Infinite Combinatorics and the theorems of Steinhaus and Ostrowski

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Abstract

We define combinatorial principles which unify and extend the classical results of Steinhaus and Piccard on the existence of interior points in the distance set. Thus the measure and category versions are derived from one topological theorem on interior points applied to the usual topology and the density topology on the line. Likewise we unify the subgroup theorem by reference to a Ramsey property. A combinatorial form of Ostrowski's theorem (that a bounded additive function is linear) permits the deduction of both the measure and category automatic continuity theorem for additive functions.

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1 Introduction

The field of infinite combinatorics has, largely under the influence of Erdős and his school, grown to have many applications, for example to Ramsey theory within combinatorics ([GRS], [Hin]), and to number theory ([TaoVu]). The theme of this paper is that an aspect of infinite combinatorics has powerful applications in analysis. The relevant concept we call *subuniversality,* because of the use of *universality* in a related context by Kestelman [Kes1]. We develop here the positive consequences of subuniversality; previously, attention had instead been focussed on negative aspects (particularly of universality, cf. [Mil1]). The applications in analysis include a unified treatment of the classical theorems of Steinhaus and Ostrowski in our title, on difference sets and on additive functions. Subuniversality is a notion of compactness, naturally linked to the notion of *shift-compactness* employed in the semigroup structures of measures under convolution (see [Par]); for a topological analysis of this insight and further applications see [BOst8] and [BOst12]. It also draws on the ambient additive combinatorics.

Recall that a function $h : \mathbb{R} \to \mathbb{R}$ is *additive* if it satisfies the Cauchy functional equation

$$h(x + y) = h(x) + h(y) \qquad \forall x, y \in \mathbb{R}.$$

Obvious examples of additive functions include *linear* functions h:

$$h(x) = cx \qquad \text{for some constant } c \in \mathbb{R}.$$

Use of Hamel bases readily shows that not all additive functions are linear (see e.g. [Kucz], V.2, [GerKucz]; cf. [CiePaw1] §5.2) in general. But continuity is enough to deduce linearity from additivity (by approximating real arguments by rational ones). One can assume much less; quite how much less is one of the questions addressed below.

In qualitative measure theory (that is, measure theory in which one is concerned only with whether the measure of a set is zero or positive, rather than with its numerical value), it is often the case that a measure-theoretic theorem has a category-theoretic analogue, in which we replace 'measurable function' by 'function having the Baire property' ([Kur-1], [Oxt] – briefly, 'Baire function' below), and 'set of positive measure' by 'non-meagre set' (or set of second category). See [Oxt] for a monograph treatment of such measure-category duality.

In previous work, on additive functions (the Ostrowski theorem below) and related results (the Steinhaus theorem below), it is the measure case that has been regarded as primary and the Baire (or topological, or category) case as secondary. As we shall see below, the reverse order is the more natural: it is the Baire case that is paramount. Indeed, we deduce the measure cases from the Baire cases, and do so by passing from the Euclidean topology to the density topology.

Recall that, with $|.|$ Lebesgue measure, a point x is a *density point* of a set A if

$$|A \cap (x-t, x+t)|/(2t) \to 1 \qquad (t \downarrow 0),$$

and that the *density topology* on $\mathbb{R}$ is defined by taking as open sets those sets all of whose points are density points. This does define a topology (see [LMZ] for a monograph treatment of density topologies in the broader context of fine topologies). That the density topology provides a bridge between the measure and category versions of the results above is indicated by the following result: a set A is Lebesgue measurable iff it has the Baire property under the density topology (see e.g. [Kech], p.119, Ex. 17.47).

In §2 below we state and prove the results (Theorems 1 and 2) that give us the necessary tools. These are from infinite combinatorics, and topology (relatives of the Baire Category Theorem). As corollaries, we obtain the theorems of Piccard and of Steinhaus. In §3, we obtain (Theorem 3) a Combinatorial Steinhaus Theorem. In §4, we obtain (Theorem 4) a corresponding Combinatorial Ostrowski Theorem, with as corollaries the classical Ostrowski and Banach-Mehdi theorems, in the measure and category cases. As a corollary of Theorem 2, we also obtain (Theorem 5) the 'No Trumps Theorem', unifying the measure and category cases for functions as the results of §2 do for sets, noting another classical corollary (the Fréchet-Banach Theorem). In §5 we also obtain (Theorem 6) a result on the sense in which the sets studied in §2 (the 'subuniversal sets') are 'big enough' (just as non-null and non-meagre sets are 'big enough'). In §6 we extend our treatment to higher dimensions (Theorem 7). We close in §7 with a number of remarks on related work and extensions.

