Frugal Colouring of Graphs

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Abstract

A k-frugal colouring of a graph G is a proper colouring of the vertices of G such that no colour appears more than k times in the neighbourhood of a vertex. This type of colouring was introduced by Hind, Molloy and Reed in 1997. In this paper, we study the frugal chromatic number of planar graphs, planar graphs with large girth, and outerplanar graphs, and relate this parameter with several well-studied colourings, such as colouring of the square, cyclic colouring, and $L(p,q)$-labelling. We also study frugal edge-colourings of multigraphs.

1 Introduction

Most of the terminology and notation we use in this paper is standard and can be found in any text book on graph theory (such as [1] or [4]). All our graphs and multigraphs will be finite. A *multigraph* can have multiple edges; a *graph* is supposed to be simple; loops are not allowed.

For an integer $k \geq 1$, a *k-frugal colouring* of a graph G is a proper vertex colouring of G (i.e., adjacent vertices get a different colour) such that no colour appears more than k times in the neighbourhood of any vertex. The least number of colours in a k-frugal colouring of G is called the *k-frugal chromatic number*, denoted $\chi_k(G)$. Clearly, $\chi_1(G)$ is the chromatic number of the square of G; and for k at least the maximum degree of G, $\chi_k(G)$ is the usual chromatic number of G.

A *k-frugal edge colouring* of a multigraph G is a (possibly improper) colouring of the edges of G such that no colour appears more than k times on the edges incident with a vertex. The least number of colours in a k-frugal edge colouring of G, the *k-frugal edge chromatic number* (or *k-frugal chromatic index*), is denoted by $\chi'_k(G)$. Remark that for $k=1$ we have $\chi'_1(G) = \chi'(G)$, the normal chromatic index of G.

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When considering the possibility that each vertex or edge has a list of available colours, we enter the area of *frugal list (edge) colourings.*

Frugal vertex colourings were introduced by Hind *et al* [13, 14], as a tool towards improving results about the *total chromatic number* of a graph. One of their results is that a graph with large enough maximum degree Δ has a $(\log^8 \Delta)$-frugal colouring using at most $\Delta + 1$ colours. They also show that there exist graphs for which a $\left(\frac{\log \Delta}{\log \log \Delta}\right)$-frugal colouring cannot be achieved using only $O(\Delta)$ colours.

Our aim in this note is to study some aspects of frugal colourings and frugal list colourings in their own right. In the first part we consider frugal vertex colourings of planar graphs. We show that for planar graphs, frugal colouring are closely related to several other aspects that have been the topic of extensive research the last couple of years. In particular, we exhibit close connections with colouring the square, cyclic colourings, and $L(p, q)$-labellings.

In the final section we derive some results on frugal edge colourings of multigraphs in general.

1.1 Further notation and definitions

Given a graph G, the *square of* G, denoted G^2, is the graph with the same vertex set as G and with an edge between any two different vertices that have distance at most two in G. We always assume that colours are integers, which allows us to talk about the "distance" $|\gamma_1 - \gamma_2|$ of two colours γ_1, γ_2.

The *chromatic number* of G, denoted $\chi(G)$, is the minimum number of colours required so that we can properly colour its vertices using those colours. A *t-list assignment L on the vertices* of a graph is a function which assigns to each vertex v of the multigraph a list $L(v)$ of t prescribed integers. The *list chromatic number* or *choice number* $ch(G)$ is the minimum value t, so that for each t-list assignment on the vertices, we can find a proper colouring in which each vertex gets assigned a colour from its own private list.

We introduced *k-frugal colouring* and the *k-frugal chromatic number* $\chi_k(G)$ in the introductory part. In a similar way we can define *k-frugal list colouring* and the *k-frugal choice number* $ch_k(G)$.

Further definitions on edge colourings will appear in the final section.

2 Frugal Colouring of Planar Graphs

In the next four sections we consider k-frugal (list) colourings of planar graphs. For a large part, our work in that area is inspired by a well-known conjecture of Wegner on the chromatic number of squares of planar graphs. If G has maximum degree Δ, then a vertex colouring of its square will need at least $\Delta + 1$ colours, but the greedy algorithm shows it is always possible with $\Delta^2 + 1$ colours. Diameter two cages such as the 5-cycle, the Petersen graph and the Hoffman-Singleton graph (see [1, page 239]) show that there exist graphs that in fact require $\Delta^2 + 1$ colours.

For planar graphs, Wegner conjectured that far less than $\Delta^2 + 1$ colours should suffice.

