A SPECTRALLY MINIMAL REALIZATION FORMULA FOR $H^\infty(\mathbb{D})$

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Abstract. In this article we prove a representation theorem for $H^\infty(\mathbb{D})$ functions, such that the realization formula is *spectrally minimal* in the following sense: the spectrum of the main operator in the realization intersects the unit circle precisely at those points where the given function has no holomorphic extension. We also extend this result to operator-valued H^∞ functions.

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1. Introduction

It is known (see for example [1, Theorem 6.5], [5, Theorem 2-1]) that every function in $H^\infty(\mathbb{D})$ can be represented as a "transfer function", that is, there exists a Hilbert space H and an operator $V : H \oplus \mathbb{C} \to H \oplus \mathbb{C}$ such that writing V as

$$V = \begin{array}{cc} & \begin{array}{cc} H & \mathbb{C} \end{array} \\ \begin{array}{c} H \\ \mathbb{C} \end{array} & \left[\begin{array}{cc} A & B \\ C & D \end{array} \right] \end{array}, \tag{1.1}$$

one has

$$g(z) = D + Cz(I - zA)^{-1}B, \quad z \in \mathbb{D}. \tag{1.2}$$

In the right-hand side of the *realization formula* (1.2), it is understood that for every nonzero $z \in \mathbb{D}$, $I - zA$ is invertible as an element of $\mathcal{L}(H)$, that is, $z^{-1} \in \rho(A)$, the resolvent set of A. The background to the realization formula (1.2) arises from relating g to the input-output map (the *transfer function*) of a discrete-time linear time-invariant system

$$\left\{ \begin{array}{rcl} x(n+1) & = & Ax(n) + Bu(n), \\ y(n) & = & Cx(n) + Du(n). \end{array} \right. \tag{1.3}$$

Here $n \in \mathbb{Z}$ can be interpreted as a time variable, $u(n)$, $x(n)$, $y(n)$ have interpretations as input signal, state vector, and output signal, respectively, at time n. Application of the Z-transform $(x(n))_n \mapsto \hat{x}(z) = \sum_{n\in\mathbb{Z}} x(n)z^n$ to all quantities in (1.3) formally leads to $\hat{y}(z) = g(z)\hat{u}(z)$, where g is as in (1.2); see [5] and [2].

However, the same transfer function can be realized by generators A with widely differing spectra. From an intuitive point of view, the question then arises if we can have a generator which reflects the singularities of the transfer function in a faithful manner, that is, we would like to have an A with the smallest possible spectrum required to model the singularities of the given transfer function. We make this precise below.

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Definition 1.1. Let V as in (1.1) be a realization of g, so that (1.2) holds. Let

$$S := \{z \in \mathbb{T} \mid g \text{ is holomorphic across } z\}.$$

We call the realization (1.1) for $g \in H^\infty(\mathbb{D})$ *spectrally minimal* if

$$S^{-1} = \rho(A) \cap \mathbb{T}.$$

Here $S^{-1} := \{z^{-1} \mid z \in S\}$.

Our main result is that a spectrally minimal realization as defined above always exists. It has been shown earlier (for instance in [5, §4], [3], [9, §9.8]), with different methods, that in certain special cases of g's from $H^\infty(\mathbb{D})$, a realization can be chosen so that the component of $\rho(A)$ containing the origin is a maximal holomorphic domain of g. On the other hand, we cover the general case when g is arbitrary in $H^\infty(\mathbb{D})$, but we study the spectrum in the closed unit disk only. Our main result is the following, which we prove in Section 3:

Theorem 1.2. *Let S be an open subset of $\mathbb{T}$, and let $g \in H^\infty(\mathbb{D})$ have a holomorphic extension across S. Then there exists a Hilbert space H and an operator $V : H \oplus \mathbb{C} \to H \oplus \mathbb{C}$ such that writing V as*

$$V = \begin{array}{c} \\ H \\ \mathbb{C} \end{array} \begin{array}{c} \begin{array}{cc} H & \mathbb{C} \end{array} \\ \left[\begin{array}{cc} A & B \\ C & D \end{array} \right] \end{array},$$

one has $S^{-1} = \mathbb{T} \cap \rho(A)$ and

$$g(z) = D + Cz(I - zA)^{-1}B, \quad z \in \mathbb{D}. \tag{1.4}$$

We begin by giving a new realization formula for $H^\infty(\mathbb{D})$ functions, without spectral minimality, in Section 2. In Section 3, we modify the construction of the realization from the Section 2 in order to make it spectrally minimal, and so we prove the main result of this article (Theorem 1.2). Finally, we give an extension of the main result in the context of operator-valued transfer functions in Section 4.

