SUCCESSIVE INFORMATION REVELATION IN 3-PLAYER INFINITELY REPEATED GAMES WITH INCOMPLETE INFORMATION ON ONE SIDE

JULIAN MERSCHEN [1]
Bonn Graduate School of Economics, University of Bonn
Adenauerallee 24-42, 53113 Bonn, Germany
email: J.Merschen@uni-bonn.de

CDAM Research Report LSE-CDAM-2007-09 - March 2007

ABSTRACT. In this work we focus on 3-player infinitely repeated games with incomplete information on one side. Renault (2001) has shown by counterexample that if in this setup there are more than 2 states of nature existence of completely revealing and joint plan equilibrium (JPE) for both informed players is not guaranteed. By defining a more general equilibrium concept, which relies on successive information revelation by the informed players, we were able to prove the existence of successive joint plan equilibria (SJPE) in the example from Renault (2001). Furthermore we were able to show that the set of possible SJPE is strictly larger than the set of "standard" JPE.

Submitted for the Master of Science in Applicable Mathematics 2006

London School of Economics and Political Science, University of London

---

[1] I wish to deeply thank my thesis adviser Dr. Robert Simon for his immense patience and his continuous support from the beginnings of my thesis until its completion. Not only would I like to thank Dr. Simon for the constructive and always understanding conversations but also for the excellent academic guidance. Furthermore I would also like to express my gratitude to Prof. Bernhard von Stengel for his superb support during my three years at the London School of Economics and Political Science. I would like to thank the entire Mathematics Department for their great help and support in every situation.

Contents

List of Figures

1 Introduction

Infinitely repeated games with incomplete information were first introduced by Aumann and Maschler in 1966 in order to study the behavior of agents in the arms race during the cold war. Since then their approach has found great relevance when analyzing long term behavior in strategic settings with incomplete information. Aumann and Maschler (1995) first showed in 1968 that 2-player infinitely repeated zero-sum games with incomplete information on one side have a value and thus an equilibrium. Later Simon, Spiez, and Torunczyk (1995) proved the existence of joint plan equilibria in all nonzero-sum games with two players. Yet the existence of an equilibrium in the case of 3-player infinitely repeated nonzero-sum games with incomplete information on one side has not been established. Renault (2001) proved by counterexample that not all 3-player games possess a completely revealing or a joint plan equilibrium.

In this thesis we briefly describe the setup of infinitely repeated games with incomplete information, discuss the results from the 2-player games and finally apply them to the 3-player case. After explaining the findings of Renault we introduce an extension to joint plans which relies on the concept of successive information revelation: in a first step one of the informed players would reveal some information about the outcome of the choice of nature, in a second step the other informed player would reveal additional information about the true state of nature. Using this concept of successive joint plans we prove the existence of successive joint plan equilibria in the counterexample used by Renault (2001).

2 Main Section

2.1 Infinitely Repeated Games with Incomplete Information

An *infinitely repeated game with incomplete information* Γ_∞ is given by $(K, p_0, G^j, N, S_i, g_i^j)_{i\in N}^{j\in K}$ where $K = \{1, ..., k\}$ is a finite set of k states of nature and each state of nature $j \in K$ represents a one shot game G^j. The initial probability distribution over the states of nature is given by $p_0 = (p_0^j)_{j\in K}$ and can be seen as a lottery where the outcome is a state $j \in K$. The set of n players is given by $N = \{1, ..., n\}$ where each player $i \in N$ has a finite set S_i of pure strategies or actions and for a given $j \in K$ a *payoff function* g_i^j is defined by $g_i^j : S \rightarrow \mathbb{R}$, where $S = \prod_{i=1}^{n} S_i$. There exist two kinds of players. The uninformed players only know the initial probability distribution over the states of nature but do not know the actual outcome of the lottery. The informed players in addition know the chosen $j \in K$. We partition the set of players N into two sets, the set of *informed players IN* and the set of *uninformed players UN*.

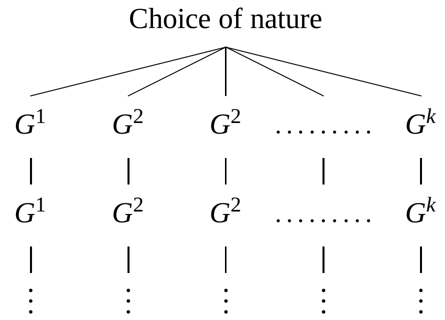

Figure 1: The Game Γ_∞

We assume that there exist at least one player in IN and one in UN and each of these players have at least two distinct actions and for all $j \in$ K, p_0^j is strictly larger than zero. The setup of the game is *common knowledge*, i.e. every player knows the setup of the game and knows that the other players know the setup of game and so on. We also assume that every player has *perfect recall* in a sense that every player remembers at every point in time what his previous choices were. These assumptions are motivated by the fact that players with only one action and states of nature with zero probability cannot affect the outcome of the game and are thus neglected.

The game is played as follows. At time t = 0 nature chooses a state $j \in K$ according to the initial probability distribution p_0. The chosen state is kept constant throughout the rest of the game. The outcome of the lottery is then communicated to the informed players but kept secret from the uninformed players. At every following stage t = 1,2,..., every player $i \in N$ chooses simultaneously an action $s_i \in S_i$. If $s_t = (s_1, ..., s_n) \in S$ is played at time t the *stage game payoff* for player i is given by $g_i^j(s_t)$ when $j \in K$ is the chosen state of nature. After each stage the chosen strategies of the players are publicly announced but the payoffs are not communicated. Note that every informed player is able to deduce his own stage game payoff with certainty as he knows the correct state of nature, contrary to the uninformed player.

The only possibility for the uniformed players to make a conjecture about the true state of nature is to consider the behavior of the informed players over time, i.e. the history of moves. For any finite $t \geq 0$ the *history* H_t is defined by $H_t = K \times (\prod_{i=1}^{n} S_i)^t$, where H_0 simply denotes the choice of nature. Furthermore the *infinite history* is given by H_∞, where $H_\infty = \lim_{t \longrightarrow \infty} H_t$.

2.1.1 Strategies, Payoff and Equilibrium concepts

A *pure strategy* ρ_i for a player i is an infinite sequence $\rho_i = (\rho_{i,1}, \rho_{i,2}, ...)$ where for each $t \geq 1$, $\rho_{i,t} \in S_i$ is chosen according the history H_{t-1} and is dependent on the true state $j \in K$ if $i \in IN$. A pure strategy for player i specifies a move at every point in the game Γ_∞. A *mixed strategy* is a randomization over the set of pure strategies. A *behavior strategy* σ_i for player i specifies a probability distributions over his actions at every stage of the game Γ_∞. Behavior strategies differ from mixed strategies in a sense that they are chosen mutually independent by the players at time t. Since the informed player can profit from the knowledge of the state $j \in K$ we define the behavior strategy for an informed player i as a k-tuple of state dependent behavior strategies. Thus σ_i is given by $\sigma_i = \left(\sigma_i^1, \ldots, \sigma_i^k\right)$, where for each $j \in K$, σ_i^j is an infinite sequence $\left(\sigma_{i,1}^j, \sigma_{i,2}^j, \ldots\right)$ such that for each time t ≥ 1 $\sigma_{i,t}^j : \{j\} \times H_{t-1} \to \Delta(S_i)$, where $\Delta(S_i)$ is defined as the set of probability distribution on S_i, i.e. $\Delta(S_i) = \left\{ (p^{s_i})_{s_i \in S_i} | \forall s_i,\ p^{s_i} \geq 0,\ \sum_{s_i} p^{s_i} = 1 \right\}$. The behavior strategy σ_i of the uninformed player i must be independent of the state and is thus given by an infinite sequence $(\sigma_{i,1}, \sigma_{i,2}, \ldots)$ such that for each time t ≥ 1 $\sigma_{i,t} : (\prod_{j=1}^{n} S_j)^{t-1} \to \Delta(S_i)$. Furthermore let B_i be the set of behavior strategies for player i and let $B = \prod_{1=1}^{n} B_i$. A *behavior strategy profile* $\sigma \in B$ is defined by $\sigma = (\sigma_1, \ldots, \sigma_n)$, that is every player i chooses the behavior strategy σ_i.

As mentioned above, behavior strategies are chosen mutually independent by the players. This has the effect that for infinitely repeated games the set of behavior strategies is countably infinite while the set of mixed strategies is uncountable. Since we are concerned with the existence of equilibria this poses the question whether there can exist equilibria which are reached with a mixed strategy, but can not be reached with a behavior strategy? The answer to this question is negative. Since we assume perfect recall we can confine the strategies of the players to behavior strategies. This follows from Aumann's result (1964) which states that in infinitely repeated games with perfect recall every mixed strategy can be expressed as a behavior strategy. In other words, if there exists an equilibrium in the game Γ_∞, then it can be reached with a behavior strategy. Note that the proof found in Aumann's paper is a generalization of Kuhn's theorem to infinitely repeated games.

Given a state $j \in K$ the behavior strategy profile $\sigma \in B$ induces an infinite sequence of stage game actions $(s_1, s_2, \ldots)$ when the players abide to σ. We can now define the payoff for the first T stages of player i as the average of the first T stage game payoffs, i.e. $\frac{1}{T} \sum_{t=1}^{T} g_i^j(s_t)$. Note that the word average refers to the average payoff per stage of the game, which is the reason for the devision with T. Furthermore if $j \in K$ is given, the behavior strategy $\sigma \in B$ induces a probability measure over the infinite history H_∞. Since the behavior strategy σ can include probability distribution over the strategies chosen by the players we consider the expectation of the average payoff with respect to σ. We can now define the expected average payoff of the players when σ is played.

Definition 2.1 (Expected Average Payoff) *Let $\sigma \in B$ be a behavior strategy profile which induces the infinite sequence of stage game action profiles $(s_1, s_2, \ldots)$. Then for all players i, all states $j \in K$ and all stages $T \geq 1$, the expected average payoff is given by:*

$$\gamma_{i,T}^{j}(\sigma) = E_\sigma \left(\frac{1}{T} \sum_{t=1}^{T} g_i^j(s_t) \right)$$

And the expected average payoff over the states of nature K is given by $\gamma_{i,T}(\sigma) = \sum_{j \in K} p_0^j \, \gamma_{i,T}^j(\sigma)$.

An alternative approach, and also the standard approach in economic models is to use a discount factor $\delta \in (0,1)$ to specify the payoffs for the first T periods. Here the payoff for player i and state j is $w_{i,T} = (1-\delta) \sum_{t=1}^{T} \delta^{t-1} g_i^j(s_t)$. In order to capture long-term behavior, however, this approach is not appropriate since for all ε larger than zero there exist a time $\hat{T} \geq 0$ such that the payoffs of the players, starting after time $\hat{T}$ do not change more than ε for any strategy of the players. This is due to the discount factor which describes real payoffs, hence after a certain time period $\hat{T}$ the payoffs, no matter how large are negligible. Real life examples for importance of long-term behavior could be bilateral trade negotiations or bargaining situations. In these situations we do not want to factor in the payoffs from the negotiation or bargaining process. We would much rather be willing to accept some lower payoffs from the negotiation process and then reap the benefits in the later stages. In other words we prefer the infinite payoff sequence $(-1,-1,-1,...,-1,1,1,1,1,...)$ over the payoff sequence of all zeros.

In order to define an equilibrium concept we introduce the following notation. If $\sigma \in B$ then $\sigma_{-i} \in \prod_{h \neq i} B_h$ is denoted as the behavior strategy for all players except for player i and $\sigma = (\sigma_{-i}, \sigma_i)$.

Definition 2.2 (Uniform Equilibrium) *The behavior strategy $\sigma \in B$ describes an uniform equilibrium of Γ_∞ if it satisfies:*

1. *For all players $i \in N$ and for all $\varepsilon > 0$ there exists a $T_0 = T_0(\varepsilon)$ such that σ is an $\varepsilon - Nash -$ equilibrium in finitely repeated games with at least T_0 stages, that is for all $T \geq T_0$ and for any alternative behavior strategy $\tau_i \in B_i$ of player i, $\gamma_{i,T}(\sigma_{-i}, \tau_i) \leq \gamma_{i,T}(\sigma) + \varepsilon$.*
2. *For all informed players $i \in IN$ and for all $j \in K$, $\lim_{T \to \infty} \gamma_{i,T}^j(\sigma) \to \gamma_i^j(\sigma)$ and for all uninformed players $i \in UN$, $\lim_{T \to \infty} \gamma_{i,T}(\sigma) \to \gamma_i(\sigma)$.*

An uniform equilibrium generates, as Hart (1985) describes, an ε-equilibrium in all long enough, but finite games Γ_T. For an uniform equilibrium σ of the game Γ_∞ we define the equilibrium payoff as follows: $(\gamma_1^j(\sigma)_{j \in K}, ..., \gamma_{|IN|}^j(\sigma)_{j \in K}, \gamma_{|IN|+1}(\sigma), ..., \gamma_{|IN|+|UN|}(\sigma))$.

The description of Γ_∞ is a generalization of the three player model as described in Renault (2001). Having defined the general model we will now turn to special cases in order to develop step by step the results that will later be important for further analysis. We first consider the case where there are two players and the games G^j for $j \in K$ are zero-sum games. In the next step we drop the second assumption and consider nonzero-sum stage games. Finally, we will turn to the case when $IN = \{1,2\}$ and $UN = \{3\}$.