2 Topology and Infinite Combinatorics

Our starting point is the Category Embedding Theorem below. At its heart is the condition below applied to a sequence of autohomeomorphisms which may be regarded as a category convergence to the identity; we call it weak to distinguish it from earlier usage as exemplified by Miller in [Mil4]. We shall see that it is satisfied in the case of the Euclidean and density topologies by shifts induced by a null sequence $z_n \to 0$, namely the functions $h_n(x) := x + z_n$. In the theorem the term 'embedding' is motivated by the applications which follow. We write ω for $\{0, 1, 2, \ldots\}$.

Definition (**weak category convergence**). A sequence of homeomorphisms h_n satisfies the *weak category convergence* condition (wcc) if:

For any non-empty open set U, there is a non-empty open set $V \subseteq U$ such that, for each k,

$$\bigcap_{n \geq k} V \backslash h_n^{-1}(V) \text{ is meagre.} \tag{wcc}$$

Equivalently, for each k, there is a meagre set M such that, for $t \notin M$,

$$t \in V \Longrightarrow (\exists n \geq k)\ h_n(t) \in V.$$

In what follows, the words *quasi everywhere (q.e.)*, or *for quasi all points,* mean *for all points off a meagre set* (see [Kah]).

Theorem 1 (Category Embedding Theorem). *Let X be a Baire space and $h_n : X \to X$ be homeomorphisms satisfying (wcc). Then, for any non-meagre Baire set T, for quasi all $t \in T$, there is an infinite set $\mathbb{M}_t$ such that*

$$\{h_m(t) : m \in \mathbb{M}_t\} \subseteq T.$$

Proof. Suppose T is Baire and non-meagre. We may assume that $T = U \backslash M$ with U non-empty and open and M meagre. Let $V \subseteq U$ satisfy (wcc).

Since the functions h_n are homeomorphisms, the set

$$M' := M \cup \bigcup_n h_n^{-1}(M)$$

is meagre. Put

$$W = \mathbf{h}(V) := \bigcap_{k \in \omega} \bigcup_{n \geq k} V \cap h_n^{-1}(V) \subseteq V \subseteq U.$$

Then $V \cap W$ is co-meagre in V. Indeed

$$V \backslash W = \bigcup_{k \in \omega} \bigcap_{k \geq n} V \backslash h_n^{-1}(V),$$

which by (wcc) is meagre.

Let $t \in (V \cap W) \backslash M'$ with $t \in T$. Now there exists an infinite set $\mathbb{M}_t$ such that, for $m \in \mathbb{M}_t$, there are points $v_m \in V$ with $t = h_m^{-1}(v_m)$. Since $h_m^{-1}(v_m) = t \notin h_m^{-1}(M)$, we have $v_m \notin M$, and hence $v_m \in T$. Thus $\{h_m(t) : m \in \mathbb{M}_t\} \subseteq T$ for t in a co-meagre set, as asserted. $\square$

The conclusion of the theorem has a natural interpretation in the case of shifts. To justify it we shall need to prove that (wcc) holds for shifts.

Definition. Say that the set S is *universal* (resp. *subuniversal*) if for any null sequence $\{z_n\} \to 0$, there are $s \in \mathbb{R}$ and a co-finite (resp. infinite) set $\mathbb{M}_s$ such that

$$\{s + z_m : m \in \mathbb{M}_s\} \subseteq S.$$

We begin with the easier of two verifications of (wcc).

Proposition E (WCC for shifts in the Euclidean topology). *Let V be an open interval in $\mathbb{R}$. For any null sequence $\{z_n\} \to 0$ and each $k \in \omega$,*

$$H_k = \bigcap_{n \geq k} V \backslash (V + z_n) \text{ is empty.}$$

Proof. Let $V = (a, b)$. Assume first that the null sequence is positive. Then, for all n so large that $a + z_n < b$, we have

$$V \cap h_n^{-1}(V) = (a, a + z_n),$$

and so, for any $k \in \omega$,

$$\bigcap_{n \geq k} V \backslash h_n^{-1}(V) \text{ is empty.}$$

The same argument applies if the null sequence is negative, but with the end-points exchanged. $\square$

Proposition D (WCC for shifts in the density topology). *Let V be measurable and non-null. For any null sequence $\{z_n\} \to 0$ and each $k \in \omega$,*

$$H_k = \bigcap_{n \geq k} V \backslash (V + z_n) \text{ is of measure zero, so meagre in the d-topology.}$$

That is, the sequence $h_n(x) := x - z_n$ satisfies (wcc) under the d-topology.

