceforth assume that G does not possess any large complete bipartite graph of either colour, so that we can apply Theorem 4 with either colour.

3 Clique-cycles

Throughout this section and the rest of the paper, when $\mathcal{U} = (U_1, \ldots, U_u)$ is a list and we refer to an element U_i, $i > u$ we mean the element $U_{i \bmod u}$.

Suppose that $\mathcal{U} = (U_1, \ldots, U_u)$, $u \geq 3$, is a list of disjoint red cliques within $V(G)$. Suppose further that there are specified disjoint red *linking paths* $u_{i,i+1 \bmod u}$ between each pair U_i and U_{i+1} whose interior vertices are not in any U_j. We call this structure an *on-colour* red *clique-cycle*. In general the linking paths will be paths on only two vertices (i.e., single red edges), and never on more than four.

Suppose that $\mathcal{V} = (V_1, \ldots, V_v)$, $v \geq 3$, is a list of disjoint blue cliques within $V(G)$, with each pair V_i, V_{i+1} spanning a red complete bipartite subgraph of G. We will call this an *off-colour* red clique-cycle.

If the U_i and the V_j are disjoint, and furthermore there exist disjoint red paths P_1 and P_2 between U_1 and V_1, each of length at most $18000 + u + v$, neither of which meet either $u_{1,2}$ or $u_{u,1}$, and whose interior vertices are not in any of the U_i or V_j, we call $(\mathcal{U}, P_1, P_2, \mathcal{V})$ a red *clique-cycle pair* (see Figure 1(iii)). We do permit one or both of the sets $\mathcal{U}$ and $\mathcal{V}$ to be empty, in which case we require that the paths P_1 and P_2 are empty.

Given a red clique-cycle pair, it is trivial to see that there exists a red cycle which passes through every vertex of the on-colour red clique-cycle, both paths P_1 and P_2, and $\min_i |V_i|$ vertices of each of the V_i.

We define similarly blue on-colour and off-colour clique-cycles and a blue clique-cycle pair.

The purpose of this section is to establish the following lemma.

Lemma 5. *When $n \geq 2^{18000}$ there exists a partition of the vertices of G into the following three parts:*

a red clique-cycle pair $(\mathcal{U}, P_1, P_2, \mathcal{V})$,

a blue clique-cycle pair $(\mathcal{X}, Q_1, Q_2, \mathcal{Y})$, and

a 'leftover set' L_1.

The leftover set has size at most $2^{17990} + \frac{n}{80} + 6(v+y)$ (where $v = |\mathcal{V}|$ and $y = |\mathcal{Y}|$), and all of the cliques in the off-colour clique-cycles have size between 8981 and 8989. Furthermore, when two of the clique-cycles are not empty we have $|L_1| \leq 2^{17990} + \frac{n}{120} + 6(v + y)$.

Proof. By Ramsey's Theorem, we can guarantee that any set of 4^{8995} vertices of G contains either a red or a blue clique of size 8995.

Thus we can find a partition of $V(G)$ into a collection $\mathcal{R} = R_1, \ldots$ of red cliques each of size 8995, a collection $\mathcal{B} = B_1, \ldots$ of blue cliques each of size 8995, and a set L_0 of size at most $2^{17990} < \frac{n}{1000}$.

We say that two red cliques are *red-adjacent* if there exists a red matching of size at least four between them, and *blue-adjacent* otherwise. This defines a two-coloured complete graph with vertex set $\mathcal{R}$.

By Theorem 1 (Gyárfás' result), there exist red and blue cycles C_r and C_b within this graph which cover $\mathcal{R}$ and which intersect in at most one member of $\mathcal{R}$ (Figure 1(i)).

We let $\mathcal{U}' = C_r - C_b$. Note that C_r and C_b may intersect in at most one clique. If they do intersect in a clique $(C_b)_j$, so that $C_r = (\ldots, U'_s, (C_b)_j, U'_{s+1}, \ldots)$, then there is a red path $u_{s,s+1}$ on either three or four vertices from U'_s to U'_{s+1} through $(C_b)_j$. Since every other pair of sets U'_i, U'_{i+1} has a red matching of size four between them, we can construct all the desired disjoint red paths $u_{i,i+1 \bmod u}$ as single red edges from the matchings. With these paths, the list $\mathcal{U}'$ becomes an on-colour red clique-cycle.

We let $\mathcal{Y}'$ be the cliques in C_b, with the exception that if $C_r \cap C_b = \{(C_b)_j\}$ we replace $(C_b)_j$ with $(C_b)_j - u_{s,s+1}$.

Now since there is no red matching of size four between any pair Y'_j, Y'_{j+1} we can remove six vertices from each Y'_j to obtain Y''_j such that each pair Y''_j, Y''_{j+1} spans a blue complete bipartite graph. The list $\mathcal{Y}''$ is an off-colour blue clique-cycle; each clique in it has size between 8987 and 8989. The two clique-cycles $\mathcal{U}'$ and $\mathcal{Y}''$ are disjoint, as in Figure 1(ii).