2.2 2-Player Infinitely Repeated Zero-Sum Games with Incomplete Information on One Side

We start by giving an example. For Example 1, Γ_∞ is given by the two games G^1 and G^2 which are chosen by equal probability, i.e. $p_0 = (\frac{1}{2}, \frac{1}{2})$. The stage game G^j can be interpreted as payoff matrices for players 1 and 2, which we will denote as A_1^j and A_2^j respectively, where $A_1^j = -A_2^j$. The informed player, namely player 1, (which we will refer to as she), chooses her actions along the rows, i.e. $S_1 = \{T, B\}$ and player 2 (he), the uninformed player, chooses his actions along the columns, i.e. $S_2 = \{L, M, R\}$. The entries in the games G^1 and G^2 represent the payoffs to player 1. The payoffs of player 2 are the negation of the payoffs of player 1, i.e. for all strategies $s \in \Delta(S_1) \times \Delta(S_2)$ we have $g_1^j(s) = -g_2^j(s)$ for both states of nature, where $g_1^j(s) = s_1 A_1^j s_2$.

$$\begin{array}{cc}
\begin{array}{cc|c|c|c|}
 & & \multicolumn{3}{c}{\text{Player 2}} \\
 & & L & M & R \\ \cline{3-5}
\text{Player 1} & T & -1 & 0 & 1 \\ \cline{3-5}
 & B & 1 & 0 & 1 \\ \cline{3-5}
 & & \multicolumn{3}{c}{G^1}
\end{array}
&
\begin{array}{cc|c|c|c|}
 & & \multicolumn{3}{c}{\text{Player 2}} \\
 & & L & M & R \\ \cline{3-5}
\text{Player 1} & T & 1 & 1 & 0 \\ \cline{3-5}
 & B & -1 & 1 & 0 \\ \cline{3-5}
 & & \multicolumn{3}{c}{G^2}
\end{array}
\end{array}$$

Figure 2: Example 1

Clearly, in this example the informed player will not want to completely reveal her information, that is player 2 could deduce with probability equal to 1 the true state of nature when he knows the strategy of player 1. This kind of strategy of player 1 is called a *completely revealing strategy* and it would yield player 1 a payoff of zero in every state. It is also not optimal for player 1 to play a *non-revealing strategy* where player 2 cannot make any conclusions about the true state of nature. Note that a non-revealing strategy would also yield a payoff of zero to player 1. We are interested in how players should behave in Example 1 and generally in the game Γ_∞ with k states of nature and thus find an expression for the equilibrium payoff $\gamma_1(\sigma) = \sum_{j \in K} p_0^j \gamma_1^j(\sigma)$ which we denote here as the *value* of the game Γ_∞. We first need to develop some underlying theory in order to be able to give an expression for the value of Γ_∞.

Given Γ_∞ with k states where the one shot games' payoff can be represented, as above, with payoff matrices A_1^j and A_2^j, define the game $\Delta_1(p_0) = \sum_{j \in K} p_0^j \, G^j$ where the payoff matrices are now given by $A_1(p_0) = \sum_{j \in K} p_0^j \, A_1^j$ and similarly for $A_2(p_0)$. The interpretation of $\Delta_1(p_0)$ is that player 1 neglects her knowledge of the true state of nature and considers only the expectation over the games G^j as a reference for her actions. For Example 1, $\Delta_1(\frac{1}{2}, \frac{1}{2})$ is given in Figure 3.

It should be clear that in the game Γ_∞ player 1 can at least guarantee the Nash Equilibrium Payoff of the game $\Delta_1(p_0)$ by simply neglecting her knowledge and playing a Nash equilibrium strategy in the game $\Delta_1(p_0)$. Define the function $a_1 : \Delta(K) \to \mathbb{R}$ for player 1 by

		Player 2		
		L	M	R
Player 1	T	0	0.5	0.5
	B	0	0.5	0.5

Figure 3: $\Delta_1(\frac{1}{2},\frac{1}{2})$ for Example 1

$$a_1(p) = \max_{s_1\in\Delta(S_1)} \min_{s_2\in\Delta(S_2)} s_1 A_1(p) s_2 = \min_{s_2\in\Delta(S_2)} \max_{s_1\in\Delta(S_1)} s_1 A_1(p) s_2$$

and $a_2 : \Delta(K) \to \mathbb{R}$ for player 2 by

$$a_2(p) = \min_{s_1\in\Delta(S_1)} \max_{s_2\in\Delta(S_2)} s_1 A_2(p) s_2 = \min_{s_2\in\Delta(S_2)} \max_{s_1\in\Delta(S_1)} s_1 A_2(p) s_2$$

The meaning of the function $a_i(p)$ as Simon (2006, lecture 6) describes, is the amount to which player i's payoff can be held down by the other player when they believe that p is the true probability distribution governing the states of nature. Hence player 1 can, by neglecting her information at least guarantee a payoff of $a_1(p_0)$. But can she do better in some cases?

2.2.1 The Optimal Strategy of the Informed Player

We have seen that player 1's payoff is closely related to the function a_1 which takes as an argument a probability distribution over the states of nature, which is believed to true by both players. If player 1 were to induce, with her actions, a new probability distribution over the states of nature then she would be able in some situations to change the value of a_1 and thus improve her payoff. For example if player 1 in Example 1 always plays B then player 2 will, conditionally on player 1 playing B, deduce that the true state of nature is state 1 (note that this strategy is not optimal). This intuition suggests that player 2 can, by considering player 1's actions, update his belief about the true state of nature with Bayes formula. Player 1 can use this concept to her advantage by considering the function $a_1(p)$ for all $p = (p^1, p^2, ..., p^k) \in \Delta(K)$, which is given in Figure 4. To be consistent, p^2 refers to the probability assigned to state 2 and not to the quadratic function.

$$a_1(p^1,p^2) = \begin{cases} p^2 & p^2 \in [0,\frac{1}{3}] \\ 1-2p^2 & p^2 \in (\frac{1}{3},\frac{1}{2}] \\ 2p^2-1 & p^2 \in (\frac{1}{2},\frac{2}{3}] \\ 1-p^2 & p^2 \in (\frac{2}{3},1] \end{cases}$$

The graph of $a_1(p)$ suggests that, in order to be at the maximum value in both states, player 1 should try to make player 2 believe that the updated probability is $p^2 = \frac{1}{3}$ when state 1 is chosen and $p^2 = \frac{2}{3}$ when state 2 is chosen. Now assume the following behavior strategy $\sigma_1 = (\sigma_1^1, \sigma_1^2)$ for player 1. In state 1 player 1 plays with $\frac{1}{3}$ probability T forever and with $\frac{2}{3}$ probability B forever. In state 2 she plays with $\frac{2}{3}$ probability

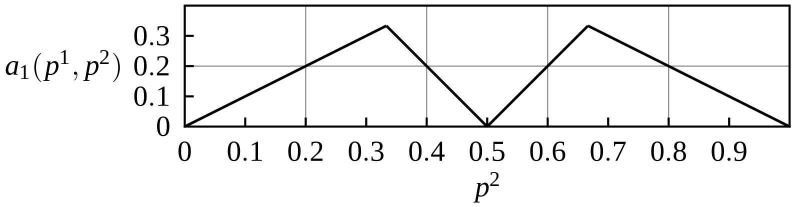

Figure 4: $a_1(p)$ for Example 1

T forever and with $\frac{1}{3}$ probability B forever. Now suppose that player 2 knows the strategy of player 1, as player 1 could communicate her strategy by using the actions as an alphabet. Then conditional on hearing the action T, player 2 will conclude with Bayes formula that the conditional probability of being in state 2 is $\frac{2}{3}$ and conditional on hearing the action B player 2 will conclude that the conditional probability of being in state 2 is $\frac{1}{3}$. In other words player 2 adjusts his belief over the states and sets $p^2 = \frac{2}{3}$ and $p^2 = \frac{1}{3}$ when hearing T and B, respectively. Player 2 thus beliefs when hearing T that he is playing the game $\Delta_1(\frac{1}{3}, \frac{2}{3})$ and when hearing B he is playing the game $\Delta_1(\frac{2}{3}, \frac{1}{3})$.

Player 2

Player 1		L	M	R
	T	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$
	B	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$

$\Delta_1(\frac{1}{3}, \frac{2}{3})$

Player 2

Player 1		L	M	R
	T	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
	B	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$

$\Delta_1(\frac{2}{3}, \frac{1}{3})$

Figure 5: $\Delta_1(\frac{1}{3}, \frac{2}{3})$ and $\Delta_1(\frac{2}{3}, \frac{1}{3})$ for Example 1

Player 2 will best respond in both games $\Delta_1(\frac{1}{3}, \frac{2}{3})$ and $\Delta_1(\frac{2}{3}, \frac{1}{3})$ with action L. Hence in both games player 1 gets an expected payoff of a $\frac{1}{3}$ and thus the overall expected payoff is also $\frac{1}{3}$. Hence we have seen in this example how player 1 can achieve a payoff strictly larger than $a_1(p_0)$ by updating the beliefs of player 2. This concept has great implications for repeated games of incomplete information. We now introduce a formal notation for updating beliefs, the proof of the theorem can be found in Simon (2006, lecture 7).

Theorem 2.3 *For $p_0 \in \Delta(K)$, let $P \subseteq \Delta(K)$ be a finite set, and $\lambda \in \Delta(P)$ s.t. $p_0 = \sum_{p \in P} \lambda_p p$. For every $j \in K$ let a $p \in P$ be chosen according to the probability q^j defined by $q^j(p) = \frac{\lambda_p p^j}{\sum_{t \in P} \lambda_t t^j}$. Then the probability distribution on K conditional on $p \in P$ being chosen is the distribution $p \in \Delta(K)$.*

Using this concept player 1, can thus induce a new probability distribution $p \in \Delta(K)$ on the states of nature by choosing some $p \in P$ according to q^j and then signaling p to player 2. The signaling process could be done by using his actions for some stages or simply to introduce a set of messages M which forms a one-to-one relationship with the states.

In order to describe how player 1 should behave in general we use the concept of concavification of functions. For a real valued function f on some convex space C the *concavification* of f, $cav(f)$, is the minimum over all concave functions $\widetilde{f}: C \to \mathbb{R}$ s.t. $\widetilde{f}(c) \geq f(c)$ for all $c \in C$. The graph of $cav(a_1)$ for Example 1 is given in Figure 6.

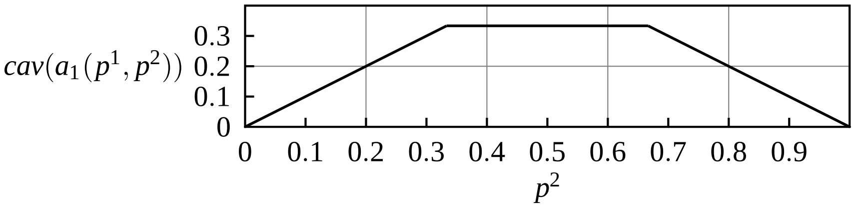

Figure 6: $cav(a_1(p))$ for Example 1

The payoff received by player 1 when using the strategy described above lies on the graph of $cav(a_1)$ and the optimal behavior strategy for player 1 is to update player 2's beliefs such that player 1 can guarantee $cav(a_1(p_0))$. This is done by choosing P s.t. $\exists\, \lambda_p \in \Delta(P)$ with $\sum\limits_{p \in P} \lambda_p = p_0$ and $cav(a_1(p_0)) = \sum\limits_{p \in P} \lambda_p a_1(p)$ where p is chosen according to the conditional probability distribution q^j at time 1. After updating the beliefs, player 1 can assure a payoff equal to $a_1(p)$ by playing a minmax strategy when $p \in \Delta(K)$ is now believed to be the true probability distribution over the states. With respect to the initial probability distribution, player 1 can assure $cav(a_1(p_0))$. This is exactly how player 1 acted in Example 1. She updated the beliefs of player 2 according to the true state of nature. If nature chose state 1 she induced a probability distribution setting $p^1 = \frac{2}{3}$ and $p^2 = \frac{1}{3}$ and similarly for state 2. This strategy guarantees her an expected payoff of $\frac{1}{3}$.

2.2.2 The Optimal Strategy of the Uninformed Player

We have already introduced the reasoning that explains how player 1 can guarantee a payoff of at least $cav(a_1(p_0))$. We will now briefly turn to player 2's strategy which guarantees him that he will pay no more than $cav(a_1(p_0))$. In other words, player 2 can get a payoff of at least $vex(a_2(p_0))$ where the *convexification* of a function is defined in analogy to the definition of its concavification: For a real valued function f on some convex space C the *convexification* of f, $vex(f)$, is the maximum over all convex function $\widetilde{f}: C \to \mathbb{R}$ s.t. $\widetilde{f}(c) \leq f(c)$ for all $c \in C$.

Since the uninformed player does not know his payoffs during the play of the game we need a different approach for player 1. As after each period of the game the played strategies are announced to all the players, player 2 can interpret his payoff at stage $t \geq 1$ as a k-dimensional vector where each entry represents the payoff to player 2 in the game G^j; that is for each behavior strategy $\sigma \in B$ and each stage $t \geq 1$ we can define r_t by $r_t = (g_2^1(s_t), ..., g_2^k(s_t))$ where s_t is the strategy induced by σ at time t. Now set v_n as the average over all vector payoffs r_i up to n, that is $v_n = \frac{1}{n} \sum\limits_{i=1}^{n} r_i$. Blackwell (1956) first studied games with vector payoffs and defined approachable payoff sets for player 2. A set S is *approachable* for player 2 if for all strategies of the other players, player 2 has a strategy that will guarantee that v_n will converge to some point in S as n goes to infinity. Furthermore, Blackwell specified the *Blackwell strategy* which describes how player 2 should behave in order to reach the approachable set. The Blackwell strategy relies on the hyperplane theorem and the fact that player 2 can base his decision at time $t+1$ on the payoff vector v_t. By using a Blackwell strategy player 2 can thus reach a set S that will guarantee him a payoff of at least $vex(a_2(p_0))$.