We will use the following standard notation.

$\mathbb{D}$, $\overline{\mathbb{D}}$, $\mathbb{T}$	the open unit disk, the closed unit disk, and the unit circle, respectively: $\mathbb{D} := \{z \in \mathbb{C} \mid \lvert z\rvert < 1\}$, $\overline{\mathbb{D}} := \{z \in \mathbb{C} \mid \lvert z\rvert \le 1\}$, $\mathbb{T} := \{z \in \mathbb{C} \mid \lvert z\rvert = 1\}$;
$H^\infty(\mathbb{D}; X)$	If X is a Banach space, then $H^\infty(\mathbb{D}; X)$ denotes the space of X-valued bounded holomorphic functions on $\mathbb{D}$, equipped with the supremum norm. The space $H^\infty(\mathbb{D}; \mathbb{C})$ will be denoted simply by $H^\infty(\mathbb{D})$.
$\mathcal{L}(X, Y)$	If X, Y are Hilbert spaces, then $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X to Y, equipped with the operator norm;
$\rho(A)$	If $A \in \mathcal{L}(X)$ is a linear transformation on a Hilbert space X, then $\rho(A)$ denotes the resolvent set of A.

2. A realization formula for $H^\infty(\mathbb{D})$ without spectral minimality

In this section, we prove the result below, which gives a realization formula (apparently new) for functions from $H^\infty(\mathbb{D})$. We will modify the realization in order to make it spectrally minimal in the next section, but for clarity of exposition, we shall just prove the following theorem now.

Theorem 2.1. *If $g \in H^\infty(\mathbb{D})$, then there exists a Hilbert space H and an operator $V : H \oplus \mathbb{C} \to H \oplus \mathbb{C}$ such that writing V as*

$$V = \begin{array}{cc} & \begin{array}{cc} H & \mathbb{C} \end{array} \\ \begin{array}{c} H \\ \mathbb{C} \end{array} & \left[\begin{array}{cc} A & B \\ C & D \end{array}\right] \end{array},$$

one has

$$g(z) = D + Cz(I - zA)^{-1}B, \quad z \in \mathbb{D}. \tag{2.1}$$

Before giving the proof, we explain the main idea behind it: assuming for the moment that $g(0) = 0$, if g were holomorphic in a domain containing $\overline{\mathbb{D}}$, then by Cauchy integral formula, we would have

$$\frac{g(z)}{z} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{g(e^{i\theta})e^{-i\theta}}{e^{i\theta} - z} ie^{i\theta} d\theta = \frac{1}{2\pi}\int_0^{2\pi} (1 - ze^{-i\theta})^{-1} \frac{g(e^{i\theta})}{e^{i\theta}} d\theta,$$

and we view this integral as $C(I - zA)^{-1}B$, where $C \in \mathcal{L}(L^2(\mathbb{T}); \mathbb{C})$, $A \in \mathcal{L}(L^2(\mathbb{T}), L^2(\mathbb{T}))$, $B \in \mathcal{L}(\mathbb{C}; L^2(\mathbb{T}))$ are given by:

$$C = \frac{1}{2\pi}\int_0^{2\pi} \bullet \, d\theta, \quad A = \text{multiplication operator on } L^2(\mathbb{T}) \text{ by } e^{-i\theta}, \quad B = (u \mapsto \frac{g(e^{i\cdot})}{e^{i\cdot}} u).$$

However, since g need not be holomorphic across $\mathbb{T}$, we first work on a circle with radius r inside the disk, and then then pass the limit as r increases to 1.

Proof of Theorem 2.1. We assume that $g(0) = 0$ (set $D = g(0)$ in the general case). The map $z \mapsto g(z)/z$ is then holomorphic in $\mathbb{D}$. Let $H = L^2(\mathbb{T})$. Suppose that $z \in \mathbb{D}$. Let r be such that $|z| < r < 1$.