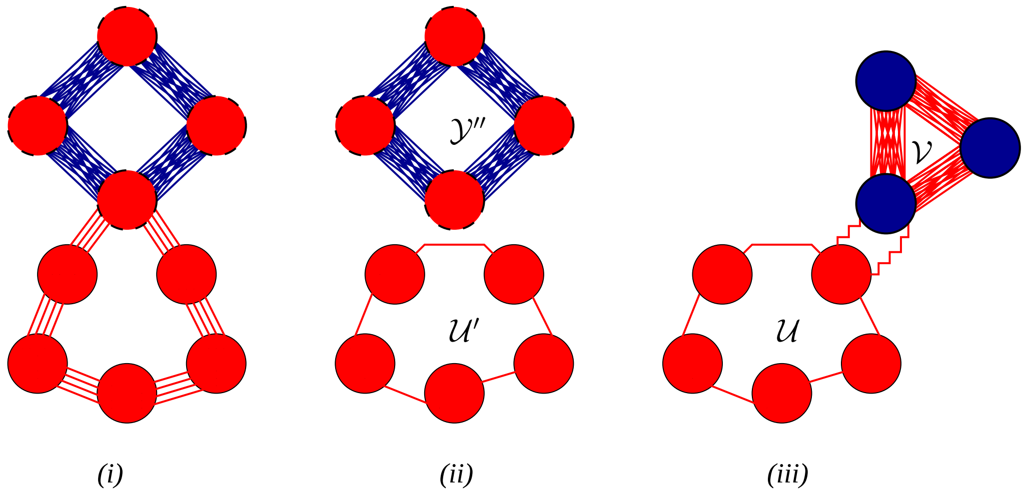

Figure 1: (i) The red cliques covered by a red and a blue cycle,
(ii) The clique-cycles $\mathcal{U}'$ and $\mathcal{Y}''$ obtained, and
(iii) A red clique-cycle pair.

Similarly we say that two blue cliques are blue-adjacent if there exists a blue matching of size at least four between them, and red-adjacent otherwise. By applying the theorem of Gyárfás in the same way to $\mathcal{B}$ we obtain the disjoint on-colour blue clique-cycle $\mathcal{X}'$ and off-colour red clique-cycle $\mathcal{V}''$.

We will now construct $\mathcal{U}$, $\mathcal{V}$, $\mathcal{X}$ and $\mathcal{Y}$.

First, if $|\bigcup \mathcal{U}'| \leq \frac{n}{240}$ then we set $\mathcal{U} = \emptyset$ and $u = 0$, and similarly for $\mathcal{V}$, $\mathcal{X}$ and $\mathcal{Y}$.

For each $1 \leq i \leq u$ either $u_{i,i+1}$ is a path on three or four vertices or we can identify a red matching of size four between U'_i and U'_{i+1} including the edge $u_{i,i+1}$. We can similarly identify blue matchings of size four between pairs in $\mathcal{X}'$. Let C_1 be the union of all the vertices in these identified matchings and the linking paths.

If both $\mathcal{U}$ and $\mathcal{V}$ are non-empty then let $A_1 = \bigcup \mathcal{U}' - C_1$, and $B_1 = \bigcup \mathcal{V}''$. Now $|C_1| \leq 8\frac{n}{8995} < \frac{17n}{18000}$, $|A_1|, |B_1| \geq \frac{n}{360}$ and the sets A_1, B_1 and C_1 are disjoint by construction. Thus we can apply Theorem 4 with $k = 180$ to obtain a minimal red path P_1 of length at most 18000 from A_1 to B_1 which does not pass through any vertices of C_1.

Note that $18000 < \frac{n}{18000}$. We let $C_2 = C_1 \cup P_1$, $A_2 = \bigcup \mathcal{U}' - C_2$ and $B_2 = \bigcup \mathcal{V}'' - C_2$. These three sets still satisfy the conditions of Theorem 4, so applying it we obtain a second minimal red path P'_2 of length at most 18000 between A_2 and B_2 which avoids the vertices of C_2.

Continuing this, if both $\mathcal{X}$ and $\mathcal{Y}$ are non-empty we obtain blue paths Q_1 and Q_2' between $\bigcup \mathcal{X}' - C_1$ and $\bigcup \mathcal{Y}''$ which are of length at most 18000 and such that the paths P_1, P_2', Q_1, Q_2' are pairwise disjoint.

We renumber the lists $\mathcal{U}'$ and $\mathcal{V}''$ if necessary such that the path P_1 goes from U_1' to V_1''. The path P_2' does not necessarily go from U_1' to V_1''. But there is a chain of sets $U_1', \ldots, U_p'$ such that P_2' terminates in U_p' and such that each pair of sets U_i', U_{i+1}', $1 \leq i < p$, spans a red matching of size four contained in C_1 (we may assume that if there is a path $u_{i,i+1}$ of length greater than one then it comes after p). In each matching one of the four red edges must be disjoint from the linking paths; thus we can find a red path P_2 extending P_2' into U_1' and into V_1'' (the latter since consecutive cliques in $\mathcal{V}''$ span red complete bipartite graphs) such that P_2 does not intersect any of P_1, Q_1, Q_2' (since these avoid C_1) or the linking paths. The path P_1 is of length at most 18000, while $|P_2| \leq 18000 + u + v$.

Similarly we can assume Q_1 goes from X_1' to Y_1'' and extend Q_2' to obtain Q_2 which also starts and ends in those sets. Again $|Q_1| \leq 18000$ and $|Q_2| \leq 18000 + x + y$.

Finally we obtain $\mathcal{U} = (U_1, \ldots, U_u)$ by letting U_i contain all the vertices in U_i' that are not interior vertices of any of the paths P_1, P_2, Q_1, Q_2, and $\mathcal{V}$, $\mathcal{X}$ and $\mathcal{Y}$ similarly. We let L_1 contain all the vertices which are not in either clique-cycle pair.