Hence when playing optimal, player 2 can guarantee a payoff of at least $vex(a_2(p_0))$ which is equal to $-cav(a_1(p_0))$. At the same time, player 1 can assure a payoff of at least $cav(a_1(p_0))$. The value of the game is thus $cav(a_1(p_0))$ when p_0 is the initial distribution of the states, as the following theorem states.

Theorem 2.4 (Aumann and Maschler) *A 2-player infinitely repeated zero-sum game with incomplete information on one side has a value which is given by cav*$(a_1(p_0))$ *where* p_0 *is the initial probability distribution on the k states of nature.*

In the next section we will apply these results to infinitely repeated nonzero-sum games with incomplete information. We will see that some concepts of zero-sum games can be transferred but we need to adjust others.

2.3 2-Player Infinitely Repeated Nonzero-Sum Game with Incomplete Information on One Side

We now drop the constraint that the one shot games, G^j for $j \in K$ are zero-sum games, but otherwise keep the setup as in the last section. Due to this change in payoff matrices, the payoffs of the players do not depend negatively on each other anymore. This asymmetry makes the aspect of cheating and punishment more important for assuring an equilibrium. The concept of cheating was not necessary in the zero-sum case since player 2's optimal response directly affected player 1's payoff. Player 1 could independently of player 2's actions guarantee to get $cav(a_1(p_0))$ and could not get a higher payoff since player 2 could guarantee to pay not more than $cav(a_1(p_0))$. This is not the case when considering the nonzero-sum case as Example 2 given in Figure 7 illustrates.

Player 2

Player 1		L	R
	T	$2,2$	$0,0$
	B	$2,2$	$0,0$

G^1

Player 2

Player 1		L	R
	T	$4,0$	$2,2$
	B	$4,0$	$2,2$

G^2

Figure 7: Example 2

When the initial distribution is given by $p_0 = (\frac{1}{3}, \frac{2}{3})$ Theorem 1.4 would suggest that the equilibrium payoff for player 1 would be $cav(a_1(p_0))$ which in this case is equal to $a_1(p_0) = \frac{2}{3}$, see Figure 8. First suppose that player 1 could use her actions to signal which state of nature is chosen. Player 1 would play a completely revealing strategy where she signals the true state of nature with her first move and then they play (T,L) and (T,R) in state 1 and 2, respectively. This would guarantee both players a payoff of 2. But this is not an equilibrium strategy since she can profit by sending the wrong message in state 2 and get a payoff of 4. This intuition leads to the conclusion that in equilibrium player 1 should not be able to gain by sending signals which are not according to protocol in any states of nature. This concept is called *incentive compatibility*.

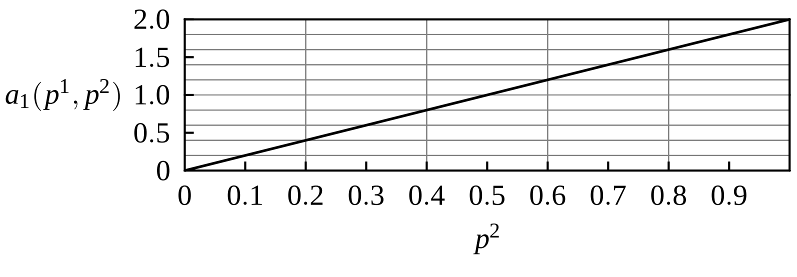

Figure 8: $a_1(p)$ for **Example 2**

Since the informed player can always play a minmax strategy in the game G^j when $j \in K$ is the true state we require that the payoffs of player 1 are *individually rational*. This means that given an equilibrium

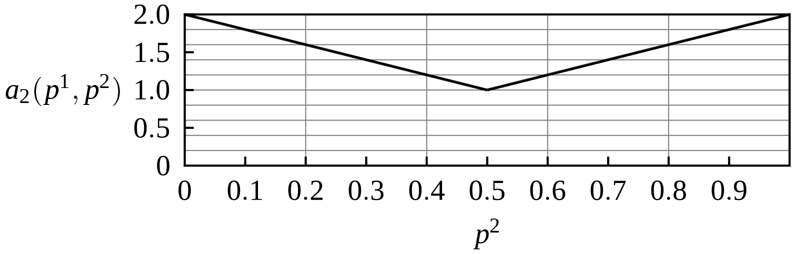

Figure 9: $a_2(p)$ for **Example 2**

strategy player 1 gets in each state of nature at least her state dependent minmax payoff. As the informed player can update the beliefs over the states we define individual rationality as follows, keeping in mind that the uninformed player can still guarantee, using a Blackwell strategy, $vex((a_2(p_0))$.

Definition 2.5 (Individual Rationality) *For player 1 and player 2:*

- $\gamma_1 = (\gamma_1^j)_{j\in K} \in \mathbb{R}^k$ *is individual rational if* $\gamma_1 \cdot p \geq a_1(p)$ *for all* $p \in \Delta(K)$.
- $\gamma_2 \in \mathbb{R}$ *is individual rational if* $\gamma_2 \geq vex(a_2(p))$, *where* $p \in \Delta(K)$ *is believed to be the true probability distribution governing the states.*

An equilibrium has to satisfy individual rationality for both players and incentive compatibility for the informed player. We now describe the slightly altered equilibrium concept.

2.3.1 Joint Plans and Joint Plan Equilibria

Joint Plans have first been described by Aumann and Maschler and we adopt the generalized form given in Renault (2001).

Definition 2.6 (Joint Plan) *For i in IN, a joint plan for player i is a tuple* (M,λ,P,z,γ_i) *where:*

1. *M is a non empty finite set of messages (or signals).*
2. $\lambda = (\lambda^j)_{j\in K}$ *is a signaling strategy such that for each state j,* $\lambda^j \in \Delta(M)$ *and* $\forall m \in M$, $\lambda_m =_{def} \sum_{j\in K} p_0^j \lambda_m^j > 0$.
3. $P = (p_m)_{m\in M}$ *such that for all* $m \in M$, $p_m \in \Delta(K)$ *is the induced probability distribution on the states of the nature given m, with* $p_m^j = p_0^j \lambda_m^j / \lambda_m$ *where* p_m^j *is the probability that is being assigned to state j conditional on hearing the message m.*
4. $z = (z_m)_{m\in M}$ *is a frequency strategy which is played after the signal is given such that for all* $m \in M$, $z_m \in \Delta(S)$.
5. $\gamma_i \in \mathbb{R}$ *is the payoff to player i such that for all states j,* $\gamma_i^j = \max_{m\in M} g_i^j(z_m)$.

The essence of a joint plan is that the informed player, who formulates the joint plan can update the beliefs of the other player by signaling a message (or messages) which is selected according to the state dependent probability distribution λ^j. These signals are then transmitted using her actions as an alphabet. Note that these signals do not directly affect the payoff function and can be regarded as cheap talk. Once the signal has been sent, the playing phase starts, where the players use a frequency strategy dependent on the sent message. If an informed player deviates from the frequency strategy she is punished to an individual rational vector in $\mathbb{R}^k$. When the uninformed player deviates the informed player punishes him to $vex(a_2(p_m))$, where p_m is the a posteriori probability distribution on states, conditional on message m being sent. Although it may seem that the informed player has the upper hand when designing the joint plan in equilibrium this is surprisingly not the case. The informed player must design the contract in such a way that the individual rationality constraint for the uninformed player is satisfied, which places restrictions on the joint plan. In the zero sum case the uninformed player could only guarantee $vex(a_2(p_0))$, where now the joint plan has to give him a payoff of $vex(a_2(p_m))$ conditional on m being sent. The reason behind this is that the informed player has to offer the uninformed player something in order to commit to the contract z_m. In equilibrium the joint plan must thus satisfy individual rationality for all players and incentive compatibility for the informed player who designs the joint plan.

Definition 2.7 (Joint Plan Equilibrium) *For player 1 a joint plan* (M,λ,P,z,γ_1) *describes an equilibrium if there exists an individual rational vector* $y_1 \in \mathbb{R}^k$ *such that for all* $m \in M$ *the following conditions are satisfied:*

1. *For all* $j \in K$ *with* $\lambda^j_m > 0$, $\gamma^j_1(z_m) = y^j_1$
2. $\sum_{j\in K} p^j_m g^j_2(z_m) \geq vex(a_2(p_m))$
3. *For all* $j \in K$ *with* $\lambda^j_m = 0$, $\gamma^j_1(z_m) \leq y^j_1$

Condition 1 and 2 are individual rationality conditions and condition 3 is the incentive compatibility condition for player i. Simon et. al (1995) proved the existence of joint plan equilibria for 2-player infinitely repeated games with incomplete information on one side with k states of nature using the concept of joint plan equilibrium. Although the existence of joint plan equilibria is known for this case, a good algorithm to find a joint plan equilibria is not known. For Example 2 there exists a simple joint plan for player 1 which is given by (M,λ,P,z,γ_1) where:

- $M = \{1,2\}$
- $\lambda = (\lambda^1,\lambda^2)$ where $\lambda^1 = (0,1)$ and $\lambda^2 = (\frac{1}{2},\frac{1}{2})$

  The signaling strategy is illustrated in Figure 10, where the leaves correspond to the messages and the numbers along the branches are probabilities for nature and the state dependent lotteries.

- $P = (p_1,p_2)$ where $p_1 = p_2 = (\frac{1}{2},\frac{1}{2})$

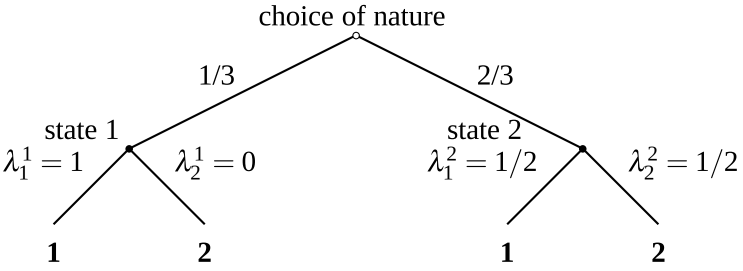

Figure 10: Signaling Strategy

- $z = (z_1, z_2)$ where $z_1 = z_2 = (T, R)$
- $\gamma_1 = (0, 2)$.

We will now show that the joint plan satisfies the condition of a joint plan equilibrium. Condition 1 is satisfied since $y = (y_1^1, y_1^2) = (0, 2)$ is an individual rational vector and since $\lambda_1^1 > 0$ we have $\gamma_1^1(z_1) = y_1^1$. As $\lambda_1^2 > 0$ and $\lambda_2^2 > 0$ we have $\gamma_1^2(z_1) = y_1^2$ and $\gamma_1^2(z_2) = y_1^2$. Condition 2 is satisfied since for message 1 we have $\sum_{j \in K} p_1^j g_2^j(z_1) = 1 = vex(a_2(p_1))$ and for message 2 we have $\sum_{j \in K} p_2^j g_2^j(z_2) = 1 = vex(a_2(p_2))$. The incentive compatibility condition is also satisfied since for $\lambda_2^1 = 0$ we have $\gamma_1^1(z_2) = 0 = y_1^1$, which completes the proof. In the next section we will add another informed player and investigate the implications in this setup.

2.4 3-Player Infinitely Repeated Nonzero-sum Games with Incomplete Information on One Side

We now consider infinitely repeated games with 3 players Γ_∞ when $IN = \{1,2\}$ and $UN = \{3\}$. An example for this situation could be that a policy maker, who has incomplete information, should make a decision based on the input of his two advisers who both have full information.

In the 2-player case the informed player, since she only plays against one uninformed player, could guarantee at least a payoff of $a_1(p_0)$. Now each informed player has to factor in that she is playing against not only one uninformed player, but also against another informed player who knows the true state. This will change the magnitude of the payoff which she can guarantee. For example, if player 1 is trying to maximize her payoff, the other informed player can reveal the true state of nature to the uninformed player and player 2 and player 3 play a state dependent minmax strategy against the player 1. Thus the informed player i can guarantee in each state $j \in K$, v_i^j where $v_i^j = \min_{x_{-i} \in \prod_{h \neq i} \Delta(S_h)} \max_{x_i \in \Delta(S_i)} g_i^j(x_i, x_{-i})$. The informed players can surely defend this payoff as they only have to play a minmax strategy in the game G^j when $j \in K$ is the true state. For the uninformed player the amount which he can guarantee coincides with the 2-player case and we define player 3's minmax payoff by the function $a_3 : \Delta(K) \to \mathbb{R}$ where $a_3(p) = \min_{(x_1,x_2) \in \Delta(S_1) \times \Delta(S_2)} \max_{x_3 \in \Delta(S_3)} \sum_{j \in K} p^j g_3^j(x_1, x_2, x_3)$. We can now give the adjusted individual rationality constraint for all players.