Proof. Suppose otherwise. Then for some k, $|H_k| > 0$. Write H for H_k. Since $H \subseteq V$, we have, for $n \geq k$, that $\emptyset = H \cap h_n^{-1}(V) = H \cap (V + z_n)$ and so a fortiori $\emptyset = H \cap (H + z_n)$. Let u be a metric density point of H. Thus for some interval $I_\delta(u) = (u - \delta/2, u + \delta/2)$ we have

$$|H \cap I_\delta(u)| > \frac{3}{4}\delta.$$

Let $E = H \cap I_\delta(u)$. For any z_n, we have $|(E + z_n) \cap (I_\delta(u) + z_n)| = |E| > \frac{3}{4}\delta$. For $0 < z_n < \delta/4$, we have $|(E+z_n) \backslash I_\delta(u)| \leq |(u+\delta/2, u+3\delta/4)| \leq \delta/4$. Put $F = (E+z_n) \cap I_\delta(u)$, then $|F| > \delta/2$. But $\delta \geq |E \cup F| = |E| + |F| - |E \cap F| \geq \frac{3}{4}\delta + \frac{1}{2}\delta - |E \cap F|$. So

$$|H \cap (H + z_n)| \geq |E \cap F| \geq \frac{1}{4}\delta,$$

contradicting $\emptyset = H \cap (H + z_n)$. This establishes the claim. □

Propositions E and D taken together with the Category Embedding Theorem yield as immediate the following result, due in this form in the measure case to Borwein and Ditor [BoDi], answering a question of Erdős [Erd] (see [Mil1] for more on this). The result was already known much earlier albeit in somewhat weaker form by Kestelman ([Kes1] Th. 3), and rediscovered by Trautner [Trau] (see [BGT] p. xix and footnote p. 10). See also [BOst10] for a homotopic generalization.

Theorem (Kestelman-Borwein-Ditor Theorem). *Let $\{z_n\} \to 0$ be a null sequence of reals. If T is measurable and non-null (resp. Baire non-meagre), then, for almost all (resp. for quasi-all) $t \in T$, there is an infinite set $\mathbb{M}_t$ such that*

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

For a useful weakening see Theorem 5, 'The No Trumps Theorem' of Section 4.

We now change from a topological to a bitopological setting ([Kel], cf. [LMZ]) in which we have two distinct but related topologies in play. This bitopological viewpoint enables us to unify the two classical theorems on interior points due in the measurable case to Steinhaus and in the Baire case to Piccard. The common generalization is of course a category theorem.

Theorem 2 (Topological, or Category, Interior Point Theorem). *Let $\{z_n\} \to 0$ be a null sequence (in the Euclidean topology). Let $\mathbb{R}$ be given a shift-invariant topology τ under which it is a Baire space and the homeomorphisms $h_n(x) = x + z_n$ satisfy* (wcc). *For S Baire and non-meagre in τ, the difference set $S - S$ contains an interval around the origin.*

Proof. Suppose otherwise. Then for each positive integer n we may select

$$z_n \in (-1/n, +1/n) \backslash (S - S).$$

Since $\{z_n\} \to 0$ (in the Euclidean topology), the Category Embedding Theorem applies, and gives an $s \in S$ and an infinite $\mathbb{M}_s$ such that

$$\{h_m(s) : m \in \mathbb{M}_s\} \subseteq S.$$

Then for any $m \in \mathbb{M}_s$,

$$s + z_m \in S \text{ , i.e. } z_m \in S - S,$$

a contradiction. □

Corollary (Piccard Theorem, Piccard [Pic1], [Pic2]). *For S Baire and non-meagre in the Euclidean topology, the difference set $S - S$ contains an interval around the origin.*

First Proof. Apply Theorem 2, since by Proposition E, the (wcc) condition holds. □

Second Proof. Suppose otherwise. Then, as before, for each positive integer n we may select $z_n \in (-1/n, +1/n) \backslash (S - S)$. Since $z_n \to 0$, by the Kestelman-Borwein-Ditor Theorem, for quasi all $s \in S$ there is an infinite $\mathbb{M}_s$ such that $\{s + z_m : m \in \mathbb{M}_s\} \subseteq S$. Then for any $m \in \mathbb{M}_s$, $s + z_m \in S$, i.e. $z_m \in S - S$, a contradiction. □