Conjecture 2.1 (Wegner [24])
For a planar graph G of maximum degree $\Delta(G) \geq 8$ we have $\chi(G^2) \leq \left\lfloor \frac{3}{2}\,\Delta(G) \right\rfloor + 1$.

Wegner also conjectured maximum values for the chromatic number of the square of planar graph with maximum degree less than eight and gave examples showing his bounds would be tight. For even $\Delta \geq 8$, these examples are sketched in Figure 1.

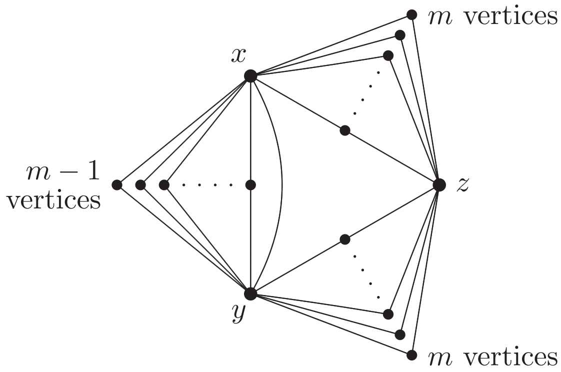

Figure 1: The planar graphs G_m.

Inspired by Wegner's Conjecture, we conjecture the following bounds for the k-frugal chromatic number of planar graphs.

Conjecture 2.2
For any integer $k \geq 1$ and planar graph G with maximum degree $\Delta(G) \geq \max\{2k, 8\}$ we have

$$\chi_k(G) \leq \begin{cases} \left\lfloor \frac{\Delta(G)-1}{k} \right\rfloor + 3, & \text{if } k \text{ is even;} \\ \left\lfloor \frac{3\,\Delta(G)-2}{3\,k-1} \right\rfloor + 3, & \text{if } k \text{ is odd.} \end{cases}$$

Note that the graphs G_m in Figure 1 also show that the bounds in this conjecture are best possible. The graph G_m has maximum degree $2\,m$. First consider a k-frugal colouring with $k = 2\,\ell$ even. We can use the same colour at most $\frac{3}{2}\,k$ times on the vertices of G_m, and every colour that appears exactly $\frac{3}{2}\,k = 2\,\ell$ times must appear exactly ℓ times on each of the three sets of common neighbours of x and y, of x and z, and of y and z. So we can take at most $\frac{1}{\ell}\,(m-1) = \frac{1}{k}\,(\Delta(G_m)-1)$ colours that are used $\frac{3}{2}\,k$ times. The graph that remains can be coloured using just three colours.

If $k = 2\,\ell + 1$ is odd, then each colour can appear at most $3\,\ell + 1 = \frac{1}{2}\,(3\,k-1)$ times, and the only way to use a colour so many times is by using it on the vertices in $V(G_m) \setminus \{x, y, z\}$. Doing this at most $\frac{3\,m-1}{(3\,k-1)/2} = \frac{3\,\Delta(G)-2}{3\,k-1}$ times, we are left with a graph that can be coloured using three colours.

We next derive some upper bounds on the k-frugal chromatic number of planar graphs. The first one is a simple extension of the approach from [11]. In that paper, the following structural lemma is derived.

Lemma 2.3 (Van den Heuvel & McGuinness [11])
Let G be a planar simple graph. Then there exists a vertex v with m neighbours $v_1, \ldots, v_m$ with $d(v_1) \leq \cdots \leq d(v_m)$ such that one of the following holds :
(i) $m \leq 2$*;*
(ii) $m = 3$ *with* $d(v_1) \leq 11$*;*
(iii) $m = 4$ *with* $d(v_1) \leq 7$ *and* $d(v_2) \leq 11$*;*
(iv) $m = 5$ *with* $d(v_1) \leq 6$*,* $d(v_2) \leq 7$*, and* $d(v_3) \leq 11$*.*

Van den Heuvel and McGuinness [11] use this structural lemma to prove that the chromatic number of the square of a planar graph is at most $2\,\Delta + 25$. Making some slight changes in their proof, it is not difficult to obtain a first bound on ch_k (and hence on χ_k) for planar graphs.

Theorem 2.4
For any planar graph G with $\Delta(G) \geq 12$ and integer $k \geq 1$ we have $ch_k(G) \leq \left\lfloor \frac{2\,\Delta(G)+19}{k} \right\rfloor + 6$.