Observe that since the paths P_1 and P_2' are of minimal length, neither path intersects any one of the red cliques $\mathcal{Y}''$ in more than two places, and by construction the paths Q_1 and Q_2 intersect each clique in at most one place. Thus for each i, $|Y_i''| - |Y_i| \leq 6$, so that each clique in $\mathcal{Y}$ has size between 8981 and 8989. The same holds for the cliques $\mathcal{V}$.

Since a vertex can only be in L_1 if it was either in L_0, or was removed from either $\mathcal{V}'$ or $\mathcal{Y}'$, or was in a clique-cycle of size at most $\frac{n}{240}$, we obtain the desired bounds on $|L_1|$.

This partition fulfills the requirements of the lemma. $\square$

4 Corrected cycle pairs

Given a partition of $V(G)$ into a red clique-cycle, a blue clique-cycle and a leftover set, as provided by Lemma 5, we would like to say that there is a red cycle which covers the red-clique-cycle and some of the leftover set and a disjoint blue cycle which covers everything else. Unfortunately this is not generally true. In this section we will define a similar concept to a red clique-cycle pair: a red *corrected cycle pair*. We will see that it can be covered by a red cycle.

First we must define some terms, in each case with respect to a given partition of G into red and blue clique-cycle pairs and a leftover set (as is provided by Lemma 5). A red *pickup path* is a red path whose start and end vertices are in the same clique in one clique-cycle, and whose interior vertices are alternately vertices within the leftover set and within other clique-cycles. We will see that disjoint pickup paths can be constructed covering every vertex of the leftover set.

A red *balance path* is a red path whose initial and final vertices are in the same clique in an off-colour clique-cycle; its purpose is to cover some excess vertices within off-colour clique-cycles.

We say that a *free* vertex is any vertex which is not contained in any pickup or balance path, any of the linking paths in the on-colour clique-cycles, or the paths P_1, P_2, Q_1, Q_2.

When S is a subset of $V(G)$, we let $\mathit{Pick}(S)$ be the number of pickup paths which start and end in S, $\mathit{Bal}(S)$ be the number of balance paths which start and end in S, and $\mathit{Free}(S)$ be the number of free vertices in the set S.

Finally, when V_i is a clique in an off-colour clique-cycle $\mathcal{V}$, we define $\mathit{Spin}(V_i)$ by

$$\begin{aligned} \mathit{Spin}(V_i) &= \mathit{Free}(V_i) + \mathit{Pick}(V_i) + \mathit{Bal}(V_i) & (i \geq 2 \text{ or } i = 1, P_1 = \emptyset) \\ \mathit{Spin}(V_1) &= \mathit{Free}(V_1) + \mathit{Pick}(V_1) + \mathit{Bal}(V_1) + 1 & (i = 1, P_1 \neq \emptyset). \end{aligned}$$

We say that the off-colour clique-cycle $\mathcal{V}$ is *balanced* if all its cliques have the same spin.

We define a red *corrected cycle pair* to be a collection $(\mathcal{U}, P_1, P_2, \mathcal{V}, J_r)$ consisting of a red clique-cycle pair $(\mathcal{U}, P_1, P_2, \mathcal{V})$ together with a set J_r of red pickup and balance paths, such that the pickup and balance paths are disjoint from each other, from the linking paths in the on-colour clique-cycle, and from the paths P_1, P_2, and such that the off-colour clique-cycle is balanced.

Lemma 6. *If G possesses a red corrected cycle pair $(\mathcal{U}, P_1, P_2, \mathcal{V}, J_r)$ then we can find a red cycle C_r in G covering exactly the vertices of the corrected cycle pair.*

Proof. We construct C_r as follows.

If neither $\mathcal{U}$ nor $\mathcal{V}$ are empty, we start at the start vertex of P_2 in U_1.

If this is the start vertex of a pickup or balance path we follow the path to its end vertex, by definition also in U_1. Now if there are any pickup or balance paths remaining in U_1 we move directly to the start vertex and then along each in turn. We then move to each free vertex in U_1 that we have not yet visited in succession, and eventually to the start vertex of the path $u_{1,2}$ and along it.

We now repeat the above procedure for each U_i, $2 \leq i \leq u$. On returning to U_1 along $u_{u,1}$ we move to the start vertex of P_1 and along it to V_1.

We now apply the following process. If the vertex in V_i we are currently at is one end of a pickup or balance path, we follow the path until we return to V_i. We now select if possible a vertex in $V_{i+1 \bmod v}$ which is the start vertex of a pickup or balance path which we have not yet visited and move to it; if this is not possible we move to any free vertex in $V_{i+1 \bmod v}$ which we have not yet visited.

We repeat this process until we are forced to stop. When this occurs, we are at a vertex in some clique V_i, having travelled every pickup and balance path and visited every free vertex in $V_{i+1 \bmod v}$. Thus we have been around the clique-cycle $\mathit{Spin}(V_{i+1 \bmod v})$ times. Since the off-colour clique-cycle is balanced, we are at V_v and have been along every pickup and balance path and through every free vertex in $\bigcup \mathcal{V}$. We move directly to the end vertex of P_2 in V_1 and along P_2 to U_1, completing the cycle C_r.

