How many times can a function be iterated?

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Abstract

Let C be a closed subset of a topological space X, and let $f : C \to X$. Let us assume that f is continuous and $f(x) \in C$ for every $x \in \partial C$.

How many times can one iterate f?

This paper provides estimates on the number of iterations and examples of their optimality. In particular we show how some topological properties of f, C, X are related to the maximal number of iterations, both in the case of functions and in the more general case of set-valued maps.

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1 Introduction

Let C be a closed subset of a topological space X, and let $f : C \to X$. We investigate the existence of finite or infinite sequences (orbits) $\{x_i\}_{i \in I}$ in X, where $I = \{0, 1, \ldots, n\}$ or $I = \mathbb{N}$, such that $x_i = f(x_{i-1})$ for every $i \in I$ with $i \geq 1$.

At this level of generality there is of course no reason for such a sequence to exist with $n > 1$. For this reason we assume two conditions on f:

- f is continuous;
- f maps ∂C back to C, namely $f(x) \in C$ for every $x \in \partial C$.

For want of a better term, we call this topic "Discrete Viability Theory". Surprisingly this problem seems to be quite new. Up to now indeed we have found little related literature, although this topic seems to come close to different areas.

First of all it is close to conventional Viability Theory (see J.P. Aubin [1]). Standard Viability Theory considers continuous-time dynamic processes with some control mechanism and the ability of these processes to stay within given sets and concerns itself with the necessary and sufficient conditions for that dynamic system to be well defined with respect to this set. Unfortunately there are relatively few theorems for discrete-time models: one example is Theorem 3.7.11 of [1] where $X = \mathbb{R}^n$, C is a convex subset, f is a multi-valued map with convex images, and the existence of a fixed point is proven.

Discrete Viability Theory is close to fixed points theorems. The trivial reason is that we can of course iterate f infinitely many times if we start from a fixed point. A more profound reason is that in many cases the ω-limit of an infinite orbit is a closed f-inviariant set, hence a fixed point of f as a function acting on the space of closed sets. Conversely, any point of a f-invariant set is the starting point of an infinite orbit.

Discrete Viability Theory is close also to discrete-time dynamic system, but there the iterations are well defined for the trivial reason that $C = X$ and the main questions concern their asymptotic behavior. Conley Index Theory is related weakly, especially by the further assumption of ours that the function is homotopic to the identity, giving our problem some relation to the study of flows [4].

In any case, Discrete Viability Theory is close to algebraic topology since that is the natural place where to find obstructions to move C to $X \setminus C$ with a finite number of iterations.

Finally, Discrete Viability Theory is close to Game Theory. There are special stochastic games called quitting games [6] for which the existence of ε-equilibria is equivalent to the existence of non-convergent orbits of correspondences defined from the δ-equilibria of the one-shot game [5]. It was the context of quitting games (and an overly optimistic approach to equilibrium existence) that led us to discover Example 4.8.

Let us come now to an explicit example of question. If C is an interval in $X = \mathbb{R}$, then f admits always a fixed point. Therefore the simplest nontrivial problem in Discrete Viability Theory is probably the following one.

Problem 1.1 *Let C be a compact and connected subset of $\mathbb{R}^2$. Let $f : C \to \mathbb{R}^2$ be a continuous function such that $f(x) \in C$ for every $x \in \partial C$.*

How many times can we iterate f starting from a suitable $x \in C$?

What motivates us the most is the search for some theorem that implies the existence of an infinite orbit for functions that neither have fixed points nor $f(C) \subseteq C$. When we started attacking this problem we were rather optimistic about the existence of such a theorem under general assumptions on f, C, X, and all the colleagues we contacted in that period shared our optimism. The first two iterations are indeed given for free, and a simple connectedness argument provides two more iterations. At a first glance it seemed also possible to reiterate the argument (see Remark 2.6) assuming only the connectedness of X and C.

Our optimism decreased when T. Wiandt [8] showed us a simple situation (see Example 4.3) where X and C are compact and connected but only four iterations are possible. That example showed us that further requirements on f, C, X were needed in order to perform further iterations of f. In order to rule out the situation of Example 4.3 we worked in two different directions: either by asking that f is homotopic to the identity in a suitable sense (see Problem 2.2), or by requiring X to be simply connected, since in that example X is the unit circle S^1. In both cases we succeeded in proving the existence of a fifth iteration (Theorem 2.7, Theorem 2.8 and Theorem 2.12). However, the argument is more involved, and surprisingly more or less the same despite of the different additional assumptions.

The little optimism left become pessimism when we found Example 4.6, where f, C, X are as in Problem 1.1, f is homotopic to the identity in the suitable sense, and nevertheless only six iterations can be computed.

In any case we are not sure that this is the end of the story, because probably further topological requirements on C can provide more iterations (see Section 5).

In this paper we present our estimates on the number of iterations, and some examples to show their optimality. In order to give a complete theory we work both with functions and with set-valued maps (a good reference on iterating set-valued maps is [3]). Many parts of the theory are similar in both cases, but there are also some remarkable differences (see Remark 2.13).

This paper is organized as follows. In Section 2 we state the questions and our results. In Section 3 we prove the results. In Section 4 we present some examples showing the optimality of our estimates. In Section 5 we state some open problems.

2 Statements

Throughout this paper, unless otherwise stated, X denotes a topological space. Given $A \subseteq X$, $\mathrm{Int}(A)$ denotes the set of interior points, $\mathrm{Clos}(A)$ the closure, ∂A the boundary of A in X. We recall that X is said to be locally connected if every $x \in X$ has a fundamental system of connected neighborhoods.

Every $Y \subseteq X$ may be regarded as a topological space itself, with the topology inherited as a subset of X. If now $A \subseteq Y$, then $\mathrm{Int}_Y(A)$, $\mathrm{Clos}_Y(A)$, $\partial_Y A$ denote, respectively, the set of interior points, the closure, and the boundary of A relative to the topological space Y.

We say that Y satisfies the fixed point property if every continuous function $g : Y \to Y$ has a fixed point. For example, any nonempty compact convex subset of $\mathbb{R}^n$ has the fixed point property.

In this paper we make a mild use of Cech-Alexander cohomology, in the sense that in some statements we assume that $\check{H}^1(X) = 0$, namely that the first Cech-Alexander cohomology group (with $\mathbb{Z}$ as coefficient group, just to fix the ideas) is trivial. For readers which are not familiar with this cohomology theory, in Lemma 3.2 we show that for reasonable spaces (*e.g.* paracompact Hausdorff spaces) this assumption implies the following: "for every open set $A \subseteq X$, if A and $X \setminus A$ are connected, then ∂A is connected". This last property is what we use in this paper. We recall also that a simple case in which $\check{H}^1(X) = 0$ is when X is locally contractible and simply connected. Good references for Cech-Alexander cohomology are Chapter 3 of [2] and Chapter 6 of [7].

2.1 DVT for functions

The following is the main question in what we called Discrete Viability Theory.

Problem 2.1 *Let X be a topological space, and let $C \subseteq X$ be a nonempty closed subset. Let $f : C \to X$ be a continuous function such that $f(x) \in C$ for every $x \in \partial C$.*

How many times can we iterate f starting from a suitable $x \in C$?

In the following problem we strengthen the assumptions on f by asking that f is homotopically equivalent to the identity map on C by a homotopy whose intermediate maps also send ∂C back to the set C.

Problem 2.2 *Let X, C, and f be as in Problem 2.1. Let us assume that there exists a function $\Phi : C \times [0, 1] \to X$ such that*

- $\Phi(x, 0) = x$ *for every $x \in C$;*
- $\Phi(x, 1) = f(x)$ *for every $x \in C$;*

- *$\Phi(x,t) \in C$ for every $x \in \partial C$ and every $t \in [0,1]$.*

How many times can we iterate f starting from a suitable $x \in C$?

In order to better investigate these problems, we introduce some notations.

Definition 2.3 *Let X, C, and f be as in Problem 2.1. We recursively define a sequence $\{C_n\}_{n\in\mathbb{N}}$ of subsets of X by*

$$C_0 := X, \qquad\qquad C_{n+1} := \{x \in C : \ f(x) \in C_n\}.$$

Then we set

$$A_n := C_n \setminus C_{n+1};$$

$$\mathrm{Iter}(f, C, X) := \sup\{n \in \mathbb{N} : \ C_n \neq \emptyset\} \in \mathbb{N} \cup \{+\infty\}.$$

The following proposition clarifies the set-theoretic properties of the notions we have just introduced (proofs are trivial).