Define $A_r \in \mathcal{L}(H)$ to be the multiplication operator by $1/(re^{i\theta})$:

$$(A_r f)(e^{i\theta}) = f(e^{i\theta})/(re^{i\theta}), \quad \theta \in [0, 2\pi), \quad f \in L^2(\mathbb{T}).$$

The spectrum of A_r is the range of $\theta \mapsto 1/(re^{i\theta})$ (see for example [6, Problem 67]), that is, $\sigma(A_r) = (1/r)\mathbb{T}$. Since $|z| < r$, it follows that $1 \in \rho(zA_r)$, and

$$((I - zA_r)^{-1} f)(e^{i\theta}) = \frac{f(e^{i\theta})}{1 - z/(re^{i\theta})}, \quad \theta \in [0, 2\pi), \quad f \in L^2(\mathbb{T}).$$

Let $B_r \in \mathcal{L}(\mathbb{C}; H)$ be defined by

$$(B_r u)(e^{i\theta}) = \frac{g(re^{i\theta})}{re^{i\theta}} u, \quad \theta \in [0, 2\pi), \quad u \in \mathbb{C}.$$

Note that $\theta \mapsto g(re^{i\theta}) \in L^\infty(\mathbb{T}) \subset L^2(\mathbb{T})$.

Let $C \in \mathcal{L}(H; \mathbb{C})$ be defined by

$$Cf = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) d\theta, \quad f \in L^2(\mathbb{T}).$$

Let γ denote the curve $\theta : [0, 2\pi] \to re^{i\theta}$. By the Cauchy integral formula, we have

$$\begin{aligned} \frac{g(z)}{z} &= \frac{1}{2\pi i}\int_\gamma \frac{g(w)/(w)}{w-z}dw \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{g(re^{i\theta})/(re^{i\theta})}{re^{i\theta}-z} rie^{i\theta}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1-z/(re^{i\theta})}\frac{g(re^{i\theta})}{re^{i\theta}}d\theta \\ &= C(I-zA_r)^{-1}B_r. \end{aligned} \tag{2.2}$$

Note that with a fixed z, the above is true for every r satisfying $|z| < r < 1$.

Define $A \in \mathcal{L}(H)$ to be the multiplication operator by $1/e^{i\theta}$:

$$(Af)(e^{i\theta}) = f(e^{i\theta})/e^{i\theta}, \quad \theta \in [0, 2\pi), \quad f \in L^2(\mathbb{T}).$$

We note that $zA_r \to zA$ as $r \nearrow 1$ in the operator norm of $\mathcal{L}(H)$. As $1 \in \rho(zA_r)$ ($|z| < r < 1$) and $1 \in \rho(zA)$, it follows that (see for example [7, Theorem 2.23, p. 206]) $(I - zA_r)^{-1} \to (I - zA)^{-1}$ in $\mathcal{L}(H)$ as $r \nearrow 1$.

Furthermore, define $B \in \mathcal{L}(\mathbb{C}, H)$ by

$$(Bu)(e^{i\theta}) = \frac{g(e^{i\theta})}{e^{i\theta}}u, \quad \theta \in [0, 2\pi), \quad u \in \mathbb{C}.$$

From [8, Theorem 17.11], $g(re^{i\cdot}) \to g(e^{i\cdot})$ in $L^2(\mathbb{T})$ as $r \nearrow 1$, and so $B_r \to B$ in $\mathcal{L}(\mathbb{C}; H)$ as $r \nearrow 1$.

Consequently, from (2.2) we obtain that

$$\frac{g(z)}{z} = C(I - zA)^{-1}B. \tag{2.3}$$

Since the choice of $z \in \mathbb{D}$ was arbitrary, (2.3) holds for all $z \in \mathbb{D}$. Defining $D = 0$, we obtain (2.1). $\square$

3. A spectrally minimal realization formula for $H^\infty(\mathbb{D})$

In this section we will prove our main result on the existence of spectrally minimal realizations in Theorem 1.2 below, but we will need the following technical lemma.