Proof We will prove that if a planar graph satisfies $\Delta(G) \leq C$ for some $C \geq 12$, then $ch_k(G) \leq \left\lfloor \frac{2\,C+19}{k} \right\rfloor + 6$. We use induction on the number of vertices, noting that the result is obvious for small graphs. So let G be a graph with $|V(G)| > 1$, choose $C \geq 12$ so that $\Delta(G) \leq C$, and assume each vertex v has a list $L(v)$ of $\left\lfloor \frac{2\,C+19}{k} \right\rfloor + 6$ colours. Take $v, v_1, \ldots, v_m$ as in Lemma 2.3. Contracting the edge vv_1 to a new vertex v' will result in a planar graph G' in which all vertices except v' have degree at most as much as they had in G, while v' has degree at most $\Delta(G)$ (for case (i)) or at most 12. (for the cases (ii) – (iv)). In particular we have that $\Delta(G') \leq C$. If we give v' the same list of colours as v_1 had (all vertices in $V(G) \setminus \{v, v_1\}$ keep their list), then, using induction, G' has a k-frugal colouring. Using the same colouring for G, where v_1 gets the colour v' had in G', we obtain a k-frugal colouring of G with the one deficit that v has no colour yet. But the number of colours forbidden for v are the colours on its neighbours, and for each neighbour v_i, the colours that already appear k times around v_i. So the number of forbidden colours is at most $m + \sum\limits_{i=1}^{m} \left\lfloor \frac{d(v_i)-1}{k} \right\rfloor$. Using the knowledge from the cases (i) – (iv), we get that $|L(v)| = \left\lfloor \frac{2\,C+19}{k} \right\rfloor + 6$ is at least one more than this number of forbidden colours, hence we always can find an allowed colour for v. $\square$

In the next section we will obtain (asymptotically) better results based on more recent work on special labellings of planar graphs.

3 Frugal Colouring and $L(p, q)$-Labelling

Let $dist(u, v)$ denote the distance between two vertices u, v in a graph. For integers $p, q \geq 0$, an *$L(p, q)$-labelling of G* is an assignment f of integers to the vertices of G such that :

- $|f(u) - f(v)| \geq p$, if $dist(u, v) = 1$, and
- $|f(u) - f(v)| \geq q$, if $dist(u, v) = 2$.

The *$\lambda_{p,q}$-number* of G, denoted $\lambda_{p,q}(G)$, is the smallest t such that there exists an $L(p, q)$-labellings of G using labels from $1, 2, \ldots, t$.[1]. Of course we can also consider the list version

[1]The definition of $\lambda_{p,q}(G)$ is not uniform across the literature. Many authors define it as the minimum distance between the largest and smallest label used, which gives a λ-value one less than with our definition. We chose our definitions since it means that $\lambda_{1,1}(G) = \chi(G^2)$, and since it fits more natural with the notion of list $L(p, q)$-labellings.

of $L(p,q)$-labellings. Given a graph G, the *list $\lambda_{p,q}$-number*, denoted $\lambda^l_{p,q}(G)$, is the smallest integer t such that, for every t-list assignment L on the vertices of G, there exists an $L(p,q)$-labelling f such that $f(v) \in L(v)$ for every vertex v.

The following is an easy relation between frugal colourings and $L(p,q)$-labellings.

Proposition 3.1
For any graph G and integer $k \geq 1$ we have $\chi_k(G) \leq \lceil \frac{1}{k}\, \lambda_{k,1}(G) \rceil$ and $ch_k(G) \leq \lceil \frac{1}{k}\, \lambda^l_{k,1}(G) \rceil$.

Proof We only prove the second part, the first one can be done in a similar way. Set $\ell = \lceil \frac{1}{k}\, \lambda^l_{k,1}(G) \rceil$, and let L be an ℓ-list assignment on the vertices of G. Using that all elements in the lists are integers, we can define a new list assignment L^* by setting $L^*(v) = \bigcup_{x \in L(v)} \{k\,x,\ k\,x+1,\ \ldots,\ k\,x+k-1\}$. Then L^* is a $(k\,\ell)$-list assignment. Since $k\,\ell \geq \lambda^l_{k,1}(G)$, there exists an $L(k,1)$-labelling f^* of G with $f^*(v) \in L^*(v)$ for all vertices v. Define a new labelling f of G by taking $f(v) = \lfloor \frac{1}{k}\, f^*(v) \rfloor$. We immediately get that $f(v) \in L(v)$ for all v. Since adjacent vertices received an f^*-label at least k apart, their f-labels are different. Also, all vertices in a neighbourhood of a vertex v received a different f^*-label. Since the map $x \mapsto \lfloor \frac{1}{k}\, x \rfloor$ maps at most k different integers x to the same image, each f-label can appear at most k times in each neighbourhood. So f is a k-frugal colouring using labels from each vertex' list. This proves that $ch_k(G) \leq \ell$, as required. □

We will combine this proposition with the following recent result.