Proposition 2.4 *Let X be a set, let $C \subseteq X$ be a nonempty subset, and let $f : C \to X$ be any function.*

Then the notions introduced in Definition 2.3 fulfil the following properties:

(1) $\mathrm{Iter}(f, C, X)$ is the maximal length of a sequence $x_0, \ldots, x_n$ such that $x_i = f(x_{i-1})$ for every $i = 1, \ldots, n$;

(2) $C_{n+1} \subseteq C_n$ for every $n \in \mathbb{N}$;

(3) if $C_{n+1} = C_n$ for some $n \in \mathbb{N}$, then $C_m = C_n$ for every $m \geq n$;

(4) if $x \in C_{n+1}$ then $f(x) \in C_n$;

(5) if $x \in A_{n+1}$ then $f(x) \in A_n$;

(6) if $\mathrm{Iter}(f, C, X) = k < +\infty$, then $A_i \neq \emptyset$ if and only if $i \leq k$.

We state now the topological properties of the sets A_n and C_n.

Proposition 2.5 *Let X, C, and f be as in Problem 2.1.*

Then for every $n \in \mathbb{N}$ we have that (for simplicity we use ∂_n instead of ∂_{C_n} to denote boundaries relative to C_n)

(1) C_n is a closed subset of X;

(2) $f(\partial_{n+1} C_{n+2}) \subseteq \partial_n C_{n+1}$;

(3) $\partial_n C_{n+1} \subseteq C_{n+2}$*;*

(4) $A_n \cup C_{n+2}$ *is a closed set.*

Remark 2.6 As a consequence of Proposition 2.4 and Proposition 2.5, by restricting the domain and the codomain, we can regard f as a function $f : C_{n+1} \to C_n$, and this restriction satisfies $f(\partial_n C_{n+1}) \subseteq C_{n+1}$. Therefore, if $f : C \to X$ satisfies the assumptions of Problem 2.1, then $f : C_{n+1} \to C_n$ satisfies the same assumptions for every $n \in \mathbb{N}$, and $\mathrm{Iter}(f, C, X) = n + \mathrm{Iter}(f, C_{n+1}, C_n)$.

If we know *a priori* that C_n is connected for every $n \in \mathbb{N}$, this leads to an inductive proof that $C_n \neq \emptyset$ for every $n \in \mathbb{N}$. But we can find no non-trivial condition that forces this to hold, and simple examples can be given where infinite orbits (and also fixed points) exist but C_n in not connected for all $n \geq 2$.

If C_n is not connected it may happen that C_{n+1} is the union of some connected components of C_n: in this case f can map C_{n+1} in the remaining connected components of C_n, causing C_{n+2} to be empty (see the examples in Section 4).

This points out once more the importance of relative boundaries in Proposition 2.5: boundaries are always defined relative to something, and that something can change at each step.

The following result provides our estimates on the number of iterations for Problem 2.1.

Theorem 2.7 *Let X, C, and f be as in Problem 2.1, and let* $\mathrm{Iter}(f, C, X)$ *be as in Definition 2.3. Then we have the following estimates.*

(1) If $\partial C \neq \emptyset$ then $\mathrm{Iter}(f, C, X) \geq 2$.

(2) If X is connected, then $\mathrm{Iter}(f, C, X) \geq 3$.

(3) If X is connected, and C is connected, then $\mathrm{Iter}(f, C, X) \geq 4$.

(4) Let us assume that C is connected, and that X is a paracompact Hausdorff space which is connected, locally connected and satisfies $\check{H}^1(X) = 0$.

Then $\mathrm{Iter}(f, C, X) \geq 5$.

(5) If ∂C is a retract of $X \setminus \mathrm{Int}(C)$, and C satisfies the fixed point property, then there exists $x \in C$ such that $f(x) = x$. In particular $\mathrm{Iter}(f, C, X) = +\infty$.

Under the assumptions of Problem 2.2 we have the following result (note that there are no topological requirements on C and X).

Theorem 2.8 *Let X, C, and f be as in Problem 2.2, and let* $\mathrm{Iter}(f, C, X)$ *be as in Definition 2.3.*

Then we have that $\mathrm{Iter}(f, C, X) \geq 5$.

Some examples in Section 4 show the optimality of these estimates.

2.2 DVT for set-valued maps

In this section we extend some parts of the theory from functions to set-valued maps. Let us begin with some notations and definitions.

Let X be a topological space, let $C \subseteq X$ be a closed subset, and let $\mathcal{P}_\star(X)$ be the set of nonempty subsets of X. A set-valued map on C with values in X is any map $f : C \to \mathcal{P}_\star(X)$.

The first thing we need is some continuity of f. There are several notions of continuity for set-valued maps, and all of them are equivalent to standard continuity in the case of single-valued maps. The notion we use in this paper is usually referred in the literature as upper semicontinuity, and it is defined as follows.

(usc) A map $f : C \to \mathcal{P}_\star(X)$ is upper semicontinuous if for every open set $U \subseteq X$ we have that $\{x \in C : \ f(x) \subseteq U\}$ is an open subset of C.

Then we need to control the behavior of f at ∂C. The assumption in Problem 2.1 can be extended to set-valued maps in a weak and in a strong sense (equivalent if f is single-valued), as follows.

(Bdr-w) For every $x \in \partial C$ we have that $f(x) \cap C \neq \emptyset$.

(Bdr-s) For every $x \in \partial C$ we have that $f(x) \subseteq C$.

Finally, simple examples (see Example 4.7) show that nothing but the trivial iterations can be expected without connectedness assumptions on the images. For this reason, we often need the following property (trivially satisfied by functions).

(Conn) For every $x \in C$ we have that $f(x)$ is connected.

We can now state the main question in Discrete Viability Theory for set-valued maps.

Problem 2.9 *Let X be a topological space, let $C \subseteq X$ be a nonempty closed subset, and let $f : C \to \mathcal{P}_\star(X)$ be a set-valued map satisfying (usc), (Bdr-w) or (Bdr-s), and (Conn).*

How many times can we iterate f starting from a suitable $x \in C$?

In order to study this problem, in analogy with the case of functions we consider the sequence of sets $\{C_n\}_{n\in\mathbb{N}}$ recursively defined by

$$C_0 := X, \qquad\qquad C_{n+1} := \{x \in C : \ f(x) \cap C_n \neq \emptyset\},$$

and then we define A_n and $\mathrm{Iter}(f, C, X)$ as in Definition 2.3.

The set-theoretic properties of these notions are analogous to the case of functions. We sum them up in the following Proposition.

Proposition 2.10 *Let X be a set, let $C \subseteq X$ be a nonempty subset, and let $f : C \to \mathcal{P}_\star(X)$.*

Then statements (2), (3), (6) of Proposition 2.4 hold true without changes. Moreover, statements (1), (4), (5) of Proposition 2.4 hold true in the following modified form:

- *(1′)* $\mathrm{Iter}(f, C, X)$ *is the maximal length of a sequence $x_0, \ldots, x_n$ such that $x_i \in f(x_{i-1})$ for every $i = 1, \ldots, n$;*
- *(4′) if $x \in C_{n+1}$ then $f(x) \cap C_n \neq \emptyset$;*
- *(5′) if $x \in A_{n+1}$ then $f(x) \subseteq A_0 \cup \ldots \cup A_n$ and $f(x) \cap A_n \neq \emptyset$.*

The topological properties of the sets A_n and C_n are analogous to the case of functions only for small values of n, as stated in the following Proposition.

Proposition 2.11 *Let X be a topological space, let $C \subseteq X$ be a closed subset, and let $f : C \to \mathcal{P}_\star(X)$ be a set-valued map satisfying (usc), (Bdr-w) and (Conn).*

Then (we use ∂_n instead of ∂_{C_n} to denote boundaries relative to C_n)

- *(1) C_n is a closed subset of X for every $n \in \mathbb{N}$;*
- *(2) A_0 and A_1 are open subsets of X;*
- *(3) $\partial_1 C_2 \subseteq C_3$;*
- *(4) $A_1 \cup C_3$ is a closed subset of X, hence $A_0 \cup A_2$ is an open subset of X;*
- *(5) $\partial_2 C_3 \subseteq C_4$;*
- *(6) $A_2 \cup C_4$ is a closed subset of X, hence $A_0 \cup A_1 \cup A_3$ is an open subset of X;*
- *(7) if $x \in \partial_3 C_4 \cap A_4$ then $f(x) \cap A_2 \neq \emptyset$ and $f(x) \cap A_3 \neq \emptyset$.*

The following result is the counterpart of Theorem 2.7 for set-valued maps.