If both $\mathcal{U}$ and $\mathcal{V}$ are empty, we set $C_r = \emptyset$. If $\mathcal{U}$ is empty but $\mathcal{V}$ is not we start at the start vertex of a pickup or balance path in V_1 if this is possible, or any free vertex in V_1 if not, and follow the above procedure until we return to the start vertex and complete the cycle C_r. If $\mathcal{V}$ is empty but $\mathcal{U}$ is not we start at the end vertex of $u_{u,1}$ in U_1 and follow the clique-cycle $\mathcal{U}$ as above until we return to that vertex, completing the cycle C_r.

By the definition of a corrected cycle pair, the red cycle C_r covers exactly the vertices of the corrected cycle pair. □

5 Correcting clique-cycle pairs

In this section we describe algorithms which take the partition of $V(G)$ into a leftover set, a red clique-cycle pair and a blue clique-cycle pair given by Lemma 5 and return the desired two-cycle partition, by way of Lemma 6.

We will need to use different algorithms depending on the sizes of the various parts $\mathcal{U}$, $\mathcal{V}$, $\mathcal{X}$ and $\mathcal{Y}$. In each case we will construct sequentially a set of pickup and balance paths on the way to giving corrected cycle pairs. We will again use the functions *Pick*, *Bal*, *Free* and *Spin* defined in the previous section; in each case with reference to the current set of pickup and balance paths at that point in the algorithm.

We will use the following lemma to obtain a set of pickup paths through all vertices of the leftover set.

Lemma 7. *Let $A_1, \ldots, A_a, B_1, \ldots, B_b, C$ be disjoint subsets of $V(G)$, where the A_i are subsets of cliques in one clique-cycle, the B_j are subsets of cliques in another, and C is a leftover set. Suppose that $2|C| \leq \min(|A_1| + \cdots + |A_a| - 4a, |B_1| + \cdots + |B_b| - 4b)$. Then there exist collections J_r and J_b of disjoint red and blue pickup paths within $A_1 \cup \cdots \cup B_b \cup C$ such that each red path starts and ends in an A_i while each blue path starts and ends in a B_j and such that every vertex in C is in one of the paths. Furthermore in any A_i or B_j the number of vertices which are in none of the paths J_r or J_b (free vertices) is greater than the number of vertices which are interior vertices of the paths J_r or J_b.*

Proof. We apply the following algorithm. First we mark all vertices as *active*. Now for each member c of C in succession, we proceed as follows.

If there are red edges between c and two active members x, y of some A_i then we record into J_r the red pickup path x, c, y and mark these vertices as *inactive*.

If there is no such red path through c, but there are blue edges between c and two active members x, y of some B_j then we record into J_b the blue pickup path x, c, y and mark these vertices as inactive.

If there are neither red nor blue pickup paths, we mark c as *remaining*.

We let the eventual set of remaining vertices be R. If it is empty, we are done. If not, then each $r \in R$ is red-adjacent to at most one active vertex in each A_i, and blue-adjacent to at most one active vertex in each B_j.

Observe that there must exist at least one pair of sets A_α and B_β which each contain at least five active vertices. Since any two-edge-colouring of $K_{5,5}$ has either a red or a blue matching of size three, we are guaranteed such between the active vertices of A_α and B_β. We assume without loss of generality that the former holds.

If $R = \{r_1\}$, then r_1 is blue-adjacent to at most one of the vertices of the red matching in B_β, so there is a red path on five vertices from A_α through r_1 and returning to A_α. We record this pickup path into J_r and are done.

If $|R| \geq 2$, then let $R = \{r_1, \dots, r_r\}$. Since we have a red matching of size three between A_α and B_β we can choose active vertices a_1, b_1, b_{r+1}, a_2 such that a_1, b_1, r_1 and r_r, b_{r+1}, a_2 are both red paths from A_α to r_1 and r_r respectively. We mark these vertices as *interior-inactive*. By the original condition on $|C|$ there remain at least $2|R| + 4b - 2$ active vertices in $B_1 \cup \cdots \cup B_b$.

Now for each $1 \leq i \leq r-1$ in succession, since there must be at least $2|R| + 4b - 1 - i$ active vertices in $B_1 \cup \cdots \cup B_b$ (one is made interior-inactive at each step) we can find an active vertex b_{i+1} in a set with at least four more active vertices than interior-inactive vertices which is red-adjacent to both r_i and r_{i+1}, and mark it as interior-inactive.

Finally we record the red pickup path $a_1, b_1, r_1, b_2, \dots, b_r, r_r, b_{r+1}, a_2$ which passes through all of R into J_r. This path is the only path which has interior vertices in any of the A_i or B_j, and by its construction each of the A_i and B_j contains more active vertices (in none of the J_r or J_b) than interior-inactive vertices. $\square$

We now give the various algorithms for constructing two-cycle partitions.

Lemma 8. *If $n \geq 2^{18000}$ and G has a partition as in Lemma 5 in which both $|\bigcup \mathcal{U}|, |\bigcup \mathcal{X}| \geq \frac{n}{20}$, then G has a two-cycle partition.*

Proof. First we modify the off-colour clique-cycles $\mathcal{V}$, $\mathcal{Y}$ (if these are not empty) by removing vertices from each clique in these clique-cycles until $|V_1| = |Y_1| = 8981$ and all the other cliques have size 8980, to obtain $\mathcal{V}'$ and $\mathcal{Y}'$. If either clique-cycle is empty we do nothing to it. We create a new leftover set L_2 as the union of L_1 and the at most $9(v+y)$ vertices removed. The modified off-colour clique-cycles are balanced. Observe that $|L_2| \leq 2^{17790} + \frac{n}{120} + 15(v+y)$.