Corollary (Pettis, [Pet]) *For S, T Baire and non-meagre in the Euclidean topology, the difference set $S - T$ contains an interval.*

Proof. First consider the special case when S, T are equal to each other, and so also to $S \cap T$, modulo meagre sets; then $S - T$ contains an interval around the origin. To prove this, argue similarly but now select $z_n \in (-1/n, +1/n) \backslash (S - T)$. Since $z_n \to 0$, by the Kestelman-Borwein-Ditor Theorem, for quasi all $t \in S \cap T$ there is an infinite $\mathbb{M}_t$ such that $\{t + z_m : m \in \mathbb{M}_s\} \subseteq S$. Then, for any $m \in \mathbb{M}_t$, $t + z_m \in S$, i.e. $z_m \in S - T$, a contradiction. The general case may be reduced to the special case by an appropriate translation, say by a, of T to $T' = T - a$, so that $S - T = (S - T') + a$. Indeed, for some interval I, S contains I modulo meagre sets and a translate of T, say $T - a$, also contains I modulo meagre sets; thus we may replace S by the smaller set $S' := S \cap I$ and T by $T' := T \cap (I + a)$ and then S' and $T' - a$ are both equal modulo meagre sets to I. $\square$

Corollary (Steinhaus' Theorem, [St]). *For S of positive measure, the difference set $S - S$ contains an interval around the origin.*

Proof. Arguing as in the first proof above, by Proposition D, the wcc holds and S, in the density topology, is Baire and non-meagre ([Kech]). The measure-theoretic form of the second proof above also applies. $\square$

Just as with the Pettis extension of Piccard's result, so too here, Steinhaus proved that for S, T non-null measurable $S - T$ contains an interval.

Unlike some of the results above, these results extend to topological groups. See e.g. [Com] Th. 4.6 p.1175 for the positive statement, and the closing remarks for a negative one.

3 More on Steinhaus' Theorem

The following corollary to Steinhaus' Theorem (and its Baire category version) is important enough to merit a name.

Theorem (Category [Measure] Subgroup Theorem). *For an additive Baire [measurable] subgroup S of $\mathbb{R}$, the following are equivalent:*
(i) $S = \mathbb{R}$,
(ii) *S is non-meagre [non-null].*

Proof. By the Topological/Category Interior Point Theorem, for some interval I,

$$I \subseteq S - S \subseteq S,$$

and hence $\mathbb{R} = \bigcup_n nI = S$. $\square$

Here we develop a combinatorial version, in the language of Ramsey theory ([GRS], [Hin]).

Definition. Say that a set S has the *Ramsey distance property* if for any convergent sequence $\{u_n\}$ there is an infinite $\mathbb{M}$ such that

$$\{u_n - u_m : m, n \in \mathbb{M}\} \subseteq S.$$

Thinking of the points of S as those having a particular colour, S has the Ramsey distance property if any convergent sequence has a subsequence all of whose pairwise distances have this colour.

Theorem 3 (Combinatorial Steinhaus Theorem). *For an additive subgroup S of $\mathbb{R}$, the following are equivalent:*
(i) $S = \mathbb{R}$,
(ii) *S is universal,*
(iii) *S is subuniversal,*
(iv) *S has the Ramsey distance property.*

We quote ([Muth], Prop. 1, cf. [MilMuth]):
Muthuvel's Infinite Index Theorem. *If S is a proper subgroup of $\mathbb{R}$, then the index of S in $\mathbb{R}$ is infinite.*
Our proof requires the following strengthening of Muthuvel's Infinite Index Theorem (see also the Remark at the end of the section).

Non-covering Lemma. *Let S be a proper additive subgroup of $\mathbb{R}$. An open interval cannot be covered by a finite union of cosets of S.*

Proof. Suppose otherwise and that for S a proper subgroup of $\mathbb{R}$ there is an interval I and distinct, and so pairwise disjoint, cosets $S, S+u, ..., S+w$ covering it. The Lemma may be reduced by a shift (say by $s+u$ when $s+u$ is in I for some s in S) to considering the case that

$$(0, \alpha) \subseteq T := S \cup (S+u) \cup ...(S+w), \tag{1}$$

with $\alpha > 0$. We claim that $\mathbb{R} \subseteq T$. This implies that the index of S in $\mathbb{R}$ is finite, thus contradicting Muthuvel's Infinite Index Theorem. Let $\sigma = \inf\{s \in S : s > 0\}$. The claim will follow once we have shown that $\sigma = 0$; see below. So, suppose on the contrary, that $\sigma > 0$. We now make the subsidiary claim that, without loss of generality, $u, ..., w$ all lie in $(0, \sigma)$. To see this in the case of u, we may first assume that $u > 0$, otherwise replace u by $-u$, appealing to the symmetry $S + u = -(S - u)$. Let