Theorem 2.12 *Let X be a topological space, let $C \subseteq X$ be a closed subset, let $f : C \to \mathcal{P}_\star(X)$ be a set-valued map, and let* $\mathrm{Iter}(f, C, X)$ *be as in Definition 2.3.*

Then we have the following estimates.

- *(1) If $\partial C \neq \emptyset$ and f satisfies (Bdr-w) then* $\mathrm{Iter}(f, C, X) \geq 2$.
- *(2) Let us assume that X is connected, and f satisfies (usc), (Bdr-w), and (Conn). Then* $\mathrm{Iter}(f, C, X) \geq 3$.

(3) Let us assume that X is connected, C is connected, and f satisfies (usc), (Bdr-w), and (Conn). Then $\mathrm{Iter}(f, C, X) \geq 4$.

(4) Let us assume that

- *X is a paracompact Hausdorff space which is connected, locally connected and satisfies $\check{H}^1(X) = 0$;*
- *C is connected;*
- *f satisfies (usc), (Bdr-s), and (Conn).*

Then $\mathrm{Iter}(f, C, X) \geq 5$.

The optimality of these estimates follows from the optimality of the corresponding estimates for functions.

Remark 2.13 Example 4.8 shows that statement (3) is the best one can expect under assumption (Bdr-w) (note that we assumed (Bdr-s) in statement (4)). In that example indeed X is $\mathbb{R}^2$, C is a contractible compact set which satisfies the fixed point property for functions, and all images of f are convex sets.

3 Proofs

3.1 Topological lemmata

The five lemmata we collect in this section are the technical core of this paper.

The first one is standard point-set topology. The statements may seem trivial: nevertheless, at least (1), (2), and (3) are false without local connectedness assumptions.

Lemma 3.1 *Let Y be a locally connected topological space.*

Then the following implications are true.

(1) If $V \subseteq Y$ is any subset, and V' is a connected component of V, then $\partial V' \subseteq \partial V$.

(2) If $V \subseteq Y$ is closed, and V' is a connected component of V, then $\partial V' = V' \cap \partial V$.

(3) For every family $\{A_i\}_{i \in I}$ of subsets of Y we have that

$$\partial \left(\bigcup_{i \in I} A_i \right) \subseteq \mathrm{Clos} \left(\bigcup_{i \in I} \partial A_i \right).$$

(4) Let us assume that Y is connected, $A \subseteq Y$ is an open subset such that $Y \setminus A$ is connected, and A' is a connected component of A. Then $Y \setminus A'$ is connected.

Proof.

Statement (1). Let $x \in \partial V'$. Then $x \in \mathrm{Clos}(V') \subseteq \mathrm{Clos}(V)$, hence either $x \in \partial V$ or $x \in \mathrm{Int}(V)$. Assume by contradiction that $x \in \mathrm{Int}(V)$. Since Y is locally connected there exists a connected neighborhood U of x contained in V. Since U is connected it is necessarily contained in V', but this implies that $x \in \mathrm{Int}(V')$ and contradicts the assumption that $x \in \partial V'$.

Statement (2). We have that $\partial V' \subseteq V'$ because V' is closed, and $\partial V' \subseteq \partial V$ because of the statement (1). The opposite inclusion $V' \cap \partial V \subseteq \partial V'$ is trivial (it holds true also without the local connectedness of Y or the closedness of V).

Statement (3). Let x be a point in the boundary of the union, and let U be any connected neighborhood of x. By assumption there exists $i_0 \in I$ such that $U \cap A_{i_0} \neq \emptyset$ and $U \setminus A_{i_0} \neq \emptyset$. By the connectedness of U this implies that $U \cap \partial A_{i_0} \neq \emptyset$. Since x has a fundamental system of connected neighborhoods, this is enough to conclude that x belongs to the closure of the union of the boundaries.

Statement (4). If A is connected the conclusion is trivial. Otherwise, let $\{A_i\}_{i\in I}$ be the set of connected components of $A \setminus A'$ so that

$$Y \setminus A' = (Y \setminus A) \cup \bigcup_{i\in I} A_i = \bigcup_{i\in I} [(Y \setminus A) \cup A_i] .$$

Thus it is enough to show that $(Y \setminus A) \cup A_i$ is connected for every $i \in I$. Since A_i is a nontrivial subset of the connected space Y, we have that $\partial A_i \neq \emptyset$, hence by statement (1)

$$\emptyset \neq \partial A_i \subseteq \partial A = \partial(Y \setminus A) \subseteq Y \setminus A.$$

Since $\mathrm{Clos}(A_i)$ is also connected it follows that $(Y \setminus A) \cup A_i = (Y \setminus A) \cup \mathrm{Clos}(A_i)$ is the union of two connected sets with nonempty intersection, hence it is connected. □

The second lemma relates the cohomological assumption on the space to the connectedness of the boundary of suitable subsets. We use this result every time we want to prove that the boundary of an open set is connected.

Lemma 3.2 *Let Y be a paracompact Hausdorff topological space such that $\check{H}^1(Y) = 0$. Let $A \subseteq Y$ be a connected open set such that $Y \setminus A$ is also connected.*

Then ∂A is connected.

Proof. We recall that a topological space is connected if and only if its 0-dimensional reduced Alexander cohomology group (with any coefficient group) is trivial.

Let us consider the long exact sequence of reduced Alexander cohomology groups for the pair $(Y, \mathrm{Clos}(A))$ (see [2, Theorem 2.13]):

$$\ldots \longrightarrow \widetilde{H}^0(\mathrm{Clos}(A)) \longrightarrow \widetilde{H}^1(Y, \mathrm{Clos}(A)) \longrightarrow \widetilde{H}^1(Y) \longrightarrow \ldots$$

In this sequence we have that $\widetilde{H}^0(\mathrm{Clos}(A)) = 0$ because $\mathrm{Clos}(A)$ is connected, and $\widetilde{H}^1(Y) = 0$. This implies that $\widetilde{H}^1(Y, \mathrm{Clos}(A)) = 0$.

By the strong excision property in paracompact Hausdorff spaces (see Exercise 6B in [2, p. 89] or Theorem 5 in [7, p. 318]) we can subtract A to both Y and $\mathrm{Clos}(A)$ obtaining that

$$\widetilde{H}^1(Y \setminus A, \partial A) = \widetilde{H}^1(Y, \mathrm{Clos}(A)) = 0.$$

Thus in the long exact sequence for the pair $(Y \setminus A, \partial A)$

$$\ldots \longrightarrow \widetilde{H}^0(Y \setminus A) \longrightarrow \widetilde{H}^0(\partial A) \longrightarrow \widetilde{H}^1(Y \setminus A, \partial A) \longrightarrow \ldots$$

we have that $\widetilde{H}^1(Y \setminus A, \partial A) = 0$ and $\widetilde{H}^0(Y \setminus A) = 0$ because $Y \setminus A$ is connected. It follows that $\widetilde{H}^0(\partial A) = 0$, which is equivalent to say that ∂A is connected. $\square$

The following result is used in the sequel every time we prove the existence of a fifth iteration. We state and prove it under the joint hypotheses of Lemma 3.1 and Lemma 3.2. We suspect it can be true also without the local connectedness assumption, but in that case the proof could be much more involved. On the contrary, the cohomological assumption is likely to be necessary.

Lemma 3.3 *Let Y be a paracompact Hausdorff locally connected topological space such that $\check{H}^1(Y) = 0$. Let K_1, K_2, U be three subsets such that*

(i) $K_1 \cap K_2 = \emptyset$;

(ii) U *is open and* $Y \setminus U$ *is connected;*

(iii) $\partial U \subseteq K_1 \cup K_2$;

(iv) $\partial U \cap K_1$ *and* $\partial U \cap K_2$ *are closed sets.*

Then U is the disjoint union of two subsets U_1 and U_2 such that $\partial U_1 \subseteq \mathrm{Clos}(K_1)$ and $\partial U_2 \subseteq \mathrm{Clos}(K_2)$.