Theorem 3.2 (Havet *et al* [9])
For each $\epsilon > 0$, there exists an integer Δ_ϵ so that the following holds. If G is a planar graph with maximum degree $\Delta(G) \geq \Delta_\epsilon$, and L is a list assignment so that each vertex gets a list of at least $(\frac{3}{2} + \epsilon)\, \Delta(G)$ integers, then we can find a proper colouring of the square of G using colours from the lists. Moreover, we can take this proper colouring so that the colours on adjacent vertices of G differ by at least $\Delta(G)^{1/4}$.

In the terminology we introduced earlier, an immediate corollary is the following.

Corollary 3.3
Fix $\epsilon > 0$ and an integer $k \geq 1$. Then there exists an integer Δ_ϵ so that if G is a planar graph with maximum degree $\Delta(G) \geq \max\{\, \Delta_\epsilon,\ k^4 \,\}$, then $\lambda^l_{k,1}(G) \leq (\frac{3}{2} + \epsilon)\, \Delta(G)$.

Combining this with Proposition 3.1 gives the asymptotically best upper bound for χ_k and ch_k for planar graphs we currently have.

Corollary 3.4
Fix $\epsilon > 0$ and an integer $k \geq 1$. Then there exists an integer $\Delta_{\epsilon,k}$ so that if G is a planar graph with maximum degree $\Delta(G) \geq \Delta_{\epsilon,k}$, then $ch_k(G) \leq \frac{(3+\epsilon)\,\Delta(G)}{2\,k}$.

In [17], Molloy and Salavatipour proved that for any planar graph G, we have $\lambda_{k,1}(G) \leq \lceil \frac{5}{3}\, \Delta(G) \rceil + 18\,k + 60$. Together with Proposition 3.1, this refines the result of Proposition 2.4 and gives a better bound than Corollary 3.4 for small values of Δ. Note that this corollary only concerns frugal colouring, and not frugal list colouring.

Corollary 3.5
For any planar graph G and integer $k \geq 1$, we have $\chi_k(G) \leq \lceil \frac{5\,\Delta(G)+180}{3\,k} \rceil + 18$.

Proposition 3.1 has another corollary for planar graphs of large girth that we describe below. The *girth* of a graph is the length of a shortest cycle in the graph.

In [23], Lih and Wang proved that for planar graphs of large girth the following holds:

- $\lambda_{p,q}(G) \leq (2\,q-1)\,\Delta(G)+6\,p+12\,q-8$ for planar graphs of girth at least six, and
- $\lambda_{p,q}(G) \leq (2\,q-1)\,\Delta(G)+6\,p+24\,q-14$ for planar graphs of girth at least five.

Furthermore, Dvořák *et al* [5] proved the following tight bound for $(k,1)$-labellings of planar graphs of girth at least seven, and of large degree.

Theorem 3.6 (Dvořák *et al* [5])
Let G be a planar graph of girth at least seven, and maximum degree $\Delta(G) \geq 190+2\,k$, for some integer $k \geq 1$. Then we have $\lambda_{k,1}(G) \leq \Delta(G)+2\,k-1$.

Moreover, this bound is tight, i.e., there exist planar graphs which achieve the upper bound.

A direct corollary of these results are the following bounds for planar graphs with large girth.

Corollary 3.7
Let G be a planar graph with girth g and maximum degree $\Delta(G)$. For any integer $k \geq 1$, we have

$$\chi_k(G) \;\leq\; \begin{cases} \left\lceil \frac{\Delta(G)-1}{k} \right\rceil + 2, & \text{if } g \geq 7 \text{ and } \Delta(G) \geq 190+2\,k; \\ \left\lceil \frac{\Delta(G)+4}{k} \right\rceil + 6, & \text{if } g \geq 6; \\ \left\lceil \frac{\Delta(G)+10}{k} \right\rceil + 6, & \text{if } g \geq 5. \end{cases}$$

4 Frugal Colouring of Outerplanar Graphs

We now prove a variant of Conjecture 2.2 for *outerplanar graphs* (graphs that can be drawn in the plane so that all vertices are lying on the outside face). For $k=1$, i.e., if we are colouring the square of the graph, Hetherington and Woodall [10] proved the best possible bound for outerplanar graphs G: $ch_1(G) \leq \Delta(G)+2$ if $\Delta(G) \geq 3$, and $ch_1(G)=\Delta(G)+1$ if $\Delta(G) \geq 6$.