Now we let A_i be the set of free vertices in U_i for each i, B_j be the set of free vertices in X_j for each j, and $C = L_2$.

Observe that $|L_2| \leq 2^{17790} + \frac{n}{120} + 15\frac{n}{8995} < \frac{23n}{2000}$. Furthermore the number of free vertices in $\mathcal{U}$ is at least $|\bigcup \mathcal{U}| - 2u - 2 > \frac{90n}{2000} + 4u$, and similarly the number of free vertices in $\mathcal{X}$ is at least $\frac{90n}{2000} + 4x$. Thus the sets A_i, B_j and C satisfy the conditions of Lemma 7, and we can apply this lemma to obtain disjoint sets J_r and J_b consisting of pickup paths which are disjoint from each other, from the linking edges and paths in $\mathcal{U}$ and $\mathcal{X}$, and from the paths P_1, P_2, Q_1, Q_2. Every vertex in L_2 is in one of these paths. We modify $\mathcal{U}$ by removing every vertex in J_b to obtain $\mathcal{U}'$, and we modify $\mathcal{X}$ similarly to obtain $\mathcal{X}'$.

Now $(\mathcal{U}', P_1, P_2, \mathcal{V}', J_r)$ forms a red corrected cycle pair, which is disjoint from the blue corrected cycle pair $(\mathcal{X}', Q_1, Q_2, \mathcal{Y}', J_b)$. The two corrected cycle pairs cover $V(G)$. By Lemma 6 their vertices form the desired two-cycle partition. $\square$

This construction was made easier by the fact that we could simply remove a small number of vertices from the off-colour clique-cycles to force them to be balanced. In the remaining cases we have to do more work to ensure this. We require the following trivial lemma.

Lemma 9. *If A and B are disjoint subsets of $V(G)$ each of size at least three, then either there exists a vertex in A red-adjacent to two vertices in B, or there exists a vertex in B blue-adjacent to two vertices in A.*

This lemma allows us to construct a balance path and so reduce the spin of the cliques containing A and B by one.

Lemma 10. *If $n \geq 2^{18000}$ and both $|\bigcup\mathcal{U}|, |\bigcup\mathcal{Y}| \geq \frac{n}{20}$ then we can find a two-cycle partition of G.*

Proof. We modify the off-colour clique-cycle $\mathcal{V}$ (if it is not empty) by removing vertices until each clique has size 8980, except for V_1 which has size 8981, to obtain $\mathcal{V}'$. We create a new leftover set L_2 consisting of L_1 together with the removed vertices. We observe that $|L_2| \leq \frac{n}{1024} + \frac{n}{120} + 15v + 6y < \frac{23n}{2000}$.

Since $|\bigcup\mathcal{U}| \geq \frac{n}{20}$ we see that there are certainly at least $4|L_2| + 130u$ free vertices in $\bigcup\mathcal{U}$.

Now choose the largest $m \leq 8980$ such that $|\bigcup\mathcal{Y}| - my \geq 2|L_2| + 4y$. Observe that $m \geq 4480$. We apply a similar algorithm to that in the previous lemma. From each clique Y_j we choose a subset B_j consisting of $|Y_j| - m$ of the free vertices. Observe that

$$|B_1 \cup \cdots \cup B_y| = |\bigcup\mathcal{Y}| - my \in [2|L_2| + 4y, 2|L_2| + 5y) .$$

We let A_i be the set of free vertices in U_i for each i. We let $C = L_2$, and apply Lemma 7 to obtain sets J_r, J_b of pickup paths covering L_2.

By the definition of the spin of a clique Y_j, when a pickup path is constructed which starts and ends in Y_j (using two vertices of Y_j) it decreases the spin of the clique by one, while from Lemma 7 the number of vertices of Y_j which are interior vertices of any pickup path is exceeded by the number of vertices which are in no pickup path. It follows that the use of Lemma 7 to create a set of pickup paths causes the spin of the clique Y_j to decrease by at most $\frac{|B_j|}{2} \leq \frac{8981-m}{2}$. Thus at this point each clique Y_j has spin at least $b = \left\lfloor \frac{8980+m}{2} \right\rfloor \geq 6730$.

We now say that a clique Y_j is *balanced* if $Spin(Y_j) = b$, and *unbalanced* otherwise. We note that an unbalanced clique must have spin greater than b. We call the difference $Spin(Y_j) - b$ the excess spin of the clique Y_j.

Since $|\bigcup\mathcal{U}| \geq \frac{n}{20}$ and not more than $2|L_2|$ vertices in $\mathcal{U}$ can be in any of the paths $J_r \cup J_b$ we observe that the number of free vertices in $\bigcup\mathcal{U}$ is still at least $2|L_2| + 130u$.

From the definitions of m and b the sum of the excess spins of all the cliques in $\mathcal{Y}$ cannot exceed

$$\frac{|\bigcup\mathcal{Y}| - my}{2} \leq \frac{2|L_2| + 5y}{2} \leq |L_2| + 3y < |L_2| + 60u .$$

We construct J_r' and J_b' by adding new balance paths sequentially to J_r and J_b as follows.