Proof. Let U' be any connected component of U. By (ii) and statement (4) of Lemma 3.1 we have that $Y \setminus U'$ is connected, and therefore from Lemma 3.2 we deduce that $\partial U'$ is connected. Due to statement (1) of Lemma 3.1 and assumption (iii) we have that $\partial U' \subseteq \partial U \subseteq K_1 \cup K_2$. We can therefore write

$$\partial U' = (\partial U' \cap K_1) \cup (\partial U' \cap K_2).$$

By assumptions (i) and (iv), the two terms in the right hand side are closed and disjoint, hence one of them must be empty. This proves that every connected component U' of U satisfies either $\partial U' \subseteq K_1$ or $\partial U' \subseteq K_2$.

Let $\{U_i\}_{i\in I}$ be the set of connected components of U whose boundary is contained in K_1, and let $\{U_j\}_{j\in J}$ be the set of connected components of U whose boundary is contained in K_2. Let us set

$$U_1 := \bigcup_{i\in I} U_i, \qquad\qquad U_2 := \bigcup_{j\in J} U_j.$$

It is clear that $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = U$. Moreover from statement (3) of Lemma 3.1 we have that

$$\partial U_1 = \partial\left(\bigcup_{i\in I} U_i\right) \subseteq \mathrm{Clos}\left(\bigcup_{i\in I} \partial U_i\right) \subseteq \mathrm{Clos}(K_1),$$

and similarly for U_2. $\square$

A first consequence of Lemma 3.3 is the following result, which is the main tool in the proof of Theorem 2.8.

Lemma 3.4 *It is not possible to decompose the unit square* $[0,1]\times[0,1]$ *as the disjoint union of subsets* $\mathcal{A}_i$ *(*$i = 0,1,2,3,4$*) satisfying the following properties:*

(A1) $(0,1)\in\mathcal{A}_2$ *and* $(1,1)\in\mathcal{A}_1$*;*

(A2) $\mathcal{A}_4$ *does not intersect the side* $[0,1]\times\{1\}$*;*

(A3) $\mathcal{A}_0$ *does not intersect the other three sides;*

(A4) $\mathcal{A}_3$*,* $\mathcal{A}_2\cup\mathcal{A}_4$*,* $\mathcal{A}_1\cup\mathcal{A}_3\cup\mathcal{A}_4$ *are closed sets;*

(A5) $\mathcal{A}_0$ *is an open set and* $\partial\mathcal{A}_0 \subseteq \mathcal{A}_2\cup\mathcal{A}_3\cup\mathcal{A}_4$*.*

Proof. Let us set for simplicity $Q := [0,1]\times[0,1]$. First of all we show that, up to modifying the sets $\mathcal{A}_0, \ldots, \mathcal{A}_4$, we can assume that they fulfil (A1) through (A5) and also the following additional property:

(A6) $Q\setminus\mathcal{A}_0$ is connected.

Let indeed P be the union of the three sides considered in (A3). By (A3) the closed set $Q\setminus\mathcal{A}_0$ contains the connected set P. Let $\mathcal{V}$ be the connected component of $Q\setminus\mathcal{A}_0$ containing P. Let us set $\widetilde{\mathcal{A}}_0 := Q\setminus\mathcal{V}$ and $\widetilde{\mathcal{A}}_i := \mathcal{A}_i\cap\mathcal{V}$ for $i = 1,2,3,4$.

It is easy to see that the sets $\widetilde{\mathcal{A}}_0, \ldots, \widetilde{\mathcal{A}}_4$ are disjoint and satisfy assumptions (A1) through (A4), and (A6). Moreover $\widetilde{\mathcal{A}}_0$ is open because $\mathcal{V}$ is closed. Finally, from statement (2) of Lemma 3.1 we have that

$$\partial\widetilde{\mathcal{A}}_0 = \partial\mathcal{V} = \partial(Q\setminus\mathcal{A}_0)\cap\mathcal{V} = \partial\mathcal{A}_0\cap\mathcal{V} \subseteq (\mathcal{A}_2\cup\mathcal{A}_3\cup\mathcal{A}_4)\cap\mathcal{V} = \widetilde{\mathcal{A}}_2\cup\widetilde{\mathcal{A}}_3\cup\widetilde{\mathcal{A}}_4,$$

which proves also (A5).

Roughly speaking, what we have done in this first part of the proof is to fill the holes of $\mathcal{A}_0$ which do not touch P, as shown in the following picture (P is the union of the lower and lateral sides of the squares).

$\mathcal{A}_0$ $\qquad\qquad\qquad\qquad$ $\widetilde{\mathcal{A}}_0$

From now on we drop tildes and we assume that $\mathcal{A}_0, \ldots, \mathcal{A}_4$ satisfy (A1) through (A6).

Since of course $\check{H}^1(Q) = 0$, we can apply Lemma 3.3 with $Y = Q$, $K_1 = \mathcal{A}_3$, $K_2 = \mathcal{A}_2 \cup \mathcal{A}_4$, $U = \mathcal{A}_0$. We obtain that $\mathcal{A}_0$ is the disjoint union of two sets $\mathcal{A}_0'$ and $\mathcal{A}_0''$ such that $\partial\mathcal{A}_0' \subseteq \mathcal{A}_3$ and $\partial\mathcal{A}_0'' \subseteq \mathcal{A}_2 \cup \mathcal{A}_4$. Together with (A4) this implies in particular that $\mathcal{A}_0' \cup \mathcal{A}_1 \cup \mathcal{A}_3 \cup \mathcal{A}_4$ and $\mathcal{A}_0'' \cup \mathcal{A}_2 \cup \mathcal{A}_4$ are closed subsets of Q.

Let us consider now the side $S := [0, 1] \times \{1\}$, which can be written in the form

$$S = [S \cap (\mathcal{A}_0' \cup \mathcal{A}_1 \cup \mathcal{A}_3 \cup \mathcal{A}_4)] \cup [S \cap (\mathcal{A}_0'' \cup \mathcal{A}_2 \cup \mathcal{A}_4)] =: S_1 \cup S_2.$$

By (A2) we have that $S \cap \mathcal{A}_4 = \emptyset$, which proves that $S_1 \cap S_2 = \emptyset$. By (A1) we have that $(1, 1) \in S_1$ and $(0, 1) \in S_2$. Since S_1 and S_2 are closed sets, this contradicts the connectedness of S. $\square$

The last lemma is the set-valued extension of a well known result for continuous functions.

Lemma 3.5 *Let X be a topological space, let $C \subseteq X$ be a closed subset, and let $f : C \to \mathcal{P}_\star(X)$. Given $A \subseteq C$, let $f(A)$ be the image of A, defined as the union of $f(x)$ when x ranges in A.*

If f satisfies (usc) and (Conn), and A is connected, then $f(A)$ is connected.

Proof. We argue by contradiction. Let us assume that U and V are open subsets of X such that $f(A) \cap U$ and $f(A) \cap V$ are nonempty disjoint sets whose union is $f(A)$. Let $x \in A$. Since $f(x)$ is connected and contained in $f(A)$, it is clear that either $f(x) \subseteq U$ or $f(x) \subseteq V$. Therefore if we now define

$$U_1 := \{x \in A : \ f(x) \subseteq U\}, \qquad\qquad V_1 := \{x \in A : \ f(x) \subseteq V\},$$

we have found two nonempty disjoint open subsets of A whose union is A. This contradicts the connectedness of A. $\square$

3.2 Proof of Proposition 2.5

As a general fact we recall that, since each C_i is a closed set, the closure $\mathrm{Clos}_i(Z)$ in C_i of any subset $Z \subseteq C_i$ coincides with the closure $\mathrm{Clos}(Z)$ of Z in X.

Statement (1) This can be easily proved by induction using the definition of C_n and the continuity of f.