Definition 2.8 (Individual Rationality) *For the informed players i ∈ IN and for the uninformed player 3:*

- *the payoff* $\gamma_i = (\gamma_i^j)_{j \in K}$ *in* $\mathbb{R}^K$ *is individual rational if* $\gamma_i^j \geq v_i^j$ *for all* $j \in K$.
- *the payoff* γ_3 *in* $\mathbb{R}$ *is individual rational if* $\gamma_3 \geq vex(a_3(p))$, *where* $p \in \Delta(K)$ *is believed to be the true probability distribution on the states.*

The added difficulty in this situation is that player 3 does not always know who deviated. Since there are different types of deviation, we will first introduce them. If any player deviates from a frequency strategy, such as specified in joint plans, the deviation is observed by every other player, including the uninformed player. Thus the deviator can be identified and is therefore punished. The second type of deviation concerns deviating from a state dependent signaling strategy. Suppose that an informed player does not adhere to the signaling strategy and sends a message m, which according to the signaling strategy would be sent with probability $\lambda_m^j \in (0,1)$ in state j, with certainty. In other words, suppose that player 1 should sent message 1 and message 2 both with probability of $\frac{1}{2}$ in state 1. Player 1 could deviate by always sending message 1. This type of deviation would not be observed by any other player. Note that this type of deviation should be ruled out by the incentive compatibility constraint. The third case is that an informed player does not adhere to the signaling strategy in some state $j \in K$ and sends a message m, which would according to the signaling strategy be sent with probability equal to zero in state j. Surely, the other informed player can observe the deviation since she knows the true state and would inform the uninformed player that a deviation has occurred. The problem is now that the uninformed player does not know which player is "lying". It could be the case that although the correct message has been sent the other informed player will say that a deviation has occurred. Furthermore it could even be the case that a deviation of this type occurs and the other informed player does not announce this as it would lead to an improvement in her payoff as well. Thus formulate a punishment strategy in this case becomes more complex in comparison to the 2-player setup. This is demonstrated in Example 3 given in Figure 11.

state 1

	L	R
T	$1,1,1^*$	$1,1,1^*$
B	$1,1,1^*$	$1,1,1^*$

X

	L	R
T	$2,2,0$	$2,2,0$
B	$2,2,0$	$2,2,0$

Y

state 2

	L	R
T	$2,2,0$	$2,2,0$
B	$2,2,0$	$2,2,0$

X

	L	R
T	$1,1,2^*$	$1,1,2^*$
B	$1,1,2^*$	$1,1,2^*$

Y

Figure 11: Example

Here player 1's strategy set is $\{T,B\}$, player 2's strategy set is $\{L,R\}$, player 3's strategy set is $\{X,Y\}$ and the initial probability distribution over the states is $p_0 = (\frac{1}{2},\frac{1}{2})$. The payoffs with added stars correspond to the Nash equilibrium payoffs in pure strategies for each game G^j with $j \in K$. To offer some intuition for the necessary constraints on the punishment strategy we suppose that there exist a completely revealing equilibrium strategy $\sigma \in B$. Hence player 3 finds out with probability equal to one which the chosen state of nature is. If state 1 is chosen by nature then player 3 plays X and if the selected state is 2, player 3 chooses Y. The equilibrium payoff vector for σ is $((1,1),(1,1),1.5)$. Note that the equilibrium payoffs for players 1 and 2 correspond to their minmax payoffs in both stages. Now suppose that one of the players deviates from the completely revealing strategy. Player 3 will not know who actually deviated. To see this let w.l.o.g. player 1 convey with his action that the true state is 2 and let player 2 convey that 1 is the true state. There are 2 cases to be considered. Either player 1 deviated and the true state is 1 or player 2 deviated and the true state is 2. This is equivalent to saying player 2 is telling the truth and state 1 is actually the true state, the latter is equivalent to player 1 telling the truth and state 2 is actually the true state. The difficulty is now to punish both players simultaneously, in other words, to find a strategy $z \in \Delta(S)$ such that $g_1^1(z) \leq \gamma_1^1(z)$ and $g_2^2(z) \leq \gamma_2^2(z)$. But this constraint is not possible to satisfy. Consider the mixed strategy $z \in \Delta(S)$ where players assign probability p,q and l to action T,L and X, respectively. In state 2 player 2 has an expected payoff of $g_2^2 = 2l + (1-l)$ and in state 1 player 1 has an expected payoff of $g_1^1 = l + 2(1-l)$ and $g_1^1(z) + g_2^2(z) = 3$. As mentioned above $v_1^1 = v_1^2 = v_2^1 = v_2^2 = 1$. Hence in case of a deviation player 3 has no viable strategy to punish both informed players at once, as at least one of the informed players gets a payoff strictly larger than the equilibrium payoff, since $g_1^1(z) + g_2^2(z) = 3$ and the equilibrium payoffs for player 1 and player 2 are 1 in both stages. We see that it is possible to gain by deviating.

Example 3 suggests that we need to introduce a more restrictive concept of rationality for the informed players. Tomala (1995) first introduced the concept of *joint rationality* for the informed players which states that player 3 needs to have a strategy $z \in \Delta(S)$ which punishes both informed players at once below or equal to their equilibrium payoff in any two states $(j, j') \in K \times K$. In other words when player 1 signals we are in state j and player 2 says "No, we are in state j'", player 3 will not know who deviated and therefore must have a strategy such that he can punish both informed players at once in states j and j'. In addition to this constraint the jointly rational payoffs must still be individually rational for informed players.

Definition 2.9 (Joint Rationality for the Informed Player) *For any couple of states $(j, j') \in K \times K$ let $JR_{1,2}(j, j') = \left\{(\gamma_1, \gamma_2) \in \mathbb{R}^k \times \mathbb{R}^k : \exists\, z \in \Delta(S)\ s.t.\ g_1^j(z) \leq \gamma_1^j \text{and } g_2^{j'}(z) \leq \gamma_2^{j'}\right\}$. For all $j \in K$ set:*

$$IR_1(j) = \left\{(\gamma_1, \gamma_2) \in \mathbb{R}^k \times \mathbb{R}^k : \gamma_1^j \geq v_1^j\right\}$$

$$IR_2(j) = \left\{(\gamma_1, \gamma_2) \in \mathbb{R}^k \times \mathbb{R}^k : \gamma_2^j \geq v_2^j\right\}$$

A payoff vector $(\gamma_1, \gamma_2) \in \mathbb{R}^k \times \mathbb{R}^k$ is jointly rational for the informed players if

$$(\gamma_1, \gamma_2) \in IR_1 \cap IR_2 \cap JR_{1,2},$$

where $IR_1 = \bigcap_{j \in K} IR_1(j)$, $IR_2 = \bigcap_{j \in K} IR_2(j)$ and $JR_{1,2} = \bigcap_{(j,j') \in K \times K} JR_{1,2}(j, j')$.

Renault (2001) characterizes completely revealing equilibrium with the notion of joint rationality for the informed players.

Proposition 2.10 *$\gamma = ((\gamma_1^j)_{j \in K}, (\gamma_2^j)_{j \in K}, \gamma_3)) \in \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}$ is a completely revealing equilibrium payoff if and only if the following conditions are satisfied:*

1. *$((\gamma_1^j)_{j \in K}, (\gamma_2^j)_{j \in K})$ is jointly rational for players 1 and 2*
2. *$\exists (z^j)_{j \in K} \in (\Delta(S))^k$ such that for all $j \in K$, $\gamma_i^j = g_i^j(z^j)$ for i = 1,2 and*
   $\gamma_3 = \sum_{j \in K} p_0^j\, g_3^j(z^j)$ and for all $j \in K$ and $g_3^j(z^j) \geq v_3^k$.

Surely the joint rationality conditions must be satisfied and player 3 can now guarantee his state dependent minmax v_3^j for all states $j \in K$ as he knows the choice of nature. Before we give an equilibrium for Example 3 we must adjust the definition of a joint plan equilibrium to fit the new individual rationality constraint of the informed players.

Definition 2.11 (Joint Plan Equilibrium) *For player $i \in IN$ the joint plan for player i $(M, \lambda, P, z, \gamma_i)$ describes an joint plan equilibrium for player i if:*

1. $\forall j \in K, \forall m \in M$ *with* $\lambda_m^j > 0, g_1^j(z_m) \geq v_1^j$ *and* $g_2^j(z_m) \geq v_2^j$
2. $\forall \in M, \sum_{j \in K} p_m^j g_3^j \geq vex(a_3(p_m))$
3. $\forall j \in K, \forall m \in M$ *with* $\lambda_m^j > 0$, $g_i^j(z_m) = \gamma_i^j$

As before conditions 1 and 2 are individual rationality for the informed player and the third condition is the incentive compatibility constraint for player i. Renault (2001) showed that the set of joint plan equilibrium payoffs is a subset of the uniform equilibrium payoffs. Furthermore if player i designs the joint plan and we set for any state $j \in K$, $\gamma_{3-i}^j = min\left\{g_{3-i}^j(z_m) : m \in M, p_m^j > 0\right\}$ then the joint plan equilibrium payoff is jointly rational for the informed players.

For Example 3 we showed on page 17 that no completely revealing equilibrium can exist. This is due to the fact the payoff vector from the completely revealing strategy, $((1,1),(1,1),1.5)$ is not a member of $JR_{1,2}(1,2)$ and thus by Proposition 2.10 this example has no completely revealing equilibrium. Although no completely revealing equilibrium exists the game has a non-revealing joint plan equilibrium for player 1 (or player 2), which is given by:

- $M = \{1\}$
- $\lambda = (\lambda^1, \lambda^2)$ where $\lambda_1^1 = \lambda_1^2 = 1$
- $P = \{p_1\}$ is given by $p_1 = (\frac{1}{2}, \frac{1}{2})$
- $z = (z_1)$ with $z_1 = z_2 = \frac{1}{4}(T,L,Y) + \frac{1}{4}(T,R,Y) + \frac{1}{4}(B,L,Y) + \frac{1}{4}(B,R,Y)$
- $\gamma_1 = (\gamma_1^1, \gamma_1^2) = (2,1)$

The equilibrium payoff is given by ((2,1),(2,1),1). The conditions for a joint plan equilibrium as we will show are satisfied. Condition 1 is satisfied since the informed players get at least their minmax payoff in both states of nature. The individual rationality constraint for player 3 is satisfied since $vex(a_3(p_1)) = a_3(p_1) = max\left\{p_1^1, 2p_1^2\right\} = 1$ for $m = 1$. The incentive compatibility for player 1 is also satisfied since in every state j, we only send one message since the joint plan is non-revealing. The message 1 can be interpreted as 'I am telling you nothing'. Thus the given joint plan describes an equilibrium. We now investigate equilibrium existence for 3-player games when $k = 2$ and when $k \geq 3$.

2.4.1 Equilibrium Existence with $k = 2$

We have seen that for Example 3 no completely revealing equilibria exist. But we have shown that in Example 3 there exists a joint plan equilibrium. This relationship between completely revealing and joint plan equilibrium holds in general as the following theorem from Renault (2001) states.

Theorem 2.12 *For a 3-player infinitely repeated game with incomplete information on one side with two states of nature, for any initial distribution $p_0 \in \Delta(K)$ there exists a completely revealing equilibrium or a joint plan equilibrium.*

Renault (2001) showed that we can concretize this theorem as follows. There exist games where no joint plan equilibria for either player exist. Furthermore there exist games where no completely revealing nor a joint plan equilibrium for one of the two players exist.

2.4.2 Equilibrium Existence with $k \geq 3$

In the case of larger state spaces, there exist games, where there no completely revealing equilibrium and no joint plan equilibrium exists for any player. We will give the proof found in Renault (2001) since it demonstrates the interdependencies of individual rationality and incentive compatibility for the players.

state 1

	L	R
T	$0,2,0$	$0,2,0$
B	$-1,1,0$	$-1,1,0$

X

	L	R
T	$0,0,1^*$	$0,0,1^*$
B	$-1,0,1$	$-1,0,1$

Y

state 2

	L	R
T	$-1,0,3$	$-1,\ \ 0,3$
B	$0,0,3^*$	$0,-1,3$

X

	L	R
T	$\frac{1}{2},\frac{1}{2},0$	$\frac{1}{2},\frac{1}{2},0$
B	$1,0,0$	$1,1,0$

Y

state 3

	L	R
T	$1,-1,0$	$\frac{1}{2},0,0$
B	$1,-1,0$	$\frac{3}{2},0,0$

X

	L	R
T	$0,-1,1$	$0,0,1^*$
B	$1,-1,1$	$0,0,1^*$

Y

Figure 12: Example 4

For the game given in Example 4 player 1's strategy set is $\{T,B\}$, player 2's strategy set is $\{L,R\}$, player 3's strategy set is $\{X,Y\}$ and the initial probability distribution over the states is given by $p_0 = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$. Players 1 and 2 are as usual the informed players. Again the payoffs labeled with a star correspond to Nash equilibrium payoffs in G^j for all $j \in K$. The minmax payoff for both informed players is zero in all states of nature. The minmax payoff for player 3 is given by $vex(a_3(p)) = a_3(p) = max\left\{3p^2, p^1 + p^3\right\}$.

Proposition 2.13 *The game in Example 4 does not have a completely revealing equilibrium.*

Proof: We prove this by contradiction. Suppose there exists a completely revealing equilibrium strategy $\sigma \in B$. When players abide to σ the informed players receive an equilibrium payoff of zero in all stages $j \in K$. These payoffs correspond to the Nash equilibrium payoffs in state j. To find a contradiction we only need to check whether the equilibrium payoffs are jointly rational. But here no (mixed) strategy that jointly punishes both players equal or below their equilibrium payoff exists. To see this, suppose there exists such a strategy $z \in \Delta(S)$ where the players assign probability p,q and l to the action T,L and X, respectively.

Then $g_1^2(z) = \frac{1}{2}p - \frac{1}{2}pl - p - l + 1$ and $g_2^1(z) = l(p+1)$ and thus $g_1^2(z) + g_2^1(z) \geq \frac{1}{2}$. Likewise we have $g_1^3(z) + g_2^2(z) \geq \frac{1}{2}$. Hence the payoffs generated by σ are neither in $JR_{1,2}(2,1)$ nor $JR_{1,2}(3,1)$ and hence are not jointly rational. Due to Proposition 2.10 we have derived a contradiction, which completes the proof.

Proposition 2.14 *For the game in Example 4 no joint plan equilibrium exist for players 1 and 2.*

Proof: Again we prove this by contradiction. Assume that a joint plan equilibrium $(M, \lambda, P, z, \gamma_i)$ exist for player i. Since we have 3 states of nature we consider 3 messages for the joint plan. Let M be given by $M = \{1,2,3\}$. By the definition of the a posteriori probability distribution we have,

$$\forall j, m \in \{1,2,3\}, \lambda_m^j > 0 \Leftrightarrow p_m^j > 0$$

Due to this relationship it is sufficient to consider the a posteriori probability distribution $P = (p_1, p_2, p_3)$ for determining whether the individual rationality constraints of the joint plan equilibrium are satisfied. Let us now consider the implications of the individual rationality constraint of the informed players.