$$\bar{n} = \max\{n : u - n\sigma \geq 0\} = [u/\sigma].$$

Put $u' = u - \bar{n}\sigma$. Thus $0 \leq u' < \sigma$. If $\sigma \in S$, then $u' > 0$ (otherwise $S + u = S + u' = S$), and so $S + u' = S + u$ with u' in $(0, \sigma)$. If $\sigma \notin S$, we may select $s > \sigma$ with $(s - \sigma)$ small enough that $u'' = u - \bar{n}s$ satisfies $-\sigma < u'' < \sigma$. Thus $S+u = S+u''$. As before u'' is non-zero, so by symmetry, we may suppose that $u'' > 0$. Thus again $S + u = S + u''$ with u'' in $(0, \sigma)$. Thus the subsidiary claim is established.

By definition of σ, S avoids the interval $(0, \sigma)$. Hence $S + u$ avoids the interval $(u, u + \sigma)$ and a fortiori (u, σ). There is thus a smallest interval $(\tau, \sigma) \subseteq (0, \sigma)$ which S, and its finitely many translates $S + u, ..., S + w$, all avoid, which contradicts (1). We have thus ruled out $\sigma > 0$. From here the claim $\mathbb{R} \subseteq T$ follows easily. Take $x \in \mathbb{R}$; we may suppose that $x > 0$, invoking the symmetry $S+u = -(S-u)$. Let $m = [x/\alpha]$, so that $m\alpha \leq x < (m+1)\alpha$. Choose $s \in S$ with $0 < s < \min\{\alpha, (m+1)\alpha - x\}$. Let n be the largest integer such that $ns < x < (n + 1)s$. Then

$$0 < x - ns < s < \alpha.$$

Thus $x - ns \in T$ and so $x \in T$, for if say $x - ns \in S + u$, then also $x \in S + u$. Thus $\mathbb{R} \subseteq T$, contradicting Muthuvel's Infinite Index Theorem. $\square$

Proof of the Combinatorial Steinhaus Theorem. It is clear that (i) implies (ii) and (ii) implies (iii). To see that (iii) implies (iv) observe that, as S is subuniversal, there are t and an infinite M such that

$$\{t + u_n : n \in \mathbb{M}\} \subseteq S.$$

But, for m and n in M,

$$u_n - u_m = (t + u_n) - (t + u_m) \in S,$$

giving (iv). To prove (iv) implies (i) we use the Non-covering Lemma. Suppose that $S \neq \mathbb{R}$. Then, as S is a subgroup, it cannot contain any (non-empty) interval. Suppose that $v_0, ..., v_{n-1}$ have been selected with $v_k < 1/(k + 1)^2$

and $v_m + ... + v_{n-1} \notin S$ for each $m < n-1$. We want to select $v_n < 1/(n+1)^2$ such that for each $m < n$

$$v_m + ... + v_n \notin S, \tag{2}$$

or equivalently

$$v_n \notin S - (v_m + ... + v_{n-1}).$$

Thus we require that

$$\begin{aligned} v_n \quad &\in \quad \bigcap_{m<n} \left(\left(0, \frac{1}{(n+1)^2}\right) \backslash S - (v_m + ... + v_{n-1}) \right) \\ &= \quad \left(0, \frac{1}{(n+1)^2}\right) \backslash \bigcup_{m<n} \left(S - (v_m + ... + v_{n-1})\right). \end{aligned}$$

If we cannot select such a v_n, then

$$\bigcup_{m<n} S - (v_m + ... + v_{n-1}) \supseteq \left(0, \frac{1}{(n+1)^2}\right).$$

This contradicts the Non-covering Lemma. Thus after all, the induction can proceed. Put $u_n := v_1 + ... + v_n$, then $\{u_n\}$ is convergent. According to (iv) there is an infinite $\mathbb{M}$ such that for m and n in $\mathbb{M}$ with $m < n$

$$v_m + ... + v_{n-1} = u_n - u_m \in S.$$

This contradicts (2). $\square$

Second Proof of the Subgroup Theorem. For S Baire non-meagre, resp. measurable non-null, S is subuniversal by the Kestelman-Borwein-Ditor Theorem, so the theorem follows as a corollary of the Combinatorial Steinhaus Theorem. $\square$