Statement (2) Since C_{n+2} is closed we have that $f(\partial_{n+1}C_{n+2}) \subseteq f(C_{n+2}) \subseteq C_{n+1}$. Moreover

$$\partial_{n+1}C_{n+2} = \partial_{n+1}(C_{n+1} \setminus C_{n+2}) = \partial_{n+1}A_{n+1} \subseteq \mathrm{Clos}_{n+1}(A_{n+1}) = \mathrm{Clos}(A_{n+1}),$$

hence

$$f(\partial_{n+1}C_{n+2}) \subseteq f(\mathrm{Clos}(A_{n+1})) \subseteq \mathrm{Clos}(f(A_{n+1})) \subseteq \mathrm{Clos}(A_n) = \mathrm{Clos}_n(C_n \setminus C_{n+1}).$$

We have thus established that $f(\partial_{n+1}C_{n+2}) \subseteq C_{n+1} \cap \mathrm{Clos}_n(C_n \setminus C_{n+1})$, which is equivalent to say that $f(\partial_{n+1}C_{n+2}) \subseteq \partial_n C_{n+1}$.

Statement (3) Let us argue by induction. The case $n = 0$ follows from the assumption that $f(\partial C) \subseteq C$. Assume now that $\partial_n C_{n+1} \subseteq C_{n+2}$ for some given n. By statement (2) and the inductive hypothesis we have that $f(\partial_{n+1}C_{n+2}) \subseteq \partial_n C_{n+1} \subseteq C_{n+2}$, which proves that $\partial_{n+1}C_{n+2} \subseteq C_{n+3}$ and completes the induction.

Statement (4) By statement (3) we have that

$$\mathrm{Clos}(A_n) = \mathrm{Clos}_n(A_n) = A_n \cup \partial_n A_n = A_n \cup \partial_n(C_n \setminus A_n) = A_n \cup \partial_n C_{n+1} \subseteq A_n \cup C_{n+2},$$

hence, since C_{n+2} is closed, $\mathrm{Clos}(A_n \cup C_{n+2}) = \mathrm{Clos}(A_n) \cup \mathrm{Clos}(C_{n+2}) \subseteq A_n \cup C_{n+2}$, which completes the proof. $\square$

3.3 Proof of Theorem 2.7

Statement (1) Trivial because $\partial C \subseteq C_2$.

Statement (2) If $C = X$, then $\mathrm{Iter}(f, C, X) = +\infty$. If C is a proper subset of the connected space X, then $\partial C \neq \emptyset$, which proves that $\mathrm{Iter}(f, C, X) \geq 2$. Assume by contradiction that it is exactly 2. This means that $X = A_0 \cup A_1 \cup A_2$. Applying statement (4) of Proposition 2.5 with $n = 0$ and $n = 1$, we deduce that both $A_0 \cup A_2$ and A_1 are nonempty closed sets and this contradicts the connectedness of X.

Statement (3) By the previous statement we know that $\operatorname{Iter}(f, C, X) \geq 3$. Assume by contradiction that it is exactly 3. This means that $C = A_1 \cup A_2 \cup A_3$. Applying statement (4) of Proposition 2.5 with $n = 1$ and $n = 2$, we deduce that both $A_1 \cup A_3$ and A_2 are nonempty closed sets and this contradicts the connectedness of C.

Statement (4) Since X and C are connected, from statement (3) we know that $\operatorname{Iter}(f, C, X) \geq 4$. Assume now by contradiction that it is exactly 4. Applying statement (4) of Proposition 2.5 with $n = 1, 2, 3$ we have that $A_1 \cup A_3 \cup A_4$, $A_2 \cup A_4$, and A_3 are nonempty closed subsets of X.

Since $\partial A_0 = \partial C \subseteq A_2 \cup A_3 \cup A_4$, we can apply Lemma 3.3 with $Y = X$, $K_1 = A_3$, $K_2 = A_2 \cup A_4$, $U = A_0$. We obtain that A_0 is the disjoint union of two sets A_0' and A_0'' such that $\partial A_0' \subseteq A_3$ and $\partial A_0'' \subseteq A_2 \cup A_4$. This implies in particular that $A_0' \cup A_1 \cup A_3 \cup A_4$ and $A_0'' \cup A_2 \cup A_4$ are closed subsets of X.

Let us consider now the connected set $f(C)$, and let us write

$$f(C) = [f(C) \cap (A_0' \cup A_1 \cup A_3 \cup A_4)] \cup [f(C) \cap (A_0'' \cup A_2 \cup A_4)] =: F_1 \cup F_2.$$

Then F_1 and F_2 are closed subsets of $f(C)$. They are also nonempty because $f(C)$ intersects A_1, A_2 and A_3. Finally, they are disjoint because $f(C) \cap A_4 = \emptyset$. This contradicts the connectedness of $f(C)$.

Statement (5) Let $r : X \setminus \operatorname{Int}(C) \to \partial C$ be a retraction, and let

$$g(x) := \begin{cases} f(x) & \text{if } f(x) \in C \\ r(f(x)) & \text{if } f(x) \notin \operatorname{Int}(C). \end{cases}$$

It is not difficult to see that $g : C \to C$ is continuous (one only needs to verify that it is well defined when $f(x) \in \partial C$). Since C satisfies the fixed point property there exists $x_0 \in C$ such that $g(x_0) = x_0$. We claim that x_0 is indeed a fixed point of f.

If $f(x_0) \in C$ then $x_0 = g(x_0) = f(x_0)$ and so x_0 is also a fixed point of f. Assume now by contradiction that $f(x_0) \notin C$. Since $f(\partial C) \subseteq C$, this implies that $x_0 \notin \partial C$. On the other hand, in this case $g(x_0) = r(f(x_0)) \in \partial C$, which is absurd. This completes the proof. $\square$

3.4 Proof of Theorem 2.8

Let C' be a connected component of C, and let X' be the connected component of X containing C'. Since f is homotopic to the identity it is easy to see that f maps C' to X'. From now on we can therefore assume that C and X are connected, so that by statement (3) of Theorem 2.7 we have that $\operatorname{Iter}(f, C, X) \geq 4$.

Assume now that it is exactly 4. Applying statement (4) of Proposition 2.5 with $n = 1, 2, 3$ we have that $A_1 \cup A_3 \cup A_4$, $A_2 \cup A_4$, and A_3 are nonempty closed subsets of X. Moreover A_0 is open and $\partial A_0 \subseteq A_2 \cup A_3 \cup A_4$.

Step 1. We prove that $\partial C \cap A_3 \neq \emptyset$.

Let us assume indeed by contradiction that $\partial C = \partial A_0 \subseteq A_2 \cup A_4$, hence in particular that $A_0 \cup A_2 \cup A_4$ is a closed set. Now we consider the connected set $f(C)$ and we write

$$f(C) = [f(C) \cap (A_1 \cup A_3 \cup A_4)] \cup [f(C) \cap (A_0 \cup A_2 \cup A_4)] =: F_1 \cup F_2.$$

Then F_1 and F_2 are closed subsets of $f(C)$. They are also nonempty because $f(C)$ intersects A_1, A_2 and A_3. Finally, they are disjoint because $f(C) \cap A_4 = \emptyset$. This contradicts the connectedness of $f(C)$.

Step 2. Let $x_0 \in \partial C \cap A_3$. We show that there exists a continuous curve $\gamma : [0, 1] \to C$ such that $\gamma(0) = x_0$ and $\gamma(1) \in \partial C \cap A_2$.

To begin with, let us consider the curve $\gamma_1 : [0, 1] \to C$ defined by $\gamma_1(t) = \Phi(x_0, t)$. This curve takes its values in C because the homotopy sends ∂C back to C. We can therefore extend it to a curve $\gamma_2 : [0, 2] \to X$ by setting

$$\gamma_2(t) := \begin{cases} \gamma_1(t) & \text{if } t \in [0, 1], \\ f(\gamma_1(t-1)) & \text{if } t \in [1, 2]. \end{cases}$$

The curve γ_2 is continuous (one only needs to check that it is well defined for $t = 1$). Moreover $\gamma_2(1) = f(x_0) \in A_2$, $\gamma_2(2) = f(f(x_0)) \in A_1$, and for every $t \in [1, 2]$ we have that $\gamma_2(t) \in f(C) \subseteq A_0 \cup A_1 \cup A_2 \cup A_3$.

We claim that $\gamma_2(t) \in A_0$ for some $t \in [1, 2]$. Assume indeed that $\gamma_2(t) \in A_1 \cup A_2 \cup A_3$ for every $t \in [1, 2]$. Then

$$[1, 2] = \{t \in [1, 2] : \ \gamma_1(t) \in A_2 \cup A_4\} \cup \{t \in [1, 2] : \ \gamma_2(t) \in A_1 \cup A_3 \cup A_4\} =: I_1 \cup I_2.$$

Thus I_1 and I_2 are closed sets, and they are nonempty because $1 \in I_1$ and $2 \in I_2$. Moreover they are disjoint because $\gamma_2([1, 2]) \cap A_4 = \emptyset$, and this contradicts the connectedness of $[1, 2]$.