- $p_m^1 > 0$, for $m \in M$ implies that player 1 has to choose T in frequency strategy z_m.
- $p_m^3 > 0$, for $m \in M$ implies that player 1 has to choose R in frequency strategy z_m.

Since the a posteriori probability distribution P is a convex combination of the initial distribution, p_1,p_2 and p_3 must be such that $p_1^1 \geq \frac{1}{3}$, $p_2^2 \geq \frac{1}{3}$ and $p_3^3 \geq \frac{1}{3}$. The implications of the individual rationality constraint for uninformed player give:

- $3p_2^2 > p_2^1 + p_2^3$ implies that player 2 will play X after hearing message 2.

Given player 3's action, players 1 and 2 will play B and L, respectively, in order to assure their minmax payoff of zero. Given the players' behavior frequency strategy, z_2 is given by $z_2 = (B, L, X)$. With z_2 we can specify the payoff matrix for players 1 and 2 when z_2 is being played,

$$\begin{bmatrix} \gamma_1^1(z_2) & \gamma_2^1(z_2) \\ \gamma_1^2(z_2) & \gamma_2^2(z_2) \\ \gamma_1^3(z_2) & \gamma_2^3(z_2) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

As we can see from the payoff matrix player 1 would never want to play z_2 in state 1 and similarly player 2 would never want to play z_2 in state 3. Individual rationality of the informed player implies that whoever designs the joint plan it will be such that $p_2 = (0,1,0)$. To see this suppose w.l.o.g. that player 1 designs the joint plan, then p_1^1 must be zero because of individual rationality of player 1 and p_1^3 must be zero due to player 2's constraint. Hence after hearing message 2, player 3 knows with certainty that the true state is never 1 or 3.

Result 1: $z_2 = (B, L, X)$ and $p_2 = (0,1,0)$

Furthermore we know for $m \in \{1,3\}$, if $p_m^1 > 0$ we must have $\gamma_2^1(z_m) \geq \frac{2}{3}$ as player 1 will play T and thus player 2 can assure himself at least $\frac{2}{3}$ in state 1 by playing R since player 3 must chose X with a probability greater than at least $\frac{1}{3}$. Similarly if $p_m^3 > 0$ we must have $\gamma_1^3(z_m) \geq \frac{1}{2}$, since player 2 will play R and thus player 1 can assure himself $\frac{3}{2} \times \frac{1}{3} = \frac{1}{2}$ in state 3 by playing B. From above we know that if $p_1^1 > 0$ then player 1 will play T in z_1 and thus we need to satisfy $\gamma_2^1(z_1) \geq \frac{2}{3}$.

The next step is to determine the signaling strategy for message 1, $\lambda_1 = (\lambda_1^1, \lambda_1^2, \lambda_1^3)$. We have four possible types of signaling strategies, as we can rule out signaling $\lambda_1 = (0,0,1)$ and $\lambda_1 = (0,1,0)$ directly, since the a posteriori probability distribution has to be a convex combination of p_0.

1. $\lambda_1 = (>0,0,0)$ can be ruled out since player 3 would play Y which violates the constraint $\gamma_2^1(z_1) \geq \frac{2}{3}$.
2. $\lambda_1 = (>0,0,>0)$ can be ruled out for the same reasons as given in case 1.
3. $\lambda_1 = (>0,>0,>0)$ implies that player 2 believes with some positive probability that conditional on hearing message 1 the true state of nature is three. Since R strictly dominates the action L, player 2 chooses R and player 3 must choose X in order to satisfy $\gamma_1^3(z_1) \geq \frac{1}{2}$. Hence z_1 would be given by $z_1 = (T,R,X)$ which gives player 1 a payoff of $\gamma_1^2(z_1) = -1$ which is less than her minmax of zero in stage 2 and thus not individually rational.
4. Finally, let $\lambda_1 = (>0,>0,0)$ and consider z_1 to be strategy when player 1 plays T, player mixes L and R, and player 3 mixes X and Y. We can write z_1 as follows,

   $$z_1 = \alpha_1(T,L,X) + \alpha_2(T,R,X) + \alpha_3(T,R,Y) + \alpha_4(T,L,Y),$$

   such that $\sum_{r\in\{1,\ldots,4\}} \alpha_r = 1$, for $r \in \{1,\ldots,4\}$, $\alpha_r \geq 0$. In order to assure that the individual rationality constraint for player 1 is satisfied in state 1 we must have $\gamma_1^2(z_1) \geq 0$, i.e. $-(\alpha_1+\alpha_2)+\frac{1}{2}(\alpha_3+\alpha_4) \geq 0$. Hence z_1 must be such that $\alpha_1+\alpha_2 \leq \frac{1}{3}$. Due to this constraint on z_1, player 2 cannot design the joint plan since z_1 would give him a payoff which is strictly larger than zero in state 2. As $\gamma_2^2(z_2) = 0$, the incentive compatibility condition of player 2 would be violated as she would want to send signal 1 in stage 2.

Result 2: Only player 1 can design the joint plan.

We must now specify α_r for $r \in \{1,\ldots,4\}$ such that $\gamma_2^1(z_1) \geq \frac{2}{3}$ which gives us $\alpha_1+\alpha_2 = \frac{1}{3}$ and the previous constraint holds with equality. Using the individual rationality constraint of player 3 we can give the a posteriori probability distribution over the states conditional that message 1 was sent. We must have that $\sum_{j\in\{1,2,3\}} p_1^j g_3^j(z_1) \geq max\{3p_1^2, p_1^1+p_1^3\}$. Since $\lambda_1^3 = 0$ we have by definition that $p_1^3 = 0$ and we can rewrite the individual rationality constraint for player 3 as $\frac{2}{3}p_1^1 + p_1^2 \geq max\{3p_1^2, p_1^1\}$. Using $p_1^1 + p_1^2 = 1$ we find the solution at $p_1 = (p_1^1, p_1^2, p_1^3) = (\frac{3}{4}, \frac{1}{4}, 0)$. With z_1 as specified we can now give the payoff matrix for players 1 and 2 when z_1 is being played

$$\begin{bmatrix} \gamma_1^1(z_1) & \gamma_2^1(z_1) \\ \gamma_1^2(z_1) & \gamma_2^2(z_1) \\ \gamma_1^3(z_1) & \gamma_2^3(z_1) \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} \\ 0 & \frac{1}{3} \\ \leq \frac{1}{3} & \leq 0 \end{bmatrix}.$$

Result 3: $z_1 = \alpha_1(T,L,X) + \alpha_2(T,R,X) + \alpha_3(T,R,Y) + \alpha_4(T,L,Y)$, where $\alpha_1+\alpha_2 = \frac{1}{3}$ and $p_1 = (\frac{3}{4}, \frac{1}{4}, 0)$

We have already specified z_1 and z_2 and now turn to the signaling strategy λ_3 and the corresponding frequency strategy z_3. This will lead to the anticipated contradiction. By similar reasoning as above we find that λ_3 can only be of the form $\lambda_3 = (0, > 0, > 0)$. We can write z_3 as follows, $z_3 = \beta_1(T,R,X) + \beta_2(B,R,X) + \beta_3(T,R,Y) + \beta_4(B,R,Y)$, such that $\sum_{r\in\{1,\ldots,4\}} \beta_r = 1$, for $r \in \{1,\ldots,4\}$, $\beta_r \geq 0$. Due to the incentive rationality condition for player 1 we need that $\gamma_1^2(z_3) = 0$, hence $-\beta_1 + \frac{1}{3}\beta_3 + \beta_4 = 0$ and we can rewrite β_1 and β_2 as $\beta_1 = \frac{1}{3}\beta_3 + \beta_4$ and $\beta_2 = 1 - \frac{4}{3}\beta_3 - 2\beta_4$. With z_3 as specified we can now give the payoff matrix for players 1 and 2 when z_3 is being played,

$$\begin{bmatrix} \gamma_1^1(z_3) & \gamma_2^1(z_3) \\ \gamma_1^2(z_3) & \gamma_2^2(z_3) \\ \gamma_1^3(z_3) & \gamma_2^3(z_3) \end{bmatrix} = \begin{bmatrix} -1 + \frac{3}{2}\beta_3 + \beta_4 & 1 - \frac{1}{2}\beta_3 \\ 0 & 2\beta_3 + 3\beta_4 - 1 \\ \frac{3}{2} - 2\beta_3 - \frac{5}{2}\beta_4 & 0 \end{bmatrix}.$$

In order to assure that the individual rationality constraint for player 2 in state 2 is satisfied we need $\gamma_2^2(z_3) \geq 0$ which implies that $1 - 2\beta_3 - 3\beta_4 \leq 0$. Using this constraint we can rewrite the payoff of player 1 in state 3 as $\frac{3}{2} - 2\beta_3 - \frac{5}{2}\beta_4 = (1 - 2\beta_3 - 3\beta_4) + (\frac{1}{2} + \frac{1}{2}\beta_4) < \frac{1}{2} + \frac{1}{2}\beta_4$. If $\beta_4 = 1$ we have $\frac{3}{2} - 2\beta_3 - \frac{5}{2}\beta_4 = -1$ and so for all $\beta_4 \in [0,1]$ we have $\frac{3}{2} - 2\beta_3 - \frac{5}{2}\beta_4 < 1$. This violates the incentive compatibility condition of player 1 as $g_1^3(z_3) < g_1^3(z_2) = 1$ and thus player 1 would prefer to send signal 2 in state 3.

Result 4: There exists no joint plan for player 1 which satisfies the conditions of a joint plan equilibrium. This completes the proof.

We have shown that no completely revealing equilibrium and no joint plan equilibrium for neither informed player in Example 4 can exist. In the next section we turn to a new equilibrium concept which relies on successive information revelation by the informed players.

2.5 Successive Information Revelation by the Informed Players

We first discuss the approach suggested by Renault (2001) and then explain our own. Renault (2001) suggests that in the game in Example 4 player 1 should send a state dependent message $\lambda = (\lambda^1, \lambda^2, \lambda^3)$ such that the a posteriori probability distribution and the frequency strategy are as follows.

- $M = \{1,2,3\}$
- $\lambda^1 = (\lambda^1_1, \lambda^1_2, \lambda^1_3) = (1,0,0)$, $\lambda^2 = (\lambda^2_1, \lambda^2_2, \lambda^2_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\lambda^3 = (\lambda^3_1, \lambda^3_2, \lambda^3_3) = (0,0,1)$
- $p_1 = (p^1_1, p^2_1, p^3_1) = (\frac{3}{4}, \frac{1}{4}, 0)$, $p_2 = (p^1_2, p^2_2, p^3_2) = (0,1,0)$ and $p_3 = (p^1_3, p^2_3, p^3_3) = (0, \frac{1}{4}, \frac{3}{4})$
- $z_1 = \frac{1}{3}(T,L,X) + \frac{1}{3}(T,L,Y)$, $z_2 = (B,L,X)$, $z_3 = \frac{2}{5}(T,R,X) + \frac{1}{5}(B,R,X) + \frac{2}{5}(B,R,Y)$

With the frequency strategies z_1, z_2 and z_3 we obtain the following payoffs for player 1 and player 2, respectively.

$$\begin{bmatrix} \gamma^1_1(z_1) & \gamma^1_2(z_1) \\ \gamma^2_1(z_1) & \gamma^2_2(z_1) \\ \gamma^3_1(z_1) & \gamma^3_2(z_1) \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} \\ 0 & \frac{1}{3} \\ \frac{1}{3} & -1 \end{bmatrix}$$

$$\begin{bmatrix} \gamma^1_1(z_2) & \gamma^1_2(z_2) \\ \gamma^2_1(z_2) & \gamma^2_2(z_2) \\ \gamma^3_1(z_2) & \gamma^3_2(z_2) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \gamma^1_1(z_3) & \gamma^1_2(z_3) \\ \gamma^2_1(z_3) & \gamma^2_2(z_3) \\ \gamma^3_1(z_3) & \gamma^3_2(z_3) \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & 1 \\ 0 & \frac{1}{5} \\ \frac{1}{2} & 0 \end{bmatrix}$$

Note that this joint plan for player 1 is not incentive compatible since $1 = \gamma^3_1(z_2) \neq \gamma^3_1(z_3) = \frac{1}{2}$. In state 3 player 1 would want to send message 2. Renault (2001) now suggests that player 2 should have an extra communication phase after player 1 sends message 2, in which player 2 can say "OK" or "NO, the true state is c". If player 1 sends sends message 2 in state 3 player 2 should say "NO" and player 2 and 3 should punish player 1. If the message is "OK" then the game goes on as before. But if the message is "NO" they then play (T,R,X) forever. Renault (2001) proves that this strategy describes an uniform equilibrium for the game in Example 4.

Although this strategy leads to an equilibrium it seems to be some what "ad hoc". In general, why should player 3 believe in the a posteriori probability distribution if the joint plan is not incentive compatible? Say that since the plan is not incentive compatible he is not going to believe in the joint plan and always play X, which would be with respect to the initial probability distribution p_0 individual rational. Concerning player 1, why would she want give control to player 2. Suppose player 1 sends message 2 when the true state is 2, in other words player 1 is not lying. If player 2 is "nasty" she would always say "NO, the true state is c" given the true state is 2. They would then play (T,R,X) forever which would give player 2 a payoff of zero and player 1 a payoff of -1. Hence player 1 should be disinclined to give up control and be at the mercy of player 2 especially when player 2 can play an individual rational action which would give player 1 a payoff worse than her minmax payoff. Since we assumed that player 2 is not “nasty” this does not pose a problem in this particular case. We now describe a different equilibrium concept which is not only individually rational but also incentive compatible.