Theorem 4.1
For any integer $k \geq 2$ and any outerplanar graph G with maximum degree $\Delta(G) \geq 3$, we have $\chi_k(G) \leq ch_k(G) \leq \left\lfloor \frac{\Delta(G)-1}{k} \right\rfloor + 3$.

Proof Esperet and Ochem [6] proved that any outerplanar graph contains a vertex u such that one of the following holds: (i) u has degree at most one; (ii) u has degree two and is adjacent to another vertex of degree two; or (iii) u has degree two and its neighbours v and w are adjacent, and either v has degree three or v has degree four and its two other neighbours (i.e., distinct from u and w) are adjacent.

Let G be a counterexample to the theorem with minimum number of vertices, and let u be a vertex of G having one of the properties described above. By minimality of G, there exists a k-frugal list colouring c of $G-u$ if the lists $L(v)$ contain at least $\left\lfloor \frac{\Delta(G)-1}{k} \right\rfloor + 3$ colours. If u has property (i) or (ii), let t be the neighbour of u whose degree is not necessarily bounded by two. It is easy to see that at most $2+\left\lfloor \frac{\Delta(G)-1}{k} \right\rfloor$ colours are forbidden for u: the colours of the neighbours of u and the colours appearing k times in the neighbourhood of t. If u has property (iii), at most $2+\left\lfloor \frac{\Delta(G)-2}{k} \right\rfloor$ colours are forbidden for u: the colours of the neighbours of u and the colours appearing k times in the neighbourhood of w. Note that if v has degree four, its two other neighbours are adjacent and the k-frugality of v is respected since $k \geq 2$. In all cases we found that at most $\lfloor \frac{\Delta(G)-1}{k} \rfloor + 2$ colours are forbidden for u. If u has a list with one more colour, we can extend c to a k-frugal list colouring of G, contradicting the choice of G. □

We can refine this result in the case of 2-connected outerplanar graphs, provided that Δ is large enough.

Theorem 4.2 *For any integer $k \geq 1$ and any 2-connected outerplanar graph G with maximum degree $\Delta(G) \geq 7$, we have $ch_k(G) \leq \lfloor \frac{\Delta(G)-2}{k} \rfloor + 3$.*

Proof In Lih and Wang [16] it is proved that any 2-connected outerplanar graphs with maximum degree $\Delta \geq 7$ contains a vertex u of degree two that has at most $\Delta - 2$ vertices at distance exactly two.

Let G be a counterexample to the theorem with minimum number of vertices, and let u be a vertex of G having the property described above, and let v and w be its neighbours. Let H be $G - u$ if the edge vw exists, or $G - u + vw$ otherwise. By minimality of G, there is a k-frugal list colouring c of H if all lists contain at least $\lfloor \frac{\Delta(G)-2}{k} \rfloor + 3$ colours. At most $\lfloor \frac{\Delta(G)-2}{k} \rfloor + 2$ colours are forbidden for u: the colours of v and w, and the colours appearing k times in their neighbourhood. So, the colouring c of H can be extended to a k-frugal list colouring of G, contradicting the choice of G. □

5 Frugal Colouring and Cyclic Colouring

In this section, we discuss the link between frugal colouring and cyclic colouring of plane graphs. A *plane graph* G is a planar graph with a prescribed planar embedding. The size (number of vertices in its boundary) of a largest face of G is denoted by $\Delta^*(G)$.

A *cyclic colouring* of a plane graph G is a vertex colouring of G such that any two vertices incident to the same face have distinct colours. This concept was introduced by Ore and Plummer [18], who also proved that a plane graph has a cyclic colouring using at most $2\,\Delta^*$ colours. Borodin [2] (see also Jensen and Toft [15, page 37]) conjectured that any plane graph has a cyclic colouring with $\lfloor \frac{3}{2}\,\Delta^* \rfloor$ colours, and proved this conjecture for $\Delta^* = 4$. The best known upper bound in the general case is due to Sanders and Zhao [20], who proved that any plane graph has a cyclic colouring with $\lceil \frac{5}{3}\,\Delta^* \rceil$ colours.