Let us set now $t_\star := \inf\{t \in [1, 2] : \ \gamma_2(t) \in A_0\}$. From the definition of infimum it is clear that $\gamma_2(t) \in C$ for every $t \in [0, t_\star]$ and $\gamma_2(t_\star) \in \partial A_0 = \partial C$. We claim that $\gamma_2(t_\star) \in A_2$. Let us consider indeed

$$[1, t_\star] = \{t \in [1, t_\star] : \ \gamma_2(t) \in A_1 \cup A_3\} \cup \{t \in [1, t_\star] : \ \gamma_2(t) \in A_2\}.$$

Once again the two sets in the right hand side are closed and disjoint, and the second one is nonempty because it contains $t = 1$. By the connectedness of $[0, t_\star]$ it follows that the first one is empty and therefore $\gamma_2(t_\star) \in A_2$.

The curve γ we are looking for is just (a reparametrization of) the restriction of γ_2 to the interval $[0, t_\star]$.

Step 3. Let γ be the curve of step 2, and let

$$\mathcal{A}_i := \{(\tau, t) \in [0,1] \times [0,1] : \ \Phi(\gamma(\tau), t) \in A_i\}$$

for $i = 0, 1, 2, 3, 4$. If we show that the $\mathcal{A}_i$'s satisfy assumptions (A1) through (A5) of Lemma 3.4 we have a contradiction.

Since $\gamma(0) \in A_3$ we have that $\Phi(\gamma(0), 1) = f(\gamma(0)) \in A_2$, hence $(0,1) \in \mathcal{A}_2$. Since $\gamma(1) \in A_2$ we have that $\Phi(\gamma(1), 1) = f(\gamma(1)) \in A_1$, hence $(1,1) \in \mathcal{A}_1$. This proves (A1).

Since the image of f is contained in $A_0 \cup A_1 \cup A_2 \cup A_3$ it follows that $\Phi(\gamma(\tau), 1) = f(\gamma(\tau)) \notin A_4$ for every $\tau \in [0,1]$, which proves (A2).

Since $\gamma(0)$ and $\gamma(1)$ belong to ∂C, and Φ sends ∂C back to C, we have that $\Phi(\gamma(0), t)$ and $\Phi(\gamma(1), t)$ are in C for every $t \in [0,1]$. Since also $\Phi(\gamma(\tau), 0) = \gamma(\tau) \in C$ for every $\tau \in [0,1]$, this proves (A3).

Finally, (A4) and (A5) follow from the continuity of $\Phi(\gamma(\tau), t)$ and the analogous properties of the A_i's. $\square$

3.5 Proof of Proposition 2.11

Statement (1) This can be easily proved by induction using the definition of C_n and the upper semicontinuity of f.

Statement (2) The set $A_0 = X \setminus C$ is open because C is closed. Now since

$$A_1 = \{x \in C : \ f(x) \cap C = \emptyset\} = \{x \in C : \ f(x) \subseteq A_0\},$$

and since f satisfies (usc), we have that A_1 is an open subset of C. In order to conclude that it is also an open subset of X it suffices to prove that $A_1 \cap \partial C = \emptyset$. This follows from (Bdr-w).

Statement (3) Let $x \in \partial_1 C_2$. Since C_2 is closed we have that $x \in C_2$, hence either $x \in C_3$ or $x \in A_2$. Let us assume by contradiction that $x \in A_2$. Then $f(x) \subseteq A_0 \cup A_1$ and $f(x) \cap A_1 \neq \emptyset$. Since A_0 and A_1 are open sets, and $f(x)$ is connected, we have that $f(x) \subseteq A_1$. This means that actually $A_2 = \{x \in C_1 : \ f(x) \subseteq A_1\}$, and thus it is an open subset of C_1 contained in C_2. Therefore if $x \in A_2$ then $x \in \mathrm{Int}_1(C_2)$, which contradicts the initial assumption that $x \in \partial_1 C_2$.

Statement (4) The argument is the same used in the proof of statement (4) of Proposition 2.5. Since C_1 is closed and $\partial_1 C_2 \subseteq C_3$ we have that

$$\mathrm{Clos}(A_1) = \mathrm{Clos}_1(A_1) = A_1 \cup \partial_1 A_1 = A_1 \cup \partial_1(C_1 \setminus A_1) = A_1 \cup \partial_1 C_2 \subseteq A_1 \cup C_3,$$

hence, since C_3 is closed, $\mathrm{Clos}(A_1 \cup C_3) = \mathrm{Clos}(A_1) \cup \mathrm{Clos}(C_3) = A_1 \cup C_3$.

Statement (5) We argue more or less as in the proof of statement (3).

Let $x \in \partial_2 C_3$. Since C_3 is closed we have that $x \in C_3$, hence either $x \in C_4$ or $x \in A_3$. Let us assume by contradiction that $x \in A_3$. Then $f(x) \subseteq (A_0 \cup A_2) \cup A_1$ and $f(x) \cap A_2 \neq \emptyset$. Since $A_0 \cup A_2$ and A_1 are open sets, and $f(x)$ is connected, we have that $f(x) \subseteq A_0 \cup A_2$. This means that actually $A_3 = \{x \in C_2 : \ f(x) \subseteq A_0 \cup A_2\}$, and thus it is an open subset of C_2 contained in C_3. Therefore if $x \in A_3$ then $x \in \mathrm{Int}_2(C_3)$, which contradicts the initial assumption that $x \in \partial_2 C_3$.

Statement (6) Same proof of statement (4) with indices increased by 1.

Statement (7) Let $x \in \partial_3 C_4 \cap A_4$. Since $x \in A_4$ we know that $f(x) \cap A_3 \neq \emptyset$ and $f(x) \subseteq A_0 \cup A_1 \cup A_2 \cup A_3$. Let us assume by contradiction that $f(x) \cap A_2 = \emptyset$, hence that $f(x)$ is contained in the open set $A_0 \cup A_1 \cup A_3$. Now consider $U := \{x \in C_3 : f(x) \subseteq A_0 \cup A_1 \cup A_3\}$. It is an open subset of C_3 which is contained in C_4 (all points in U lie indeed in C_4). Since $x \in U$, we conclude that $x \in \mathrm{Int}_3(C_4)$, which contradicts the initial assumption that $x \in \partial_3 C_4$. $\square$

3.6 Proof of Theorem 2.12

Statement (1) Trivial because $\partial C \subseteq C_2$.

Statement (2) If $C = X$, then $\mathrm{Iter}(f, C, X) = +\infty$. If C is a proper subset of the connected space X, then $\partial C \neq \emptyset$, which proves that $\mathrm{Iter}(f, C, X) \geq 2$. Assume by contradiction that it is exactly 2. This means that $X = A_0 \cup A_1 \cup A_2$. By statements (2) and (4) of Proposition 2.11 we know that both $A_0 \cup A_2$ and A_1 are nonempty open sets and this contradicts the connectedness of X.

Statement (3) By the previous statement we know that $\mathrm{Iter}(f, C, X) \geq 3$. Assume by contradiction that it is exactly 3. This means that $C = A_1 \cup A_2 \cup A_3$. By statements (4) and (6) of Proposition 2.11 we know that in this case both $A_1 \cup A_3$ and A_2 are nonempty closed sets and this contradicts the connectedness of C.

Statement (4) Since X and C are connected, from statement (3) we know that $\mathrm{Iter}(f, C, X) \geq 4$. Assume now by contradiction that it is exactly 4. From statements (1), (4), and (6) of Proposition 2.11 we know that $A_1 \cup A_3 \cup A_4$, $A_2 \cup A_4$ and $A_3 \cup A_4 = C_3$ are closed sets, but we don't know whether A_3 is closed or not.