2.5.1 Successive Joint Plans and Successive Joint Plan Equilibria

In this section we consider plans where one of the informed players, say player *i*, sends some state dependent signal or message which is then followed by an additional information revelation by player $3-i$. Conditional on the message the players then play a message dependent frequency strategy, as usual. We call such a plan a *successive joint plan*. If a successive joint plan is such that the signaling strategies of both players are incentive compatible and the payoffs of all players are individually rational then the successive joint plan describes an successive joint plan equilibrium. We will now define successive joint plans and the corresponding successive joint plan equilibrium.

Definition 2.15 (Successive Joint Plan) *For i and* $3-i$ *in IN, a successive joint plan is a tuple* $(M_i, \lambda, P(\lambda), b, M_{3-i}, \mu, P(\lambda, \mu$
where:

1. M_i *is a non empty finite set of messages (or signals) for player i, who will be the first to reveal information.*

2. $\lambda = (\lambda^j)_{j \in K}$ *is a signaling strategy, such that for each state j,* $\lambda^j \in \Delta(M_i)$ *and* $\forall m \in M$, $\lambda_m =_{def} \sum_{j \in K} p_0^j \lambda_m^j > 0$.

3. $P(\lambda) = (p_m(\lambda))_{m \in M_i}$ *such that for all* $m \in M$, $p_m(\lambda) \in \Delta(K)$ *is the induced probability distribution on the states of the nature given m, with* $p_m^j(\lambda) = p_0^j \lambda_m^j / \lambda_m$ *where* $p_m^j(\lambda)$ *is the probability that is assigned to state j conditional on hearing message m from player i.*

4. $b = (b_1, ..., b_{|M_i|}) \in 2^{|M_i|}$ *is defined such that if* $b_m = 1$ *then player* $3-i$ *will reveal some additional information if she hears the message* $m \in M_i$. *If* $b_m = 0$ *then conditional on hearing message* $m \in M_i$, *player* $3-i$ *will reveal no further information.*

5. M_{3-i} *is a finite set of messages (or signals) for the player* $3-i$ *who will be able to send some message* $\widehat{m} \in M_{3-i}$ *if and only if a message* $m \in M_i$ *was sent by player i that satisfies* $b_m = 1$.

6. $\mu = (\mu^j)_{j \in K}$ *is a signaling strategy such that for each state state j, where player i can send a message* $m \in M_i$ *with* $b_m = 1$ *with positive probability,* $\mu^j \in \Delta(M_{3-i})$ *and* $\forall \widehat{m} \in M_{3-i}, \mu_{\widehat{m}} =_{def} \sum_{j \in K} p^j(\lambda) \mu_{\widehat{m}}^j >$ 0.

7. $P(\lambda) = (p_m(\lambda)_{m \in M_i,\, b_m = 0}, p_{m\widehat{m}}(\lambda, \mu)_{m \in M_i,\, b_m = 1,\, \widehat{m} \in M_{3-i}})$ *such that for all* $m \in M_i$ *with* $b_m = 0$, $p_m(\lambda) \in \Delta(K)$ *is the induced probability distribution on the states of the nature given m with* $p_m^j(\lambda) = p_0^j \lambda_m^j / \lambda_m$ *where* $p_m^j(\lambda)$ *is the probability that is being assigned to state j conditional on hearing*

the message m from player i. For all $m \in M_i$ with $b_m = 1$, $p_{m\widehat{m}}(\lambda,\mu) \in \Delta(K)$ is the induced probability distribution on the states of the nature given m and $\widehat{m}$ with $p^j_{m\widehat{m}}(\lambda,\mu) = p^j_m(\lambda)\mu^j_{\widehat{m}}/\mu_{\widehat{m}}$ where $p^j_{m\widehat{m}}(\lambda,\mu)$ is the probability that is being assigned to state j, conditional on hearing the message m from player i and $\widehat{m}$ from player $3-i$.

8. *$z = ((z_m)_{m\in M_i}, (z_{m\widehat{m}})_{m\in M_i,\widehat{m}\in M_{3-i}})$ is a frequency strategy which is played after the signal is given such that for all $m \in M_i$ with $b_m = 0$, $z_m \in \Delta(S)$ and for all $m \in M_i$ with $b_m = 1$, for all $\widehat{m} \in M_{3-i}$, $z_{m\widehat{m}} \in \Delta(S)$.*
9. *$\gamma_i \in \mathbb{R}^k$ is the payoff to player i where $\forall j \in K$, $\gamma^j_i = max \left\{ \max_{m\in M_i} g^j_i(z_m), \sum_{\widehat{m}\in M_{3-i}} \mu^j_{\widehat{m}} g^j_i(z_{m\widehat{m}}) \right\}$*
10. *$\gamma_{3-i} \in \mathbb{R}^k$ is the payoff to player $3-i$ where for all stages $j \in K$ conditional on hearing any message $m \in M$, with $b_m = 1$ the payoff in state j is $\gamma_{3-i}|m = \max_{\widehat{m}\in M_{3-i}} g^j_i(z_{m\widehat{m}})$.*

As usual, if an informed player deviates from a frequency strategy in state j the other informed player will reveal the true state of nature and she will then together with player 3 punish the deviator to her state dependent minmax. Furthermore if any informed player's signal is not according to protocol, she is punished to her state dependent minmax. If the uninformed player deviates from any frequency strategy z_m or $z_{m\widehat{m}}$ the informed players punish the uninformed player such that he will receive a payoff of $vex(a_3(p_m(\lambda))$ or $vex(a_3(p_{m\widehat{m}}(\lambda,\mu))$, respectively. We will now define a successive joint plan equilibrium.

Definition 2.16 (Successive Joint Plan Equilibrium) *The successive joint plan $(M_i,\lambda,P(\lambda),b,M_{3-i},\mu,P(\lambda,\mu),z,\gamma_i,\gamma_{3-i})$ describes a successive joint plan equilibrium if:*

1. *$\forall j \in K, \forall m \in M_i$ with $b_m = 0$ and $\lambda^j_m > 0$, $g^j_i(z_m) \geq v^j_i$ and $g^j_{3-i}(z_m) \geq v^j_{3-i}$*
2. *$\forall j \in K, \forall m \in M_i$ with $b_m = 1$ and $\lambda^j_m > 0$, $\forall\ \widehat{m} \in M_{3-i}$ and $\mu^j_{\widehat{m}} > 0$,*
   $g^j_i(z_{m\widehat{m}}) \geq v^j_i$ and $g^j_{3-i}(z_{\widehat{m}}) \geq v^j_{3-i}$.
3. *$\forall m \in M_i$ with $b_m = 0$, $\sum_{j\in K} p^j_m(\lambda) g^j_3(z_m) \geq vex(a_3(p_m(\lambda)))$*
4. *$\forall \widehat{m} \in M_{3-i}$, $\sum_{j\in K} p^j_{m\widehat{m}}(\lambda,\mu)\ g^j_3(z_{m\widehat{m}}) \geq vex(a_3(p_{m\widehat{m}}(\lambda,\mu)))$*
5. *$\forall j \in K, \forall m \in M$ with $b_m = 0$ and $\lambda^j_m > 0$, $g^j_i(z_m) = \gamma^j_i$*
6. *$\forall j \in K, \forall m \in M$ with $b_m = 1$ and $\lambda^j_m > 0$ $g^j_i(z_{m\widehat{m}}) = \gamma^j_i$*
7. *$\forall j \in K, \forall m \in M$ with $b_m = 1$ and $\lambda^j_m > 0$, $\forall \widehat{m} \in M_{3-i}$ with $\mu^j_{\widehat{m}} > 0$, $g^j_{3-i}(z_{m\widehat{m}}) = \gamma^j_{3-i}|m$*

Condition 1 is the individual rationality condition for players i and $3-i$ in case player i sends a message after which player $3-i$ sends no further message. Condition 2 is the individual rationality condition for players i and $3-i$ in case player i sends a message after which player $3-i$ sends some further message. Conditions 3 and 4 are the individual rationality conditions for player 3. Conditions 5 and 6 are the incentive compatibility conditions for player i, conditions 5 in case she sends a message $m \in M_i$ with $b_m = 0$ and condition 6 when $b_m = 1$. Condition 7 is the incentive compatibility condition for player $3-i$.

The motivation for using successive joint plans is that we can split the revelation process in two steps. This allows us to avoid certain incentive compatibility problems as we have seen in Example 4, where it was impossible to define a joint plan for player 1 that is incentive compatible. Using two phases of revelation we simply let player 2 reveal the information that was not incentive compatible for player 1.

2.5.2 Properties of Successive Joint Plans

We will now analyze the properties of successive joint plans and successive joint plan equilibria. Similar to joint plan equilibria, every successive joint plan equilibrium describes a uniform equilibrium. The proof for successive joint plan equilibria is a generalization of the proof for standard joint plan equilibria, which can be found in Renault (2001, p. 235). Since successive joint plans are an extension of joint plans, Lemma 2.17 follows directly.

Lemma 2.17 *Let the joint plan* $(M, \lambda, P, z, \gamma_i)$ *for player* $i \in IN$ *satisfy the conditions of a joint plan equilibrium, then there exists a successive joint plan* $(M_i, \lambda, P(\lambda), b, M_{3-i}, \mu, P(\lambda, \mu), z, \gamma_i, \gamma_{3-i})$ *which satisfies the conditions of a successive joint plan equilibrium.*

Proof: Let the successive joint plan be given by $(M, \lambda, P, (0, ..., 0), M_{3-i}, \mu, P, z, \gamma_i, \gamma_{3-i})$. Since $b = (0, ..., 0)$ player $3-i$ will never send additional information, the choice of μ and M_{3-i} is thus irrelevant. Furthermore the payoff of player $3-i$ in state $j \in K$ is given by $\gamma^j_{3-i} = \sum_{m \in M} p^j_m(\lambda) g^j_{3-i}(z_m)$. We now need to show that the successive joint plan satisfies the conditions of a successive joint plan equilibrium. Since $b = (0, ..., 0)$ conditions 2, 4, 6 and 7 do not have to checked since for all $m \in M$, $b_m = 0$. Since the conditions 1,2 and 3 of the joint plan equilibrium are satisfied this implies that conditions 1,3 and 5 of a successive joint plan equilibrium are satisfied. This completes the proof.

Note that the following statement is not necessarily satisfied. If there exists a successive joint plan $(M_i, \lambda, P(\lambda), b, M_{3-i}, \mu, P(\lambda, \mu), z, \gamma_i, \gamma_{3-i})$ which satisfies the conditions of a successive joint plan equilibrium then there exists a joint plan $(M, \lambda, P, z, \gamma_i,)$ for player $i \in IN$ or $3-i$ that satisfies the conditions of a joint plan equilibrium.

Proposition 2.18 *For the game* Γ *given in Example 4 there does not exist a joint plan equilibrium for player 1 and no joint plan equilibrium for player 2, but there exists a successive joint plan* $(M_i, \lambda, P(\lambda), b, M_{3-i}, \mu, P(\lambda, \mu), z, \gamma_i, \gamma_{3-i})$ *which satisfies the conditions of a successive joint plan equilibrium.*

Proof: The first part was profen in Renault (2001) and was also given in an earlier section. We now give a successive joint plan for Example 4, where first player 1 sends some state dependent message and then player 2, conditional on the message of player 1, reveals additional information about the true state of nature.

Let $(M_1, \lambda, P(\lambda), b, M_2, \mu, P(\lambda, \mu), z, \gamma_1, \gamma_2)$ be given by:

- $M_1 = \{1, 2\}$
- $\lambda^1 = (\lambda_1^1, \lambda_2^1) = (1, 0)$, $\lambda^2 = (\lambda_1^2, \lambda_2^2) = (\frac{1}{3}, \frac{2}{3})$ and $\lambda^3 = (\lambda_1^3, \lambda_2^3) = (0, 1)$

This gives us the following a posteriori probability distribution $P(\lambda) = (p^1(\lambda), p^2(\lambda))$ on the states of nature.

- $P(\lambda) = (p_1(\lambda), p_2(\lambda))$ where $p_1(\lambda) = (p_1^1(\lambda), p_1^2(\lambda), p_1^3(\lambda)) = (\frac{3}{4}, \frac{1}{4}, 0)$ and $p_2(\lambda) = (p_2^1(\lambda), p_2^2(\lambda), p_2^3(\lambda)) = (0, \frac{2}{5}, \frac{3}{5})$

In other words after hearing message 1 we know that we are with $\frac{3}{4}$ probability in state 1 and with $\frac{1}{4}$ in state 2. This makes player 3 indifferent between choosing X and Y. After hearing message 2 we know that we are with $\frac{2}{5}$ probability in state 2 and with $\frac{3}{5}$ probability in state 3 but never in state 1. After hearing message 1 player 2 does not reveal any additional information but conditional on hearing message 2 player 2 will reveal additional information with the following signaling strategy $\mu = (\mu^2, \mu^3)$. Since message 2 is sent with zero probability in state 1 we can neglect to specify μ^1. The combined signaling strategy is given in Figure 13. The leaves correspond to the messages. The numbers along the branches correspond to the probabilities that a certain state or message gets chosen.