Lemma 3.1. *Let O be an open set in $\mathbb{R}^2$ containing the segment $(0, 1)$. Then there exists a $f \in C^\infty(\mathbb{R}; \mathbb{R})$ such that for all $x \in (0, 1)$, $f(x) > 0$ and $(x, y) \in O$ for every $y \in [0, f(x)]$, and furthermore $f(x) = 0$ if $x \notin (0, 1)$.*

Proof. We first define f on $(0, 1/2)$. For $n \geq 3$, let $g_n \in C^\infty(\mathbb{R})$ be such that

$$g_n(x) = \begin{cases} 0 & x \leq \frac{1}{n} \\ 1 & x \geq \frac{1}{n-1} \\ \in (0,1) & x \in (\frac{1}{n-1}, \frac{1}{n}) \end{cases}.$$

For $n \geq 3$, choose $a_n > 0$ such that $a_n \|g_n\|_{C^n} < 1/2^n$. For $n \geq 3$, pick $r_n > 0$ such that $(x, y) \in O$ for $x \in [1/n, 1/2]$ and $y \in [0, r_n]$, also ensuring that the sequence of r_n's satisfies $r_{n+1} < r_n$ for all $n \geq 3$ and $r_n \to 0$ as $n \to \infty$. For $n \geq 3$, define $b_n = \min\{a_n, r_n - r_{n+1}\}$. Let

$$f_1 = \sum_{n=3}^{\infty} b_n g_n.$$

For any $N < \infty$, the series converges in C^N:

$$\|f_1\|_{C^N} \leq \sum_{n=3}^{\infty} b_n \|g_n\|_{C^N} \leq \sum_{n=3}^{\infty} a_n \|g_n\|_{C^N} < \sum_{n=3}^{N-1} a_n \|g_n\|_{C^N} + \sum_{n=N}^{\infty} \frac{1}{2^n} < +\infty. \tag{3.1}$$

So $f_1 \in C^\infty$. Let now $x \in (0, 1/2]$. Then $x \in (1/(n+1), 1/n]$ for some $n \geq 2$, and hence $g_k(x) = 0$ for $k \leq n$ (we define $g_k := 0$ for $k < 3$), so

$$0 < b_{n+1} g_{n+1}(x) < f_1(x) \leq \sum_{k=n+1}^{\infty} b_k \leq \sum_{k=n+1}^{\infty} (r_k - r_{k+1}) = r_{n+1}.$$

If $y \in [0, f_1(x)]$, then $y \in [0, r_{n+1}]$ and hence then $(x, y) \in O$.

From (3.1), we also have that $f_1(x) < 1$ for all $x \in (0, 1)$ (even $\|f_1\|_{C^0} \leq \sum_{k=3}^{\infty} 2^{-k} = 1/4$).

Similarly, we can construct a $f_2 \in C^\infty$ such that for all $x \in [1/2, 1)$ and all $y \in [0, f_2(x)]$, we have $(x, y) \in O$, and furthermore, $1 > f_2(x) > 0$ for $x \in (0, 1)$. Defining $f = f_1 f_2$, we are done. □

We are now ready to prove our main result.

Proof of Theorem 1.2. We assume that $g(0) = 0$ (set $D = g(0)$ in the general case).

Let Ω be a simply connected domain containing $\mathbb{D} \cup S$ such that $\partial\Omega \cap \mathbb{T} = \mathbb{T} \setminus S$, and such that g is holomorphic and bounded in Ω.

We note that it can be arranged that the boundary of Ω is smooth, that is C^1, and it has a C^1 parameterization $\theta \mapsto R(\theta)e^{i\theta}$. Indeed we observe that S can be written as a disjoint union of open arcs (see for instance [10, Theorem 1.3]), and then use Lemma 3.1).

The map $z \mapsto g(z)/z$ is holomorphic in $\mathbb{D}$. Let $H = L^2(\mathbb{T})$. Suppose that $z \in \mathbb{D}$. Let r be such that $|z| < r < 1$.

Define $A_r \in \mathcal{L}(H)$ to be the multiplication operator by $1/(rR(\theta)e^{i\theta})$:

$$(A_r f)(e^{i\theta}) = f(e^{i\theta})/(rR(\theta)e^{i\theta}), \quad \theta \in [0, 2\pi), \quad f \in L^2(\mathbb{T}).$$

The spectrum of A_r is the range of $\theta \mapsto 1/(rR(\theta)e^{i\theta})$. Since $|z| < r$, it follows that $1 \in \rho(zA_r)$, and

$$((I - zA_r)^{-1} f)(e^{i\theta}) = \frac{f(e^{i\theta})}{1 - z/(rR(\theta)e^{i\theta})}, \quad \theta \in [0, 2\pi), \quad f \in L^2(\mathbb{T}).$$