Let us prove that in any case $A_3 \cap \partial C$ is closed. Indeed, since

$$\mathrm{Clos}(A_3) = \mathrm{Clos}_3(A_3) = A_3 \cup \partial_3 A_3 = A_3 \cup \partial_3(C_3 \setminus A_3) = A_3 \cup \partial_3 C_4,$$

and since $\partial_3 C_4 \subseteq C_4 = A_4$, we have that $A_3 \cap \partial C$ is closed if and only if $\partial_3 C_4 \cap \partial C = \emptyset$. Let us assume by contradiction that there exists $x \in \partial_3 C_4 \cap \partial C$. By (Bdr-s) we have that $f(x) \subseteq C$, hence $f(x) \cap A_0 = \emptyset$ and therefore

$$f(x) = [f(x) \cap (A_1 \cup A_3 \cup A_4)] \cup [f(x) \cap (A_2 \cup A_4)] =: F_1 \cup F_2.$$

Thus F_1 and F_2 are closed subsets of $f(x)$. Moreover, since $x \in \partial_3 C_4 = \partial_3 C_4 \cap A_4$, from statement (7) of Proposition 2.11 we deduce that F_1 and F_2 are nonempty. Finally, they are disjoint because $f(x) \cap A_4 = \emptyset$. This contradicts the connectedness of $f(x)$.

Once we know that $A_3 \cap \partial C$ is closed we can proceed as in the case of functions. We apply Lemma 3.3 with $Y = X$, $K_1 = A_3$, $K_2 = A_2 \cup A_4$, $U = A_0$ and we obtain that A_0 is the disjoint union of two sets A_0' and A_0'' such that $\partial A_0' \subseteq \mathrm{Clos}(A_3) \subseteq A_3 \cup A_4$ and $\partial A_0'' \subseteq \mathrm{Clos}(A_2 \cup A_4) = A_2 \cup A_4$. This implies in particular that $A_0' \cup A_1 \cup A_3 \cup A_4$ and $A_0'' \cup A_2 \cup A_4$ are closed subsets of X.

Now we consider $f(C)$, which is a connected set because of Lemma 3.5, and we write

$$f(C) = [f(C) \cap (A_0' \cup A_1 \cup A_3 \cup A_4)] \cup [f(C) \cap (A_0'' \cup A_2 \cup A_4)].$$

Since $f(C) \cap A_4 = \emptyset$, the two sets in brackets in the right hand side are disjoint. They are also nonempty because $f(C)$ intersects A_1, A_2 and A_3. Finally, they are closed subsets of $f(C)$.

This contradicts the connectedness of $f(C)$. $\square$

4 Examples

The first four examples show that the estimates of $\mathrm{Iter}(f, C, X)$ given in the first four statements of Theorem 2.7 are optimal.

Example 4.1 Let $X := \{0\} \cup [2, 4]$ with the topology inherited as a subset of the real line, let $C := \{0, 4\}$, and let $f : C \to X$ be defined by $f(0) = 3$ and $f(4) = 0$.

Then X, C, and f satisfy the assumptions of Problem 2.1 (in this case indeed $\partial C = \{4\}$), and $\mathrm{Iter}(f, C, X) = 2$.

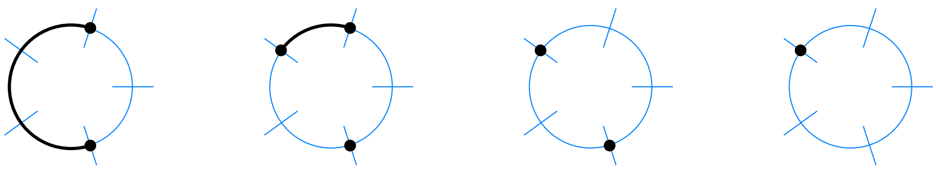
Example 4.2 Let $X := \mathbb{R}$ with the usual topology, let $C := \{0\} \cup [2,4]$, and let $f : C \to X$ be defined by $f(x) = (x-2)(x-4)/3$.

The function f maps 2 and 4 to 0, then it maps 0 inside $(2,4)$, and finally it maps the open interval $(2,4)$ outside C.

Therefore X, C, and f satisfy the assumptions of Problem 2.1. Moreover X is connected, C is not connected, $C_2 = \partial C = \{0,2,4\}$, $C_3 = \{2,4\}$, and $C_4 = \emptyset$. In particular $\mathrm{Iter}(f,C,X) = 3$.

Example 4.3 Let $X = \{(x,y) \in \mathbb{R}^2 : \ x^2+y^2 = 1\}$ be the circle, which we parametrize as usually with the angles in $[0,2\pi]$. Let $C := [2\pi/5, 8\pi/5]$ (namely 3/5 of the circle), and let $f : C \to X$ be the counterclockwise rotation by $4\pi/5$ (namely 2/5 of the way around the circle).

It turns out that X, C, and f satisfy the assumptions of Problem 2.1 (in this case indeed ∂C consists of the two points corresponding to $2\pi/5$ and $8\pi/5$). Moreover X and C are connected, and it is not difficult to see that $\mathrm{Iter}(f,C,X) = 4$. The sets $C_1, \ldots, C_4$ are represented in the following picture.



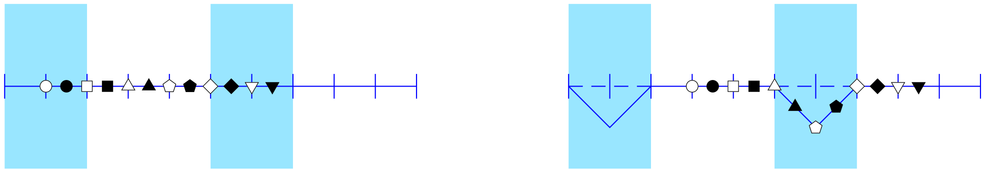
Example 4.4 Let us consider the following subsets of the real plane:

$$C := \{(x,0) \in \mathbb{R}^2 : \ x \in \mathbb{R}\}, \qquad X_1 := \bigcup_{k \in \mathbb{Z}} ([5k, 5k+2] \times \mathbb{R}).$$

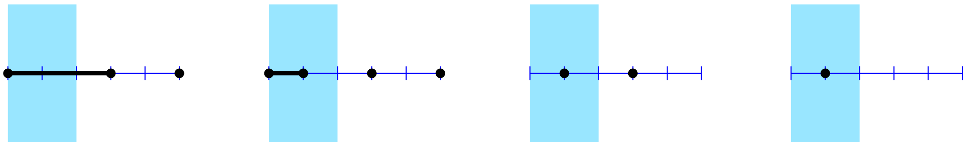
Let $X := X_1 \cup C$. Clearly both X and C, with the topology inherited as subsets of $\mathbb{R}^2$, are connected, simply connected, contractible. Let $f : C \to X$ be defined by

$$f(x,0) := \begin{cases} (x+2, 0) & \text{if } x \in [5k, 5k+3] \text{ for some } k \in \mathbb{Z}, \\ (x+2, |5k+4-x|-1) & \text{if } x \in [5k+3, 5k+5] \text{ for some } k \in \mathbb{Z}. \end{cases}$$

Roughly speaking, C is the x axis, X is the union of C and some periodically arranged vertical stripes, f is a translation by 2 in the x direction followed by a vertical bending inside the stripes. The following picture shows the action of f on some points of C.



The boundary of C in X is the union of the segments of the form $[5k, 5k+2] \times \{0\}$ (the intersection of C with the vertical stripes). The function f just translates these segments in the x direction, keeping them inside C. Therefore all the assumptions of Problem 2.1 are satisfied. It is not difficult to check that $\mathrm{Iter}(f, C, X) = 5$, and the sets $C_2, \ldots, C_5$ are those represented in the following picture (we represent only one period, of course).



C_2 C_3 C_4 C_5

The function f is also homotopic to the identity in the sense of Problem 2.2, since both are homotopic to the translation by 2 in the x direction. Therefore also the assumptions of Problem 2.2 are satisfied, and this shows the optimality of the estimate of $\mathrm{Iter}(f, C, X)$ given in Theorem 2.8.

Note that in Example 4.4 above the set C is not compact. At the present we have no example of a function $f : C \to X$ satisfying the assumptions of Problem 2.1 with X simply connected, C compact and connected, and $\mathrm{Iter}(f, C, X) = 5$.

As we have seen, Example 4.4 above shows also the optimality of Theorem 2.8. We now give another example, in which the subset C is not only closed, but also compact.