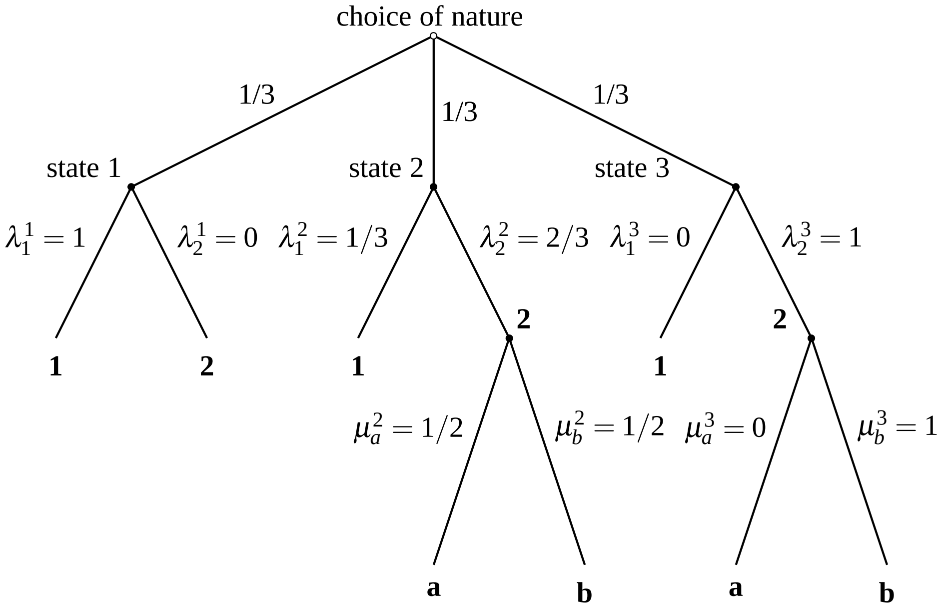

Figure 13: Successive Signaling Strategy

The signaling strategy for player 2 is given by:

- $b = (b_1, b_2) = (0, 1)$
- $M_2 = \{a, b\}$
- $\mu = (\mu^2, \mu^3)$ where $\mu^2 = (\mu_a^2, \mu_b^2) = (\frac{1}{2}, \frac{1}{2})$ and $\mu^3 = (\mu_a^3, \mu_b^3) = (0, 1)$

Hence after hearing message 2, player 2 sends conditional on being in state 2 with $\frac{1}{2}$ probability message a and likewise with $\frac{1}{2}$ probability message b. If player 2 hears message 2 and the true state is state 3 she will always send message b. This signaling behavior gives us the following final a posteriori probability distribution $P(\lambda,\mu) = (p^1(\lambda), p^2(\lambda,\mu), p^3(\lambda,\mu))$.

- $P(\lambda,\mu) = (p_1(\lambda), p_{2a}(\lambda,\mu), p_{2b}(\lambda,\mu))$ where
  $p_1(\lambda) = (p_1^1(\lambda), p_1^2(\lambda), p_1^3(\lambda)) = (\frac{3}{4}, \frac{1}{4}, 0)$,
  $p_{2a}(\lambda,\mu) = (p_{2a}^1(\lambda,\mu), p_{2a}^2(\lambda,\mu), p_{2a}^3(\lambda,\mu)) = (0,1,0)$ and
  $p_{2b}(\lambda,\mu) = (p_{2b}^1(\lambda,\mu), p_{2b}^2(\lambda,\mu), p_{2b}^3(\lambda,\mu)) = (0, \frac{1}{4}, \frac{3}{4})$

We now specify the frequency strategies z_1, z_{2a} and z_{2b}.

- $z = (z_1, z_{2a}, z_{2b})$ where $z_1 = \frac{1}{6}(T,R,X) + \frac{5}{6}(T,R,Y)$, $z_{2a} = (B,R,X)$ and
  $z_{2b} = \frac{1}{2}(B,R,X) + \frac{1}{2}(B,R,Y)$

With the frequency strategies z_1, z_{2a} and z_{2b} we obtain the following payoffs for player 1 and player 2, respectively.

$$\begin{bmatrix} \gamma_1^1(z_1) & \gamma_2^1(z_1) \\ \gamma_1^2(z_1) & \gamma_2^2(z_1) \\ \gamma_1^3(z_1) & \gamma_2^3(z_1) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{5}{12} \\ \frac{1}{12} & 0 \end{bmatrix}$$

$$\begin{bmatrix} \gamma_1^1(z_{2a}) & \gamma_2^1(z_{2a}) \\ \gamma_1^2(z_{2a}) & \gamma_2^2(z_{2a}) \\ \gamma_1^3(z_{2a}) & \gamma_2^3(z_{2a}) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \gamma_1^1(z_{2b}) & \gamma_2^1(z_{2b}) \\ \gamma_1^2(z_{2b}) & \gamma_2^2(z_{2b}) \\ \gamma_1^3(z_{2b}) & \gamma_2^3(z_{2b}) \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & 0 \\ \frac{3}{4} & 0 \end{bmatrix}$$

- The payoff for player 1 is given by $\gamma_1 = (0, \frac{1}{4}, \frac{3}{4})$
- The payoff for player 2 is given by $\gamma_2 = (\frac{1}{3}, 0, 0)$

We will now show that $(M_1, \lambda, P(\lambda), b, M_2, \mu, P(\lambda,\mu), z, \gamma_1, \gamma_2)$ as specified above satisfy the definition of a successive joint plan equilibrium. We know from before that the minmax payoff for the informed players is zero in all states; that is for all $i \in IN$ and for all $j \in K$ we have $v_i^j = 0$. For player 3 the minmax payoff is given by $vex(a_3(p)) = a_3(p) = max\{3p^2, p^1 + p^3\}$ when $p = (p^1, p^2, p^3) \in \Delta(K)$ is believed to be the true probability distribution governing the states of nature. We first show that condition 1 holds. Since message 1 is the only message in M_1 which satisfies $b_1 = 0$ we need to check that the individual rationality conditions hold in case message 1 is being sent with positive probability.

- $\lambda_1^1 > 0$: $g_1^1(z_1) = 0 = v_1^1$ and $g_2^1(z_1) = \frac{1}{3} > v_2^1$

- $\lambda_1^2 > 0$: $g_1^2(z_1) = \frac{1}{4} > v_1^1$ and $g_2^2(z_1) = \frac{5}{12} > v_2^2$

  Message 2 is being sent with positive probability in states 2 and 3 and satisfies $b_2 = 1$. Hence player 2 will further reveal information conditional on hearing message 2 by player 1. Condition 2 holds since,

- $\mu_a^2 > 0$: $g_1^2(z_{2a}) = 0 = v_1^2$ and $g_2^2(z_{2a}) = 0 = v_2^2$

- $\mu_b^2 > 0$: $g_1^2(z_{2b}) = \frac{1}{2} > v_1^2$ and $g_2^2(z_{2b}) = 0 = v_2^2$

- $\mu_b^3 > 0$: $g_1^3(z_{2b}) = \frac{3}{4} > v_1^3$ and $g_2^3(z_{2b}) = 0 = v_2^3$

We now check conditions 3 and 4. The individual rationality conditions for player 3 are satisfied since:

- Message 1 satisfies $b_1 = 0$ and with $p_1(\lambda) = (p_1^1(\lambda), p_1^2(\lambda), p_1^3(\lambda)) = (\frac{3}{4}, \frac{1}{4}, 0)$ we have $\sum_{j\in K} p_1^j(\lambda) g_3^j(z_1) = \frac{3}{4} = max\left\{3p_1^2(\lambda), p_1^1(\lambda) + p_1^3(\lambda)\right\} = vex(a_3(p_1(\lambda)))$

  Message 2 satisfies $b_2 = 0$ so

- for message a with $p_{2a}(\lambda,\mu) = (p_{2a}^1(\lambda,\mu), p_{2a}^2(\lambda,\mu), p_{2a}^3(\lambda,\mu)) = (0,1,0)$ we have $\sum_{j\in K} p_{2a}^j(\lambda,\mu) g_3^j(z_{2a}) = 3 > max\left\{3p_{2a}^2(\lambda,\mu), p_{2a}^1(\lambda,\mu) + p_{2a}^3(\lambda,\mu)\right\} = vex(a_3(p_{2a}(\lambda,\mu)))$

- for message b with $p_{2b}(\lambda,\mu) = (p_{2b}^1(\lambda,\mu), p_{2b}^2(\lambda,\mu), p_{2b}^3(\lambda,\mu)) = (0, \frac{1}{4}, \frac{1}{4})$ we have $\sum_{j\in K} p_{2b}^j(\lambda,\mu) g_3^j(z_{2b}) = \frac{3}{4} = max\left\{3p_{2b}^2(\lambda,\mu), p_{2b}^1(\lambda,\mu) + p_{2b}^3(\lambda,\mu)\right\} = vex(a_3(p_{2b}(\lambda,\mu)))$

Conditions 5 and 6 are satisfied since:

- In state 1 player 1 sends message 1 with probability equal to one, i.e. $\lambda_1^1 = 1$ and $g_1^1(z_1) = \gamma_1^1$. Player 1 would also not want to send message 2 in state 1 since z_{2a} and z_{2b} would give her a payoff of $g_1^1(z_{2a}) = -1$ and $g_1^1(z_{2b}) = -1$, respectively.

- In state 2 player 1 sends both messages with positive probability, i.e. $\lambda_1^2 = \frac{1}{3}$ and $\lambda_2^2 = \frac{2}{3}$. The payoff from sending message 1 is $g_1^2(z_1) = \frac{1}{4} = \gamma_1^2$ and the expected payoff from sending message 2 is $\frac{1}{2}g_1^2(z_{2a}) + \frac{1}{2}g_1^2(z_{2b}) = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} = \gamma_1^2$. Hence player 1 is indifferent between sending message 1 and message 2.

- In state 3 player 1 sends message 2 with probability equal to one, i.e. $\lambda_2^3 = 1$ (after which player 2 sends message b with probability equal to one) and $g_1^3(z_{2b}) = \gamma_1^1$. Player 1 would also not want to send message 1 in state 3 since z_1 would give her a payoff of $g_1^3(z_1) = 0$.

The incentive compatibility constraint for player 1 is thus satisfied without player 2 having to say "OK" or "NO". Concerning player 2 we need to check that conditional on hearing message 2 the incentive compatibility constraint holds:

- In state 2 player 2 sends message a and b with equal probability, i.e. $\mu_{2a}^2 = \mu_{2b}^2 = \frac{1}{2}$ and $g_2^2(z_2a) = g_2^2(z_2a) = \gamma_2^2|2 = 0$.

- In state 3 player 2 sends message b with probability equal to one, i.e. $\mu_{2b}^3 = 1$ and $g_2^3(z_{2b}) = \gamma_2^3|2 = 0$. In state 3 Player 2 would also not want to send message a with positive probability since z_{2a} would give her a payoff of $g_2^3(z_{2a}) = -1$.

We have shown that all payoffs are individually rational and that the incentive compatibility constraints for players 1 and 2 are satisfied. Thus the successive joint plan describes a successive joint plan equilibrium with the equilibrium payoff vector of $((0, \frac{1}{4}, \frac{3}{4}), (\frac{1}{3}, 0, 0), 1)$. This completes the proof.

We have seen that the set of possible successive joint plan equilibria is strictly larger than the set of “standard” joint plan equilibria. Furthermore we have shown that the game in Example 4 has an equilibrium realized by successive information revelation. The nice property of this equilibrium is that player 2 does not have to announce that player 1 is lying which can pose problems; that is player 1 could be telling the truth but player 2 could announce that player 1 is lying. In any case, when player 2 announces that player 1 is lying, player 3 would not necessarily know who is deviating. This problem is avoided in this case since individual rationality holds for both players. We see that incentive compatibility is a strong and desirable property of an equilibrium.

The question is now whether there exists a successive joint plan equilibrium for Example 4 where player 2 is the first one the reveal some information followed by player 1. The answer to this question is yes. The relevance of this result will be briefly discussed in the conclusion. We now give a successive joint plan $(M_2, \lambda, P(\lambda), b, M_i, \mu, P(\lambda, \mu), z, \gamma_2, \gamma_1)$ and then prove that this successive joint plans satisfies the conditions of a successive joint plan equilibrium. Due to its length we shall give the successive joint plan and the proof in a more compact form, but which is closely related to the proof of Proposition 2.18.