If Y_j is an unbalanced clique, then $\mathit{Spin}(Y_j) > 6730$, so $\mathit{Free}(Y_j) + \mathit{Pick}(Y_j) + \mathit{Bal}(Y_j) \geq 6730$. But each pickup or balance path contributing to $\mathit{Pick}(Y_j)$ or $\mathit{Bal}(Y_j)$ uses two vertices from Y_j, and $|Y_j| \leq 8981$. Thus certainly $\mathit{Free}(Y_j) \geq 3$. Since $|\bigcup \mathcal{U}| \geq 4|L_2| + 130u$ there must be a clique U_i with $\mathit{Free}(U_i) \geq 3$. By Lemma 9 there exists either a red balance path from the free vertices of U_i to a free vertex in V_j or a blue balance path from the free vertices of V_j to a free vertex in U_i. We record the red balance path into J_r' if it exists, otherwise the blue balance path into J_b'. This procedure causes $\mathit{Spin}(Y_j)$ to decrease by one. We repeat this until every clique Y_i is balanced.

Finally we modify $\mathcal{U}$ by removing all vertices in J_b' to obtain $\mathcal{U}'$ and $\mathcal{Y}$ by removing all vertices in J_r' to obtain the balanced off-colour clique-cycle $\mathcal{Y}'$.

Now $(\mathcal{U}', P_1, P_2, \mathcal{V}', J_r')$ and $(\mathcal{X}, Q_1, Q_2, \mathcal{Y}', J_b')$ are disjoint corrected cycle pairs covering $V(G)$, and the result follows by Lemma 6. $\square$

In the next case we have to balance simultaneously two off-colour clique-cycles; this case requires the most care.

Lemma 11. *If $n \geq 2^{18000}$ and both $|\bigcup \mathcal{V}|, |\bigcup \mathcal{Y}| \geq \frac{n}{20}$ then there exists a two-cycle partition of G.*

Proof. We begin similarly to the previous lemma. Choose the largest $m_r, m_b \leq 8980$ such that $|\bigcup \mathcal{V}| - m_r v \geq 2|L_1| + 4v$ and $|\bigcup \mathcal{Y}| - m_b y \geq 2|L_1| + 4y$. Note that $m_r, m_b \geq 5300$.

For each i, choose a set A_i consisting of $|V_i| - m_r$ of the free vertices of V_i; for each j choose a set B_j of free vertices of Y_j of size $|Y_j| - m_b$. Let $C = L_1$, and apply Lemma 7 to obtain sets J_r, J_b of pickup paths covering every vertex in L_1. By an identical argument to that in the previous lemma, the spin of any clique V_i has decreased by at most $\frac{8989-m_r}{2}$ so is at least $\frac{8980+m_r}{2} \geq 7100$. Similarly each clique Y_j now has spin at least $\frac{8980+m_b}{2} \geq 7100$. There remain at least $2|L_1| + 45y$ free vertices in $\mathcal{V}$, and at least $2|L_1| + 45v$ free vertices in $\mathcal{Y}$.

Now we must balance both off-colour clique-cycles. We must choose the parameters b_r and b_b which will be the spins of cliques in the red and blue off-colour clique-cycles in our eventual corrected cycle pairs.

We define the excess spin of the clique-cycle $\mathcal{V}$ by

$$\mathit{Excess}(\mathcal{V}, b_r) = \sum_{k=1}^{v} (\mathit{Spin}(V_k) - b_r) \ .$$

Since the cliques V_1 and Y_1 each have at least three free vertices, we can identify either a red or a blue matching between them of size two. Assume without loss of generality that it is a red matching (α, β), $(\gamma, \delta) \in V_1 \times Y_1$.

Let $b_r, b_b \leq 7097$ be the largest values such that

$$\mathit{Excess}(\mathcal{Y}, b_b) + 10 \leq \mathit{Excess}(\mathcal{V}, b_r) \leq \mathit{Excess}(\mathcal{Y}, b_b) + v + 20 \ .$$

Since $20v \geq y \geq \frac{v}{20}$ we are guaranteed to find that one of b_r and b_b is between 7077 and 7097.

Since $\mathit{Excess}(\mathcal{Y}, 7097)$ cannot exceed $|L_1|$ we are guaranteed to find also that $b_r \geq 5000$, and similarly for b_b. We say that a clique in $\mathcal{V}$ is balanced if its spin is b_r, and similarly for $\mathcal{Y}$. Observe that an unbalanced clique must have at least three free vertices; at this point every clique in $\mathcal{V}$ has spin at least $b_r + 3$, and similarly for $\mathcal{Y}$.

If $\mathit{Excess}(\mathcal{V}, b_r) - \mathit{Excess}(\mathcal{Y}, b_b) = s$ is even, choose a free vertex ϵ in Y_1 not in the red matching of size two. Note that $v + 20 < 2v - 4$, so that $s < 2v - 4$ and $2 + \frac{s}{2} < v$.

Now choose from each clique $V_2, \ldots, V_{2+\lfloor \frac{s}{2} \rfloor}$ two free vertices, and let B be a red balance path which starts and ends in the chosen vertices in $V_{2+\lfloor \frac{s}{2} \rfloor}$ and whose interior vertices are the other chosen vertices, α, β, γ, δ and if s is even ϵ. We record this red balance path along with the paths J_r to create J'_r.