Example 4.5 Let us consider polar coordinates (ρ, θ) in the Euclidean plane. Let

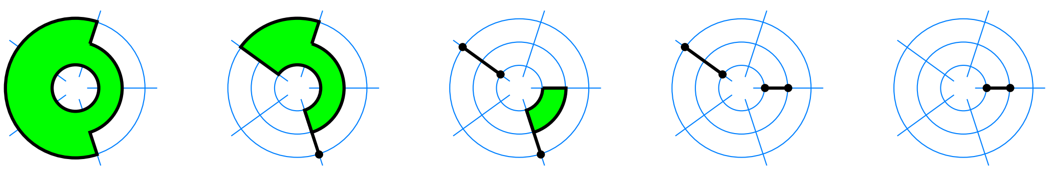
$$\begin{aligned} X &:= \{(\rho\cos\theta, \rho\sin\theta) \in \mathbb{R}^2 : \ 1 \le \rho \le 3, \ \theta \in [0, 2\pi]\}, \\ Y_1 &:= \{(\rho\cos\theta, \rho\sin\theta) \in \mathbb{R}^2 : \ 1 \le \rho \le 2, \ \theta \in [0, 2\pi]\}, \\ Y_2 &:= \{(\rho\cos\theta, \rho\sin\theta) \in \mathbb{R}^2 : \ 2 \le \rho \le 3, \ 2\pi/5 \le \theta \le 8\pi/5\}. \end{aligned}$$

Let $C := Y_1 \cup Y_2$, and let $f : C \to X$ be the function represented in polar coordinates by

$$(\rho, \theta) \to \left(\frac{5}{2}, \theta + \frac{4\pi}{5}\right).$$

We claim that X, C, and f satisfy the assumptions of Problem 2.2. Indeed, due to our choice of X, ∂C contains only the arc with $\rho = 2$ and $\theta \in [-2\pi/5, 2\pi/5]$, and the two line segments with $\rho \in [2,3]$ and $\theta \in \{-2\pi/5, 2\pi/5\}$. Therefore the function f sends ∂C in the points with $\rho = 5/2$ and $\theta \in [2\pi/5, 6\pi/5]$, hence inside C. As for the required homotopy, roughly speaking it can be constructed in three steps: reduction to the level $\rho = 2$, rotation, reduction to the level $\rho = 5/2$.

After the first iteration all points have radius equal to $5/2$, while with regard to the angle we have the identical situation of Example 4.3. It is now simple to see that $\mathrm{Iter}(f, C, X) = 5$ and the sets $C_1, \ldots, C_5$ are those represented in the following picture.



C_1 C_2 C_3 C_4 C_5

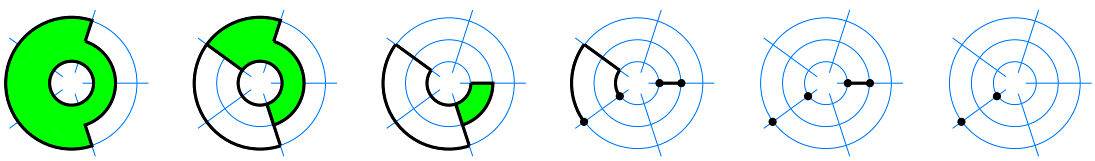
The following Example refers to the Euclidean case of Problem 1.1. It is probably the main example of this paper.

Example 4.6 Let $X := \mathbb{R}^2$ be the Euclidean plane, and let C be as in Example 4.5. Let $f : C \to X$ be represented in polar coordinates by

$$(\rho, \theta) \to \left(\frac{5 - |\rho - 2|}{2}, \theta + \frac{4\pi}{5}\right).$$

It is clear that X and C are connected, and X is simply connected. We claim that X, C, and f satisfy the assumptions of Problem 2.1. In this case indeed ∂C contains also the points in C with $\rho = 1$ and $\rho = 3$, but the image of these points is contained in the level $\rho = 2$, hence inside C. Moreover, the function f is homotopic to the identity in the sense of Problem 2.2 (as in Example 4.5 the homotopy can be realized through the level $\rho = 2$).

After two iterations all points have a radius strictly between 2 and 3, while with regard to the angle we have the identical situation of Example 4.3. It is not difficult to see that $\mathrm{Iter}(f, C, X) = 6$ and the sets $C_1, \ldots, C_6$ are those represented in the following picture.



C_1 C_2 C_3 C_4 C_5 C_6

Now we present two examples concerning set-valued maps. The first one shows that without connectedness assumptions on the images only trivial iterations are guaranteed.

Example 4.7 Let $X := \mathbb{R}$ with the usual topology, and let $C := [0, 4]$. Let $f : C \to \mathcal{P}_\star(X)$ be defined by

$$f(x) := \begin{cases} \{2\} & \text{if } x \in [0, 1) \cup (3, 4], \\ \{2, 5\} & \text{if } x \in \{1, 3\}, \\ \{5\} & \text{if } x \in (1, 3). \end{cases}$$

It is easy to show that X and C are connected (and even contractible), f satisfies (usc) and (Bdr-s), and $\operatorname{Iter}(f, C, X) = 2$.

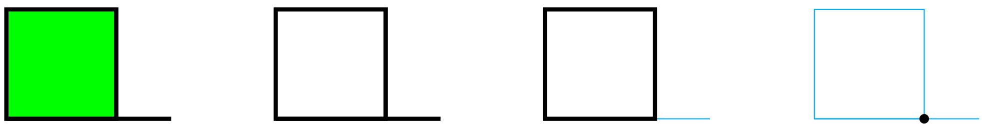
The following Example shows the optimality of statement (3) in Theorem 2.12.

Example 4.8 Let $X := \mathbb{R}^2$ with the usual topology, let Q be the square $[0, 2] \times [0, 2]$, let S be the segment with endpoints $(2, 0)$ and $(3, 0)$, and let $C := Q \cup S$. Let $d(x, y)$ denote the distance of the point (x, y) from the boundary of Q. Let $f : C \to \mathcal{P}_\star(X)$ be defined in the following way:

- if $(x, y) = (2, 0)$, then $f(x, y)$ is the segment with endpoints $(1, 1)$ and $(3, 0)$;
- if $(x, y) \in S \setminus Q$, then $f(x, y)$ is the singleton $\{(1, 1)\}$;
- if $(x, y) \in Q \setminus S$, then $f(x, y)$ is the singleton $\{(3, d(x, y))\}$.

In a few words, f is single-valued except at $(2, 0)$: it send $\partial Q \setminus S$ to $(3, 0)$, in turn $(3, 0)$ and the rest of $S \setminus Q$ are sent inside the square at $(1, 1)$, and the interior of Q is sent outside C. Finally, the image of $(2, 0)$ is the minimal convex set for which the resulting function turns out to be upper semicontinuous.

Therefore f satisfies (usc), (Bdr-w), (Conn). Moreover $\operatorname{Iter}(f, C, X) = 4$, and the sets C_1, ..., C_4 are those represented in the following picture.



C_1 C_2 C_3 C_4

5 Open problems

As mentioned in the introduction, the following is probably the main question in Discrete Viability Theory.

Open Problem 5.1 *Find nontrivial sufficient conditions on f, C, X in order to have that* $\operatorname{Iter}(f, C, X) = +\infty$.

Here "nontrivial" means that these conditions should be satisfied by reasonable classes of functions f without fixed points and with $f(C) \nsubseteq C$.

A first step in this direction could be to understand whether strengthening the topological assumptions on f, C, X guarantees further iterations. This leads to the second question.

Open Problem 5.2 *Under the assumptions of Problem 2.1 find intermediate results between statement (4) and statement (5) of Theorem 2.7.*

Example 4.4 shows that 5 iterations is the most one can expect even when C and X are contractible, hence as simple as possible from the topological point of view. This seems to be the tombstone on the search of further iterations. Nevertheless, we point out once again that in that example C is not compact. So a new frontier is understanding the role played by compactness in this subject, even in the simpler case.

Open Problem 5.3 *Find the maximal number of iterations which are assured under the assumptions of Problem 1.1.*

We know that this number is at least 5 and at most 6. We also know that if this number is 6 it is a matter of compactness. If this is the case, then we can ask ourselves what happens with further topological requirements on C, for example if $\check{H}^1(C) = 0$ (just to rule out Example 4.6).

Finally, a technical point for topologists.

Open Problem 5.4 *Find the minimal assumptions on Y under which Lemma 3.3 can be proved.*

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