Remark. The essence of Muthuvel's Infinite Index Theorem is illustrated best in a proof that the index cannot be 2. So suppose that $\mathbb{R}$ is partitioned into two cosets, S and $u + S$, where $u \notin S$. Note that $2u \in S$, for otherwise $2u \in u + S$, so $u \in S$, a contradiction. As $\mathbb{R}$ is a field, we have $\mathbb{R} = 2\mathbb{R}$. But $2\mathbb{R} \subseteq S$, because, for $x \in \mathbb{R}$, we may show that $2x \in S$. This is clear if $x \in S$,whereas if $x \in u + S$, we have that $2x \in 2u + 2S \subseteq S$. Thus $\mathbb{R} = S$, again a contradiction.

4 Ostrowski's Theorem

Theorem 4 (Combinatorial Ostrowski Theorem). *If $f : \mathbb{R} \to \mathbb{R}$ is additive and bounded (locally, above or below) on a subuniversal set S, then f is locally bounded and hence linear.*

Proof. Suppose that f in not locally bounded at the origin. Then we may choose $z_n \to 0$ such that $f(z_n) \geq n$, without loss of generality (otherwise replace f by $-f$). But there are $s \in S$ and an infinite $\mathbb{M}_s$ such that

$$\{s + z_m : m \in \mathbb{M}_s\} \subseteq S,$$

implying that f is unbounded on S locally at s, a contradiction. It follows from local boundedness at the origin that f is continuous. Indeed choose $\delta > 0$ and M such that, for all t with $|t| < \delta$, we have

$$|f(t)| < M.$$

For $\varepsilon > 0$ arbitrary, choose any integer N with $N > M/\varepsilon$. Now provided $|t| < \delta/N$, we have

$$N|f(t)| = |f(Nt)| < M, \text{ or } |f(t)| < M/N < \varepsilon. \qquad \square$$

A weaker result, with the condition S subuniversal strengthened to S universal is in [Kes2]. As immediate corollaries of the above and of the Kestelman-Borwein-Ditor Theorem we have the following pair of classical results.

Corollary (Ostrowski Theorem [Ostr], cf. [Kes2]). *If $f : \mathbb{R} \to \mathbb{R}$ is additive and bounded (locally, above or below) on a set of positive measure S, then f is locally bounded and hence linear.*

Corollary (Banach-Mehdi Theorem, [Ban-T] Th. 4 pg. 35, [Meh]). *If $f : \mathbb{R} \to \mathbb{R}$ is additive and bounded (locally, above or below) on a non-meagre Baire set S, then f is locally bounded and hence linear.*

Corollary (Fréchet [Frech], Banach [Ban-T]). *If $f : \mathbb{R} \to \mathbb{R}$ is additive and measurable or Baire, then f is continuous and so linear.*

The boundedness conditions above lead naturally to a consideration of the level sets of a function and their combinatorial properties. Here we go beyond the null sequences.

Definitions. 1. For the function $h : \mathbb{R} \to \mathbb{R}$, the (symmetric) *level sets* of h are defined by

$$H^r := \{t : |h(t)| < r\}.$$

2. We write $\mathbf{NT}(\{T_k : k \in \omega\})$ to means that, for every bounded/convergent sequence $\{u_n\}$ in $\mathbb{R}$, some T_k contains a translate of a subsequence of $\{u_n\}$, i.e. there is $k \in \omega$, infinite $\mathbb{M} \subseteq \omega, t \in \mathbb{R}$ such that

$$\{t + u_n : n \in \mathbb{M}\} \subseteq T_k.$$

The term appears in [BOst5] on subadditive functions, in the measurable and Baire cases (see §7 Remark 9 for background). Specializing to the case when $T_k = S$ for all k, we see that S is subuniversal iff $\mathbf{NT}(S)$ holds. This allows a formulation of when a function may be regarded as having 'nice' level sets. Thus, since $\mathbb{R}$ is the union of the level sets of a function, we have as an immediate corollary of the Kestelman-Borwein-Ditor Theorem:

Theorem 5 (No Trumps Theorem). *For $h : \mathbb{R} \to \mathbb{R}$, measurable/Baire $\mathbf{NT}(\{H^k : k \in \omega\})$ holds.*

As an illustration of how useful this weakened form is, we use it to derive a strengthening of two classical results above.