$(M_2, \lambda, P(\lambda), b, M_i, \mu, P(\lambda, \mu), z, \gamma_2, \gamma_1)$ is given by:

- $M_2 = \{1, 2\}$
- $\lambda^1 = (\lambda_1^1, \lambda_2^1) = (1, 0)$, $\lambda^2 = (\lambda_1^2, \lambda_2^2) = (\frac{2}{3}, \frac{1}{3})$ and $\lambda^3 = (\lambda_1^3, \lambda_2^3) = (0, 1)$
- $P(\lambda) = (p_1(\lambda), p_2(\lambda))$ where $p_1(\lambda) = (\frac{3}{5}, \frac{2}{5}, 0)$ and
  $p_2 = (p_2^1(\lambda), p_2^2(\lambda), p_2^3(\lambda)) = (0, \frac{1}{4}, \frac{3}{4})$
- $b = (1, 0)$
- $M_1 = \{a, b\}$
- $\mu = (\mu^1, \mu^2)$ where $\mu^1 = (\mu_a^1, \mu_b^1) = (1, 0)$ and $\mu^2 = (\mu_a^2, \mu_b^2) = (\frac{1}{2}, \frac{1}{2})$

  The successive signaling strategy is illustrated in Figure 14.
- $P(\lambda, \mu) = (p_{1a}(\lambda, \mu), p_{1b}(\lambda, \mu)), p_2(\lambda))$ where $p_{1a}(\lambda, \mu) = (\frac{3}{4}, \frac{1}{4}, 0)$, $p_{1b}(\lambda, \mu) = (0, 1, 0)$ and $p_2(\lambda) = (0, \frac{1}{4}, \frac{3}{4})$,
- $z = (z_{1a}, z_{1b}, z_2)$ where $z_{1a} = \frac{1}{3}(T, R, X) + \frac{2}{3}(T, R, Y)$, $z_{1b} = (B, L, X)$ and
  $z_2 = \frac{5}{12}(B, R, X) + \frac{7}{12}(B, R, Y)$
- The payoff for player 2 is given by $\gamma_2 = (\frac{2}{3}, \frac{1}{6}, 0)$

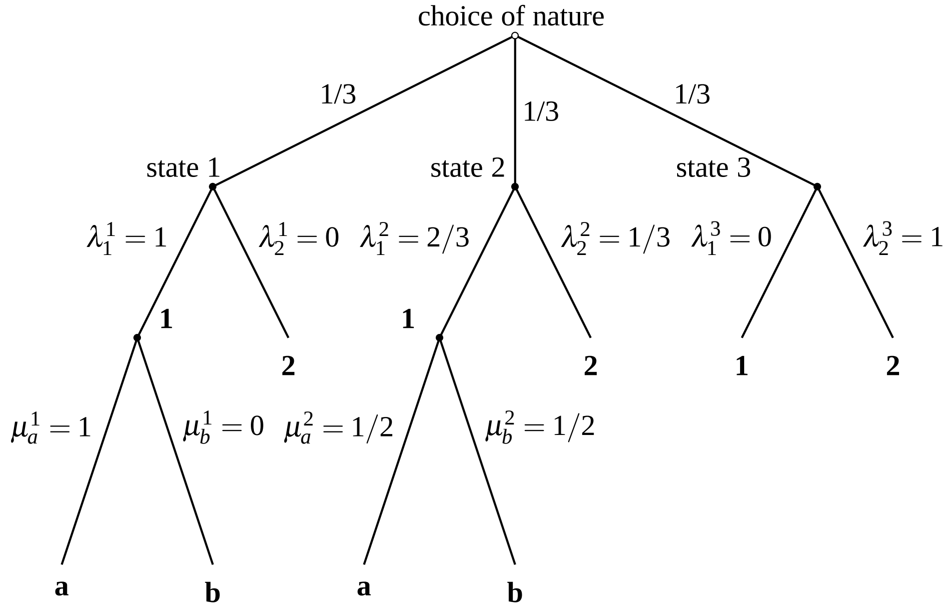

Figure 14: Successive Signaling Strategy

- The payoff for player 1 is given by $\gamma_1 = (0, \frac{7}{36}, \frac{5}{8})$

With the frequency strategies z_{1a}, z_{1b} and z_2 we obtain the following payoffs for players 1 and player 2 respectively.

$$\begin{bmatrix} \gamma_1^1(z_{1a}) & \gamma_2^1(z_{1a}) \\ \gamma_1^2(z_{1a}) & \gamma_2^2(z_{1a}) \\ \gamma_1^3(z_{1a}) & \gamma_2^3(z_{1a}) \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} \\ 0 & \frac{1}{3} \\ \frac{1}{6} & 0 \end{bmatrix}$$

$$\begin{bmatrix} \gamma_1^1(z_{1b}) & \gamma_2^1(z_{1b}) \\ \gamma_1^2(z_{1b}) & \gamma_2^2(z_{1b}) \\ \gamma_1^3(z_{1b}) & \gamma_2^3(z_{1b}) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \gamma_1^1(z_2) & \gamma_2^1(z_2) \\ \gamma_1^2(z_2) & \gamma_2^2(z_2) \\ \gamma_1^3(z_2) & \gamma_2^3(z_2) \end{bmatrix} = \begin{bmatrix} -1 & \frac{5}{12} \\ \frac{7}{12} & \frac{1}{6} \\ \frac{5}{8} & 0 \end{bmatrix}$$

We now prove that $(M_2, \lambda, P(\lambda), b, M_1, \mu, P(\lambda, \mu), z, \gamma_2, \gamma_1)$ as given above satisfies the conditions of a successive joint plan equilibrium.

We first show that condition 1 holds. Since message 2 is the only message in M_2 which satisfies $b_2 = 0$ we need to check that the individual rationality conditions holds in case message 2 is being sent.

- $\lambda_2^2 > 0$: $g_1^2(z_2) = \frac{7}{12} > v_1^2$ and $g_2^2(z_2) = \frac{1}{6} > v_2^2$
- $\lambda_2^3 > 0$: $g_1^3(z_2) = \frac{5}{8} > v_1^3$ and $g_2^3(z_2) = 0 = v_2^2$

  Message 1 is being sent with positive probability in states 1 and 2 and satisfies $b_1 = 1$. Hence player 1 will reveal further information conditional on hearing message 1 by player 2. Condition 2 holds since:

- $\mu_a^1 > 0$: $g_1^1(z_{1a}) = 0 = v_1^1$ and $g_2^1(z_{1a}) = \frac{2}{3} > v_2^1$
- $\mu_a^2 > 0$: $g_1^2(z_{1a}) = 0 = v_1^2$ and $g_2^2(z_{1a}) = \frac{1}{3} > v_2^2$
- $\mu_b^2 > 0$: $g_1^2(z_{1b}) = 0 = v_1^2$ and $g_2^2(z_{1b}) = 0 = v_2^2$

We now check conditions 3 and 4. The individual rationality conditions for player 3 are satisfied since:

- Message 2 satisfies $b_2 = 0$ and with $p_2(\lambda) = (0, \frac{1}{4}, \frac{3}{4})$ we have $\sum_{j \in K} p_2^j(\lambda) g_3^j(z_2) = \frac{3}{4} = max\{3p_2^2(\lambda), p_2^1(\lambda) + p_2^3(\lambda)\} = vex(a_3(p_2(\lambda)))$

  Message 1 satisfies $b_1 = 1$ so

- for message a with $p_{1a}(\lambda,\mu) = (\frac{3}{4}, \frac{1}{4}, 0)$ we have
  $\sum_{j \in K} p_{1a}^j(\lambda,\mu) g_3^j(z_{1a}) = \frac{3}{4} = max\{3p_{1a}^2(\lambda,\mu), p_{1a}^1(\lambda,\mu) + p_{1a}^3(\lambda,\mu)\} = vex(a_3(p_{1a}(\lambda,\mu)))$
- for message b with $p_{1b}(\lambda,\mu) = (0,1,0)$ we have
  $\sum_{j \in K} p_{1b}^j(\lambda,\mu) g_3^j(z_{1b} = 3 > max\{3p_{1b}^2(\lambda,\mu), p_{1b}^1(\lambda,\mu) + p_{1b}^3(\lambda,\mu)\} = vex(a_3(p_{1b}(\lambda,\mu)))$

Conditions 5 and 6 are satisfied since:

- In state 1 player 2 sends message 1 with probability equal to one, i.e. $\lambda_1^1 = 1$ (after which player 1 sends message a with probability equal to one) and $g_2^1(z_{1a}) = \gamma_2^1$. Player 2 would also not want to send message 2 in state 1 since z_2 would give her a payoff of $g_2^1(z_2) = \frac{5}{12}$ which is less that $g_1^2(z_{1a}) = \frac{2}{3}$.
- In state 2 player 1 sends both messages with positive probability, i.e. $\lambda_1^2 = \frac{2}{3}$ and $\lambda_1^2 = \frac{1}{3}$. The payoff from sending message 2 is $g_2^2(z_2) = \frac{1}{6} = \gamma_2^2$ and the expected payoff from sending message 1 is $\frac{1}{2}g_2^2(z_{1a}) + \frac{1}{2}g_2^2(z_{1b}) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot 0 = \frac{1}{6} = \gamma_2^2$. Hence player 1 is indifferent between sending message 1 and message 2.
- In state 3 player 2 sends message 2 with probability equal to one, i.e. $\lambda_2^3 = 1$ and $g_2^3(z_2) = \gamma_2^3$. Player 1 would also not want to send message 1 in state 3 since z_{1a} and z_{1b} would give her a payoff of $g_2^3(z_{1a}) = 0$ and $g_2^3(z_{1b}) = -1$, respectively.

Condition 7 is satisfied since:

- In state 1 player 1 sends message a with probability equal to one, i.e. $\mu_{1a}^1 = 1$ and $g_1^1(z_{1a}) = \gamma_1^1|1 = 0$. In state 1 player 1 would also not want to send message b with positive probability since z_{1b} would give her a payoff of $g_1^1(z_{1b}) = -1$.
- In state 2 player 1 sends message a and b with equal probability, i.e. $\mu_{1a}^2 = \mu_{1b}^2 = \frac{1}{2}$ and $g_1^2(z_{1a}) = g_1^2(z_{1a}) = \gamma_1^2|1$.

We have shown that all payoffs are individual rationally and that the incentive compatibility constraints for players 1 and 2 are satisfied. Thus the successive joint plan describes a successive joint plan equilibrium with the equilibrium payoff vector of $((0, \frac{7}{36}, \frac{5}{8}), (\frac{2}{3}, \frac{1}{6}, 0), 1)$. This completes the proof.

Due to the existence of successive joint plan equilibria for Example 4, where neither a completely revealing equilibrium nor a joint plan equilibrium for player 1 nor a joint plan equilibrium for player 2 exist, the question arises if this result can be generalized to all 3-player infinitely repeated games with incomplete information on one side and any finite number of states. In other words does there always exist a successive joint plan equilibrium for this class of games?

3 Conclusion

We have shown that there exist 3-player infinitely repeated games with incomplete information on one side and three states of nature where no joint plan equilibria exist for neither player 1 nor player 2. We have then introduced and defined a more general concept, successive joint plans, where both informed players take part in the revelation process. Using successive joint plans we have then profen the existence of equilibria for a particular game where no joint plan equilibrium exist. We have found two successive joint plant equilibria, one where player 1 is the first to reveal some information and the other one where player 2 is the first player to send some signal.

The fact that there exist two successive joint plan equilibria, as described above, not only supports our approach but also gives a positive outlook that the answer to the posed question will be a positive one. If the existence of successive joint plan equilibria for all 3-player infinitely repeated games with incomplete information on one side and with arbitrary many states can be profen, the general proof should give the existence of two successive joint plan equilibria, one where player 1 is the first to reveal some information and the other one where player 2 is the first player to send some signal.

We now give a possible approach to answer the question posed in the last chapter. If the number of states is 2 then there exist either a completely revealing equilibrium or a joint plan equilibrium for one of the two informed players. For an arbitrary number of states we should first check whether there exists a completely revealing equilibrium. If the existence of completely revealing equilibria fails Proposition 2.10 imposes restrictions on the equilibrium payoffs. Keeping this restriction and the individual rationality constraints in mind we now could define a correspondence $F : \Delta(K) \to \mathbb{R}^k \times \mathbb{R}^k$ by $F(p) = \left\{ \begin{bmatrix} g_1^1(z) - v_1^1 & g_2^1(z) - v_2^1 \\ \vdots & \vdots \\ g_1^k(z) - v_1^k & g_2^k(z) - v_2^k \end{bmatrix} : z \in \Delta(S), \gamma_3 > vex(a_3(p)) \right\}$. If there exists a $z \in \Delta(S)$ such that $F(p_0) \cap \begin{bmatrix} \geq 0 & \geq 0 \\ \vdots & \vdots \\ \geq 0 & \geq 0 \end{bmatrix} \neq \emptyset$ then there exists a joint plan with one message and only one stage of revelation for one of the informed players. Now let C be the connected component of $\left\{ p \in \Delta(K) : F(p) \cap \begin{bmatrix} \geq 0 & \geq 0 \\ \vdots & \vdots \\ \geq 0 & \geq 0 \end{bmatrix} = \emptyset \right\}$ containing p_0. We can characterize C by setting $C = \bigcup_i C_i$ where C_i is of the form $C_i = \left\{ p \in C : F(p) \cap \begin{bmatrix} < 0 & \geq 0 \\ > 0 & \geq 0 \\ \vdots & \vdots \\ \geq 0 & < 0 \end{bmatrix} \neq \emptyset \right\}$ In all sets C_i the individual rationality constraint is violated for some informed player. But by considering the relative frontier of some of the C_i and using the convexity of F we should be able to prove the existence of a successive joint plan equilibrium.

4 Bibliography

Aumann, R.(1964) . Mixed and Behavior Strategies in Infinite Extensive Games, in *Advances in Game Theory, Annals of Mathematics Study* **52**, edited by M. Dresher, L. S. Shapley, and A. W. Tucker, Princeton University Press, 627-650.

Aumann, R. and Maschler, M. (1995). *Repeated Games with Incomplete Information. With the collaboration of R. Stearns.* Cambridge, MA, M.I.T. Press.

Blackwell, D. (1956). An analog of the minimax theorem for vector payoffs. *Pacific Journal of Mathematics* **6**, no. 1, 1-8.

Hart, S. (1985). Nonzero-sum two-person repeated games with incomplete information. *Mathematics of Operations Research* **10**, 117-153.

Renault, J. (2001). 3-player repeated games with lack of information on one side. *International Journal of Game Theory* **30**, issue 2, 221-245.

Simon, R. (2006). Lecture Notes - MA413: Games of Incomplete Information [Internet] London School of Economics and Political Science, London. Available from:
<http://www.maths.lse.ac.uk/Courses/MA413> [Accessed 14th August 2006].

Simon, R., Spiez, S. and Torunczyk, H. (1995). The Existence of Equilibria in Certain Games, Separation for Families of Convex Functions and a Theorem of Borsuk-Ulam Type. *Israel Journal of Mathematics* **92**, 1-21.

Tomala, T. (1995). Nash Equilibria of Repeated Games with Observable Payoff Vectors. *Games and Economic Behavior* **28**, no 2, 310-324.

Zamir, S. (1992), Repeated Games of Incomplete Information: Zero-Sum, in R.J. Aumann and S. Hart, ed., *Handbook of Game Theory with Economic Applications*, First Edition 1, Volume 1, Chapter 5. North-Holland, Elsevier.