There appears to be a strong connection between bounds on colouring the square of planar graphs and cyclic colourings of plane graphs. One should only compare Wegner's conjecture in Section 2 with Borodin's conjecture above, and the successive bounds obtained for each of these connections. Nevertheless, the similar looking bounds for these types of colourings have always required independent proofs. No explicit relation that would make it possible to translate a result on one of the types of colouring into a result for the other type, has ever been derived.

In this section we show that if there is an even $k \geq 4$ so that Borodin's conjecture holds for all plane graphs with $\Delta^* \leq k$, and our Conjecture 2.2 is true for the same value k, then Wegner's conjecture is true up to an additive constant factor.

Theorem 5.1
Let $k \geq 4$ be an even integer such that every plane graph G with $\Delta^*(G) \leq k$ has a cyclic colouring using at most $\frac{3}{2}\,k$ colours. Then, if G is a planar graph satisfying $\chi_k(G) \leq \big\lfloor \frac{\Delta(G)-1}{k} \big\rfloor + 3$, we also have $\chi(G^2) = \chi_1(G) \leq \big\lfloor \frac{3}{2}\,\Delta(G) \big\rfloor + \frac{9}{2}\,k - 1$.

Proof Let G be a planar graph with a given embedding and let $k \geq 4$ be an even integer such that $t = \chi_k(G) \leq \big\lfloor \frac{\Delta(G)-1}{k} \big\rfloor + 3$. Consider an optimal k-frugal colouring c of G, with colour classes $C_1, \ldots, C_t$. For $i = 1, \ldots, t$, construct the graph G_i as follows: Firstly, G_i has vertex set C_i, which we assume to be embedded in the plane in the same way they were for G. For each vertex $v \in V(G) \setminus C_i$ with exactly two neighbours in C_i, we add an edge in G_i between these two neighbours. For a vertex $v \in V(G) \setminus C_i$ with $\ell \geq 3$ neighbours in C_i, let $x_1, \ldots, x_\ell$ be those neighbours in C_i in a cyclic order around v (determined by the plane embedding of G). Now add edges $x_1x_2, x_2x_3, \ldots, x_{\ell-1}x_\ell$ and $x_\ell x_1$ to G_i. These edges will form a face of size ℓ in the graph we have constructed so far. Call such a face a *special face*. Note that since C_i is a colour class in a k-frugal colouring, this face has size at most k.

Do the above for all vertices $v \in V(G) \setminus C_i$ that have at least two neighbours in C_i. The resulting graph is a plane graph with some faces labelled special. Add edges to triangulate all faces that are not special. The resulting graph is a plane graph with vertex set G_i and every face size at most k. From the first hypothesis it follows that we can cyclicly colour each G_i with $\frac{3}{2}\,k$ new colours. Since every two vertices in C_i that have a common neighbour in G are adjacent in G_i or are incident to the same (special) face, vertices in C_i that are adjacent in the square of G receive different colours. Hence, combining these t colourings, using different colours for each G_i, we obtain a colouring of the square of G, using at most $\frac{3}{2}\,k \cdot \big(\big\lfloor \frac{\Delta(G)-1}{k} \big\rfloor + 3\big) \leq \big\lfloor \frac{3}{2}\,\Delta \big\rfloor + \frac{9}{2}\,k - 1$ colours. □

Since Borodin [2] proved his cyclic colouring conjecture in the case $\Delta^* = 4$, we have the following corollary.

Corollary 5.2
If G is a planar graph so that $\chi_4(G) \leq \big\lfloor \frac{\Delta(G)-1}{4} \big\rfloor + 3$, then $\chi(G^2) \leq \big\lfloor \frac{3}{2}\,\Delta(G) \big\rfloor + 17$.

6 Frugal Edge Colouring

An important element in the proof in [9] of Theorem 3.2 mentioned earlier is the derivation of a relation between (list) colouring square of planar graphs and edge (list) colourings of multigraphs. Because of this, it seems to be opportune to have a short look at a frugal variant of edge colourings of multigraphs in general.

If we need to properly colour the edges of a multigraph G, the minimum number of colours required is the *chromatic index*, denoted $\chi'(G)$. The *list chromatic index* $ch'(G)$ is defined analogously as the minimum length of list that needs to be given to each edge so that we can use colours from each edge's list to give a proper colouring.

A *k-frugal edge colouring* of a multigraph G is a (possibly improper) colouring of the edges of G such that no colour appears more than k times on the edges incident with a vertex. The least number of colours in a k-frugal edge colouring of G, the *k-frugal edge chromatic number* (or *k-frugal chromatic index*), is denoted by $\chi'_k(G)$.