Corollary (**Generalized Fréchet-Banach Theorem**). *If $h : \mathbb{R} \to \mathbb{R}$ is additive and its level sets H^k satisfy $\mathbf{NT}(\{H^k : k \in \omega\})$, then h is locally bounded and so continuous and linear.*

Proof. Suppose that h in not locally bounded at the origin. Then we may choose $z_n \to 0$ such that $h(z_n) \geq n$, without loss of generality (if not replace h by $-h$). But there are $s \in \mathbb{R}$, $k \in \omega$ and an infinite $\mathbb{M}_s$ such that

$$\{s + z_m : m \in \mathbb{M}_s\} \subseteq H^k,$$

so

$$h(s + z_m) = h(s) + h(z_m) > h(s) + m,$$

so that h is unbounded on H^k, a contradiction as $|h| < k$ on H^k. Thus h is locally bounded. An interval is subuniversal and so by the Combinatorial Ostrowski Theorem h is continuous and so linear. □

This result embraces its classical counterpart for h measurable or Baire.

One also has a restatement of the Ostrowski Theorem, now by reference to functions with 'nice' level sets.

Theorem 4' (Second Combinatorial Ostrowski Theorem). *For $h(x)$ an additive function, $h(x)$ is continuous and $h(x) = cx$ for some constant c iff* $\mathbf{NT}(\{H^k : k \in \omega\})$ *holds.*

The Subgroup Theorem may also be similarly restated. For this, we need a variant on the $\mathrm{NT}(S)$ in which subuniversal is strengthened to universal; the corresponding notation is $\mathrm{NT}_A(S)$, where the suffix A denotes 'almost all', i.e. 'for all but a finite number of'.

Theorem 3' (Combinatorial Steinhaus Theorem Restated). *For an additive subgroup S of $\mathbb{R}$, the following are equivalent:*
(i) $S = \mathbb{R}$,
(ii) $\mathbf{NT}_A(S)$,
(iii) $\mathbf{NT}(S)$,
(iv) *S has the Ramsey distance property.*

5 Subuniversal Sets

The Combinatorial Ostrowski Theorem, together with the Ostrowski and Banach-Mehdi theorems, shows that subuniversality serves as a condition to make a set 'big enough', in the context of additive functions. The next two results give senses in which subuniversal and universal sets are 'big enough', in general.

Theorem 6. *A subuniversal set is uncountable.*

Proof. Suppose not: then $S, T := S - S$ are countable. We select v_m with the aim to guaranteeing inductively, for $m < n$, that

$$v_m + \ldots + v_n \notin T,$$

or, applying a shift, that

$$v_n \notin T - (v_m + \ldots + v_{n-1}).$$

So, suppose that $v_0, ..., v_{n-1}$ have been selected with $v_k < 1/(k+1)^2$ and $v_m, ..., v_{n-1} \notin S$ for each $m < n-1$. Now choose

$$v_n \in \left(0, \frac{1}{(n+1)^2}\right) \Big\backslash \bigcup_{m<n} \left(T - (v_{n-1} + ... + v_m)\right).$$

This is again possible since we have to avoid only a countable set. Now consider the sequence

$$u_n = v_0 + v_1 + ... + v_{n-1},$$

with $u_0 = 0$. Suppose for some t and infinite $\mathbb{M}$ that

$$\{t + u_n : n \in \mathbb{M}\} \subseteq S.$$

Then, for pairs $n > m$ that are in $\mathbb{M}$, we have the contradiction

$$v_m + ... + v_{n-1} = (t + u_n) - (t + u_m) \in S - S = T. \qquad \square$$

A further, very striking result holds, shown by Kestelman ([Kes1], Theorem 6), namely:

Theorem. *For S universal, S', the set of limit points of S, contains an interval.*

6 Higher Dimensions

The Subgroup Theorem holds in $\mathbb{R}^N$ – this is a matter only of a change in vocabulary: one needs only replace open intervals by open balls throughout. However, more interestingly, the Combinatorial Steinhaus Theorem actually implies its own higher-dimensional analogue.