Now the spin of any clique (with respect to the new sets J'_r, J_b) has decreased by at most three, so each clique in $\mathcal{V}$ has spin at least b_r and each clique in $\mathcal{Y}$ has spin at least b_b. The creation of B has decreased $\mathit{Excess}(\mathcal{Y}, b_b)$ by either two or three, depending on whether s is odd or even (the vertices β, δ and, if s is even, ϵ in Y_1 are no longer free). The creation of B has also decreased $\mathit{Excess}(\mathcal{V}, b_r)$ by either $s+2$ or $s+3$, again depending on whether s is odd or even (two vertices in each clique $V_1, \ldots, V_{2+\lfloor \frac{s}{2} \rfloor}$ are no longer free and one balance path has been created in $V_{2+\lfloor \frac{s}{2} \rfloor}$). Thus the creation of B gives us $\mathit{Excess}(\mathcal{Y}, b_b) = \mathit{Excess}(\mathcal{V}, b_r)$.

We apply Lemma 9 repeatedly to construct balance paths on three vertices between the free vertices of unbalanced pairs of cliques V_i and Y_j, each decreasing the spin of both V_i and Y_j by one. Eventually every clique in both off-colour clique-cycles is balanced. We let J''_r be the union of J'_r and the red balance paths just constructed, and J'_b be the union of J_b and the blue balance paths just constructed. We modify $\mathcal{V}$ and $\mathcal{Y}$ to obtain the balanced clique-cycles $\mathcal{V}'$ and $\mathcal{Y}'$ by removing all vertices in J'_b and J''_r respectively. Now $(\mathcal{U}, P_1, P_2, \mathcal{V}', J''_r)$ and $(\mathcal{X}, Q_1, Q_2, \mathcal{Y}', J'_b)$ are disjoint corrected cycle pairs covering $V(G)$, and the result follows. ☐

Finally we consider the possibility that one of the two clique-cycle pairs is small.

Lemma 12. *If $n \geq 2^{18000}$ and $|\bigcup \mathcal{X}|, |\bigcup \mathcal{Y}| \leq \frac{n}{20}$ then we have a two-cycle partition of G.*

Proof. We let $L_2 = L_1 \cup \bigcup \mathcal{X} \cup \bigcup \mathcal{Y} \cup Q_1 \cup Q_2$. Observe that

$$|L_2| \leq 2^{17990} + \frac{n}{80} + 6(v + y) + \frac{2n}{20} + 2(18000 + x + y) \leq \frac{12n}{100} .$$

Now either $|\bigcup \mathcal{U}| \geq \frac{42n}{100}$ or $|\bigcup \mathcal{V}| \geq \frac{42n}{100}$.

In the former case, we create L_3 by removing at most $8v$ vertices from $\mathcal{V}$ to obtain a balanced clique-cycle $\mathcal{V}'$. Then, for each $\ell \in L_3$ sequentially, we apply the following process to obtain a set J of pickup paths.

If ℓ is red-adjacent to two free vertices j_1, j_2 in any clique U_i then record into J the pickup path j_1, ℓ, j_2. Otherwise mark ℓ as remaining.

Let the set of remaining vertices be $R = \{r_1, \ldots, r_r\}$. Each vertex is red-adjacent to at most one free vertex in any clique U_i. Since $|\bigcup \mathcal{U}| \geq \frac{42n}{100}$ and each vertex in $L_3 - R$ has given a path in J which uses up two vertices from $\bigcup \mathcal{U}$, the number of free vertices remaining in $\bigcup \mathcal{U}$ exceeds $|R| + 3u$. We can follow the same logic as in Lemma 7 to greedily construct a blue cycle C_b whose vertices are alternately the members of R and free vertices from $\bigcup \mathcal{U}$.

We modify $\mathcal{U}$ by removing all the vertices in C_b to obtain $\mathcal{U}'$. Then $(\mathcal{U}', P_1, P_2, \mathcal{V}', J)$ is a red corrected cycle pair which covers exactly the vertices of $V(G)$ not in C_b, so by Lemma 6 it is covered by a red cycle C_r.

In the latter case, let $m \leq 8981$ be the greatest number such that

$$\sum_{i=1}^{v} (Spin(V_i) - m - 5) \geq |L_2| \ .$$

Since $|L_2| \leq \frac{12n}{100}$ and $|\bigcup \mathcal{V}| \geq \frac{42n}{100}$ we certainly have that $m > 5000$. Thus any clique with spin greater than m must have at least 100 free vertices. For each $\ell \in L_2$ we apply the following process.

If ℓ is red-adjacent to two free members j_1, j_2 of a clique V_i with $Spin(V_i) \geq m + 5$ then we record the red pickup path j_1, ℓ, j_2. If not, we mark ℓ as remaining.

Let the set of remaining vertices be R. Let

$$Excess(\mathcal{V}) = \sum_{i=1}^{v} (Spin(V_i) - m) \ .$$

Now every clique V_i has spin at least $m + 4$, and $Excess(\mathcal{V}) \geq |R|$. We say that a clique is balanced if it has spin m, and unbalanced otherwise.

We construct red balance paths on three vertices between the free vertices of pairs of unbalanced cliques V_i, V_j $(i, j \neq 1)$ until either $|R| \leq Excess(\mathcal{V}) \leq |R| + 1$ or there remain no red balance paths on three vertices between free vertices of pairs of unbalanced cliques. Observe that each balance path constructed reduces $Excess(\mathcal{V})$ by two.