Note that a k-frugal edge colouring of G is not the same as a k-frugal colouring of the vertices of the line graph $L(G)$ of G. Since the neighbourhood of any vertex in the line graph $L(G)$ can be partitioned into at most two cliques, every proper colouring of $L(G)$ is also a k-frugal colouring for $k \geq 2$. A 1-frugal colouring of $L(G)$ (i.e., a vertex colouring of the square of $L(G)$) would correspond to a proper edge colouring of G in which each colour class induces a matching. Such colourings are known as *strong edge colourings*, see, e.g., [7].

The list version of k-frugal edge colouring can also be defined in the same way: given lists of size t for each edge of G, one should be able to find a k-frugal edge colouring such that the colour of each edge belongs to its list. The smallest t with this property is called the *k-frugal edge choice number*, denoted $ch'_k(G)$.

Frugal edge colourings and its list version were studied under the name *improper edge-colourings* and *improper L-edge-colourings* by Hilton *et al* [12].

It is obvious that the chromatic index and the edge choice numbers are always at least the maximum degree Δ. The best possible upper bounds in terms of the maximum degree only are given by the following results.

Theorem 6.1

(a) *For a simple graph G we have $\chi'(G) \leq \Delta(G) + 1$.* **(Vizing [22])**

(b) *For a multigraph G we have $\chi'(G) \leq \left\lfloor \frac{3}{2}\,\Delta(G) \right\rfloor$.* **(Shannon [21])**

(c) *For a bipartite multigraph G we have $ch'(G) = \Delta(G)$.* **(Galvin [8])**

(d) *For a multigraph G we have $ch'(G) \leq \left\lfloor \frac{3}{2}\,\Delta(G) \right\rfloor$.* **(Borodin *et al* [3])**

We will use Theorem 6.1 (c) and (d) to prove two results on the k-frugal chromatic index and the k-frugal choice number. The first result shows that for even k, the maximum degree completely determines the values of these two numbers. This result was earlier proved in [12] in a slightly more general setting, involving a more complicated proof.

Theorem 6.2 (Hilton *et al* [12])

Let G be a multigraph, and let k be an even integer. Then we have $\chi'_k(G) = ch'_k(G) = \left\lceil \frac{1}{k}\,\Delta(G) \right\rceil$.

Proof It is obvious that $ch'_k(G) \geq \chi'_k(G) \geq \left\lceil \frac{1}{k}\,\Delta \right\rceil$, so it suffices to prove $ch'_k(G) \leq \left\lceil \frac{1}{k}\,\Delta \right\rceil$.

Let $k = 2\,\ell$. Without loss of generality, we can assume Δ is a multiple of k and G is a Δ-regular multigraph. (Otherwise, we can add some new edges and, if necessary, some new vertices. If this larger multigraph is k-frugal edge choosable with lists of size $\left\lceil \frac{1}{k}\,\Delta \right\rceil$, then so is G.) As k, and hence Δ, is even, we can find an Euler tour in each component of G. By given these tours a direction, we obtain an orientation D of the edges of G such that the in-degree and the out-degree of every vertex is $\frac{1}{2}\,\Delta$. Let us define the bipartite multigraph $H = (V_1 \cup V_2, E)$ as follows: V_1, V_2 are both copies of $V(G)$. For every arc (a, b) in D, we add an edge between $a \in V_1$ and $b \in V_2$.

Since D is a directed multigraph with in- and out-degree equal to $\frac{1}{2}\,\Delta$, H is a $(\frac{1}{2}\,\Delta)$-regular bipartite multigraph. That means we can decompose the edges of H into $\frac{1}{2}\,\Delta$ perfect matchings $M_1, M_2, \ldots, M_{\Delta/2}$. Define disjoint subgraphs $H_1, H_2, \ldots, H_\ell$ as follows: for $i = 0, 1, \ldots, \ell - 1$ set $H_{i+1} = M_{\frac{i}{k}\Delta+1} \cup M_{\frac{i}{k}\Delta+2} \cdots \cup M_{\frac{i+1}{k}\Delta}$. Notice that each H_i is a bipartite multigraph of regular degree $\frac{1}{k}\,\Delta$.