Theorem 7 (N-dimensional Combinatorial Steinhaus Theorem). *For an additive subgroup T of $\mathbb{R}^N$, the following are equivalent:*
(i) $T = \mathbb{R}^N$,
(ii) $\mathbf{NT}_A(T)$,
(iii) $\mathbf{NT}(T)$
(iv) *T has the Ramsey distance property.*

Proof. Again we need only prove (iv) implies (i). So suppose that (iv) holds for a subgroup $T \subseteq \mathbb{R}^N$. For any non-zero vector v in $\mathbb{R}^N$, let $S = T \cap Lin\{v\}$. We claim that Theorem 3 implies $S = Lin\{v\}$. Thereupon $T = \mathbb{R}^N$ is immediate. Now, up to homorphism, S is a subgroup of $\mathbb{R}$, so to establish the claim it suffices to observe that property (iv) for T implies the corresponding property (iv) of Theorem 3 for S, now regarded as a subgroup of $Lin\{v\}$. Indeed, given a convergent $\{u_n\} \subseteq Lin\{v\}$, there is an infinite $\mathbb{M}$ such that

$$\{u_n - u_m : m, n \in \mathbb{M}\} \subseteq T.$$

But for $m, n \in \mathbb{M}$ we trivially have

$$u_n - u_m \in Lin\{\mathbf{v}\} \cap T = S.$$

So the claim is established, and hence too our theorem. $\square$

7 Remarks

1. Topological groups and shifts.

Just as we generalized the Combinatorial Steinhaus Theorem from one to higher dimensions above, some of the results here can be generalized to topological groups; see [BOst12] for details. We point out, however, that some of the work above does not extend in this way. For, we have made use of the density topology to unify the measure and category cases. But it is known that the real line cannot be made into a topological group under the density topology, a result of Heath and Poerio ([HePo]).

In the above we work with shifts, so fixing one variable in the Cauchy functional equation and reducing the effective dimension from two to one.

2. Namioka's theorem.

The dimension reduction just mentioned is relevant to the relationship between separate and joint continuity for functions of two variables. The prototypical result here is Namioka's theorem ([Nam]; [Piot]), that separate continuity implies joint continuity, not everywhere but generically – off a large set.

3. Negligibles.

The meagre and null sets in the work above may be thought of as negligible. One generalization is in the theory of sigma-ideals ([Kech] §15.C,

[KeSo]); another is in the work of Fremlin [Frem] on measure spaces with negligibles.

4. Quantitative versus qualitative measure theory.

As mentioned in the Introduction, we work largely with qualitative rather than quantitative measure theory here. The only place where we use quantitative measure theory is in the proof of Proposition D. The distinction between the two is related to the limits of measure-category duality. For background on this, see e.g. [Oxt].

5. Dichotomy.

The theme of the Cauchy functional equation, and in particular of the Ostrowski and Banach-Mehdi theorems, is that an additive function satisfying mimimal regularity conditions is continuous (and so linear) – that is, that it is either very good or very bad (but see [CiePaw2] for positive results in this case). Such dichotomy theorems hold in other contexts; see in particular [BOst12] for normed groups.

6. Automatic continuity.

A related property is automatic continuity, where a function satisfying appropriate weak conditions is proved to be necessarily continuous. For automatic continuity for Banach algebras, see [Dal]; for other automatic properties, see [BOst6].

7. Converse Ostrowski theorem.

Our Combinatorial Steinhaus Theorem is clearly best-possible, in that it gives a set of equivalences. By contrast, our Combinatorial Ostrowski Theorem is not, and it is natural to wonder whether it may be given a best-possible form. It turns out that it may. The relevant result is topological, and hinges on the idea of shift-compactness; see [BOst8].

8. Regular variation.

The Steinhaus and Ostrowski theorems play a central role in the theory of regular variation (see e.g. the standard work on the subject, [BGT], where they are respectively Theorems 1.1.1 and 1.1.4). The unified and extended treatment of these theorems achieved here can be carried over to the theory of regular variation; this is done in a companion paper. In this regard, note the contrast between quantitative and qualitative measure theory in the several proofs given in [BGT] of the Uniform Convergence Theorem, the main result of the subject. See [BOst11] for a derivation of this result from the Category Embedding Theorem.

9. No Trumps.

The term No Trumps in Theorem 5, a combinatorial principle, is used in close analogy with earlier combinatorial principles, in particular Jensen's Diamond $\Diamond$ [Je] and Ostaszewski's Club $\clubsuit$ [Ost]. The argument in the proof of the No Trumps Theorem is implicit in [CsEr] and explicit in [BG1], p.482 and [BGT], p.9. The intuition behind our formulation may be gleaned from forcing arguments in [Mil1], [Mil2], [Mil3].

10. Models of set theory.

The programme begun here has implications for models of set theory, such as that of Solovay [Sol] in which all sets of reals are measurable (and have the Baire property), or Shelah's model [She] in which all sets of reals have the Baire property. In subuniversality one has a unifying concept for measurability and the Baire property; one may, for example, ask for models of set theory in which all uncountable sets are subuniversal.

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