In the first case, since the spin of V_1 is at least $m+4$ we have $|R| \geq 3$ and we can greedily construct a blue cycle C_b passing through all members of R and either $|R|$ (by choosing vertices alternately from R and $\mathcal{V}$) or $|R|+1$ (by having an extra edge in V_1 in the cycle) free vertices in the unbalanced cliques of $\mathcal{V}$, as appropriate. Then we construct $\mathcal{V}'$ by removing all vertices of C_b from $\mathcal{V}$; this is a balanced clique-cycle.

In the second case, we have a collection of unbalanced cliques $V_1, V_{r_1}, \ldots$ such that any pair V_{r_i}, V_{r_j} do not have any red balance path between their free vertices. Since each clique has at least 100 free vertices, certainly there are blue edges between the free vertices of any such pair.

If $|R| \leq 1$ then we can find further red balance paths between the free vertices of V_1 and of the V_{r_i} until either all the cliques are balanced or any pair of our remaining unbalanced cliques have blue edges between their free vertices. In either case we can find a blue cycle C_b

which passes through $Spin(V_i) - m$ of the free vertices of each such V_i; if $|R| = 0$ it passes through no other vertices, while if $|R| = 1$ it passes through the vertex in R also.

If $|R| \geq 2$ then we can find a blue cycle C_b covering exactly $Spin(V_i) - m$ free vertices of each unbalanced clique; between one free vertex in an unbalanced clique and the next along the cycle we may either have a blue edge or a blue path of length two passing through a member of R, as appropriate to cover all the members of R and to guarantee being able to pass from V_1 to the V_{r_i}.

In either case, we let J be the set of red balance paths we constructed and modify $\mathcal{V}$ by removing all vertices in C_b to obtain the balanced clique-cycle $\mathcal{V}'$. Then $(\mathcal{U}, P_1, P_2, \mathcal{V}', J)$ is a corrected clique-cycle which must be covered by a red cycle C_r, and C_b covers exactly the vertices of G not in it. $\square$

6 Proof of Theorem 2

Let $n \geq 2^{18000}$. Suppose that G is a two-edge-coloured complete graph on n vertices.

If G possesses a large bipartite subgraph satisfying the conditions of Theorem 3 then it possesses a two-cycle partition.

If G does not possess such a large bipartite subgraph then we may apply Lemma 5 to obtain a partition of $V(G)$ into disjoint red clique-cycle pair $(\mathcal{U}, P_1, P_2, \mathcal{V})$, blue clique-cycle pair $(\mathcal{X}, Q_1, Q_2, \mathcal{Y})$ and a leftover set.

At least one of $\mathcal{U}$, $\mathcal{V}$, $\mathcal{X}$ and $\mathcal{Y}$ must cover at least $\frac{n}{20}$ vertices, since the leftover set is not larger than $2^{17790} + \frac{n}{120} + 6(v + y) < \frac{n}{100}$ by choice of n. Without loss of generality, assume that one of $\mathcal{U}$ or $\mathcal{V}$ covers at least $\frac{n}{20}$ vertices. If also either $\mathcal{X}$ or $\mathcal{Y}$ covers at least $\frac{n}{20}$ vertices then we may apply one of Lemmas 8, 10 (which of course also gives a result when $\mathcal{X}$ and $\mathcal{V}$ are large) or 11 to find that there exists a two-cycle partition of G. If on the other hand neither $\mathcal{X}$ nor $\mathcal{Y}$ covers $\frac{n}{20}$ vertices then we may apply Lemma 12 to discover a two-cycle partition of $V(G)$. $\square$

7 Further thoughts

It is not hard to find minor improvements to the proof above, which we do not give in the interests of a shorter and more readable proof. In particular, we can define red-adjacency in Lemma 5 with a matching of size only three; we can argue that the leftover set should always be much smaller, and so on. However even making the most optimistic assumptions – that there is some way to pick up vertices from the leftover set in long paths rather than one at a time, that the correct exponent in Ramsey's Theorem should be 2 and so on, it seems impossible that this method could be made work with cliques of size smaller than 10 (and so with graphs of around 1000 vertices). On the other hand, it is already out of the question to check by brute force computation graphs on even 100 vertices, so while Lehel's conjecture certainly seems reasonable this method will not prove it in full.

The proof of Gyárfás' Theorem 1 is a linear time algorithm. We can read the proof of Theorem 4 as an algorithm which either produces the desired path on at most 18000 vertices (in quadratic time) or returns the large complete bipartite graph required for Theorem 3. The proof by Łuczak, Rödl and Szemerédi of that theorem is again a polynomial time algorithm finding the red and blue cycles explicitly, and it is easy to check that all our proofs amount to polynomial time algorithms, so that we have a polynomial time algorithm which returns the two-cycle partition of G (if it exists).

It seems reasonable that there should exist an extension of this result for larger numbers of colours: if the edges of K_n are k-coloured then we can find a partition of its vertices into k differently coloured monochromatic cycles. This is a slight strengthening of a conjecture of Erdős, Gyárfás and Pyber [1]: if the edges of K_n are k-coloured then we can find a partition of its vertices into k monochromatic cycles. However the methods in this paper do not seem to be easily extended to dealing with even three colours. We can certainly apply Ramsey's theorem in a similar way to obtain a partition into small monochromatic cliques and a leftover set, and then describe two red cliques as red-adjacent if joined by a small red matching. But we would then have to define blue- and green-adjacency between two red cliques; and the obvious way to do this (colouring by the majority colour of edges) does not even allow us to construct blue paths along blue-adjacent paths of red cliques.

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