Now, suppose that each edge comes with a list of colours of size $\frac{1}{k}\,\Delta$. (If we had to add edges to make Δ a multiple of k or the multigraph Δ-regular, then give arbitrary lists to these edges.) Each subgraph H_i has maximum degree $\frac{1}{k}\,\Delta$, so by Galvin's theorem we can find a proper edge colouring of each H_i such that the colour of each edge is inside its list. We claim that the same colouring of edges in G is k-frugal. For this we need the following observation:

Observation *Let M be a matching in H. Then the set of corresponding edges in G form a subgraph of maximum degree at most two.*

To see this, remark that each vertex has two copies in H: one in V_1 and one in V_2. The contribution of the edges of M to a vertex v in the original multigraph is then at most two, at most one from each copy of v.

To conclude, we observe that each colour class in H is the union of at most ℓ matchings, one in each H_i. So at each vertex, each colour class appears at most two times the number of H_i's, i.e., at most $2\,\ell = k$ times. This is exactly the k-frugality condition we set out to satisfy. $\square$

For odd values of k we give a tight upper bound of the k-frugal edge chromatic number.

Theorem 6.3
Let k be an odd integer. Then we have $\lceil \frac{\Delta(G)}{k} \rceil \le \chi'_k(G) \le ch'_k(G) \le \lceil \frac{3\,\Delta(G)}{3\,k-1} \rceil$.

Proof Again, all we have to prove is $ch'_k(G) \le \lceil \frac{3\,\Delta(G)}{3\,k-1} \rceil$.

Let $k = 2\,\ell + 1$. Since $3\,k - 1$ is even and not divisible by three, we can again assume, without loss of generality, that Δ is even and divisible by $3\,k - 1$, and that G is Δ-regular. Set $\Delta = m\,(3\,k - 1) = 6\,\ell\,m + 2\,m$. Using the same idea as in the previous proof, we can decompose G into two subgraphs G_1, G_2, where G_1 is $(6\,\ell\,m)$-regular and G_2 is $(2\,m)$-regular. (Alternatively, we can use Petersen's Theorem [19] that every even regular multigraph has a 2-factor, to decompose the edge set in 2-factors, and combine these 2-factors appropriately.) Since $\frac{1}{2\,\ell} \cdot 6\,\ell\,m = \frac{3}{3\,k-1}\,\Delta$, by Theorem 6.2 we know that G_1 has a $2\,\ell$-frugal edge colouring using the colours from each edge's lists. Similarly we have $\frac{3}{2} \cdot 2\,m = \frac{3}{3\,k-1}\,\Delta$, and hence Theorem 6.1 (d) guarantees that we can properly colour the edges of G_2 using colours from those edges' lists. The combination of these two colourings is a $(2\,\ell + 1)$-frugal list edge colouring, as required. $\square$

Note that Theorem 6.3 is best possible: For $m \ge 1$, let $T^{(m)}$ be the multigraph with three vertices and m parallel edges between each pair. If $k = 2\,\ell + 1$ is odd, then the maximum number of edges with the same colour a k-frugal edge colouring of $T^{(m)}$ can have is $3\,\ell + 1$. Hence the minimum number of colours needed for a k-frugal edge colouring is $\left\lceil \frac{3\,m}{3\,\ell+1} \right\rceil = \left\lceil \frac{3}{3\,k-1}\,\Delta(T^{(m)}) \right\rceil$.

7 Discussion

As this is one of the first papers on frugal colouring, many possible directions for future research are still open. An intriguing question is inspired by the results on frugal edge colouring in the previous section. These results demonstrate an essential difference between even and odd k as far as k-frugal edge colouring is concerned. Based on what we think are the extremal examples of planar graphs for k-frugal *vertex* colouring, also our Conjecture 2.2 gives different values for even and odd k. But for frugal vertex colourings of planar graphs in general we have not been able to obtain results that are different for even and odd k. Most of our results for vertex colouring of planar graphs are consequences of Proposition 3.1 and known results on $L(k, 1)$-labelling of planar graphs, for which no fundamental difference between odd and even k has ever been demonstrated. Hence, a major step would be to prove that Proposition 3.1 is far from tight when k is even.

A second line of future research could be to investigate which classes of graphs have k-frugal chromatic number equal to the minimum possible value $\lceil \frac{\Delta}{k} \rceil + 1$. Corollary 3.7 and Theorems 4.1 and 4.2 give bounds for planar graphs with large girth and outerplanar graphs with large maximum degree that are very close to the best possible bound. We conjecture that, in fact, planar graphs with large enough girth and outerplanar graphs of large enough maximum degree do satisfy $\chi_k(G) = \lceil \frac{\Delta(G)}{k} \rceil + 1$ for all $k \geq 1$.

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