Let $B_r \in \mathcal{L}(\mathbb{C}; H)$ be defined by

$$(B_r u)(e^{i\theta}) = \frac{g(rR(\theta)e^{i\theta})(R(\theta) - iR'(\theta))}{r(R(\theta))^2 e^{i\theta}} u, \quad \theta \in [0, 2\pi), \quad u \in \mathbb{C}.$$

Let $C \in \mathcal{L}(H; \mathbb{C})$ be defined by

$$Cf = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta, \quad f \in L^2(\mathbb{T}).$$

Let γ denote the curve $\theta : [0, 2\pi] \to rR(\theta)e^{i\theta}$. By the Cauchy integral formula, we have

$$\begin{aligned}
\frac{g(z)}{z} &= \frac{1}{2\pi i} \int_\gamma \frac{g(w)/(w)}{w - z} dw = \frac{1}{2\pi} \int_\gamma \frac{1}{1 - z/w} \frac{-ig(w)\frac{dw}{d\theta}/re^{i\theta}}{w^2} dw \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1 - z/(rR(\theta)e^{i\theta})} \frac{g(rR(\theta)e^{i\theta})(R(\theta) - iR'(\theta))}{r(R(\theta))^2 e^{i\theta}} d\theta \\
&= C(I - zA_r)^{-1} B_r. \qquad (3.2)
\end{aligned}$$

Note that with a fixed z, the above is true for every r satisfying $|z| < r < 1$.

Define $A \in \mathcal{L}(H)$ to be the multiplication operator given by:

$$(Af)(e^{i\theta}) = f(e^{i\theta})/(R(\theta)e^{i\theta}), \quad \theta \in [0, 2\pi), \quad f \in L^2(\mathbb{T}).$$

We note that $zA_r \to zA$ as $r \nearrow 1$ in the operator norm of $\mathcal{L}(H)$. As $1 \in \rho(zA_r)$ ($|z| < r < 1$) and $1 \in \rho(zA)$, it follows that $(I - zA_r)^{-1} \to (I - zA)^{-1}$ in $\mathcal{L}(H)$ as $r \nearrow 1$.

Furthermore, define $B \in \mathcal{L}(\mathbb{C}, H)$ by

$$(Bu)(e^{i\theta}) = \frac{g(R(\theta)e^{i\theta})(R(\theta) - iR'(\theta))}{(R(\theta))^2 e^{i\theta}} u, \quad \theta \in [0, 2\pi), \quad u \in \mathbb{C}.$$

We also note that $B_r \to B$ in $\mathcal{L}(\mathbb{C}; H)$ as $r \nearrow 1$. To see this, it is enough to prove that $g(rR(\cdot)e^{i\cdot}) \to g(R(\cdot)e^{i\cdot})$ as $r \nearrow 1$ in $L^2(\mathbb{T})$. But this follows from the Lebesgue dominated convergence theorem, if we prove that the functions converge pointwise almost everywhere. If $R(\theta) = 1$, then this follows from the fact that the radial limits exist almost everywhere for functions in $H^\infty(\mathbb{D})$ [8, Theorem 17.11]. If $R(\theta) > 1$, then this follows from the fact that g is continuous at $R(\theta)e^{i\theta}$ (this can be ensured by having chosen Ω suitably at the outset, that is, by shrinking it enough so that its boundary outside $\mathbb{D}$ lies in the region where g is holomorphic and so continuous to the boundary).

Consequently, from (3.2) we obtain that

$$\frac{g(z)}{z} = C(I - zA)^{-1}B. \tag{3.3}$$

Since the choice of $z \in \mathbb{D}$ was arbitrary, (3.3) holds for all $z \in \mathbb{D}$. Defining $D = 0$, we obtain (1.4).

Finally, we observe that $\sigma(A)$ is the range of $1/(R(\theta)e^{i\theta})$. Since $\partial\Omega \cap \mathbb{T} = \mathbb{T} \setminus S$, we have $S^{-1} = \mathbb{T} \cap \rho(A)$. $\square$

4. Operator-valued case

Theorem 4.1. *Let U, Y be Hilbert spaces, S be an open subset of $\mathbb{T}$, and $g \in H^\infty(\mathbb{D}; \mathcal{L}(U, Y))$ have a holomorphic extension across S. Then there exists a Hilbert space H and an operator $V : H \oplus U \to H \oplus Y$ such that writing V as*

$$V = \begin{array}{cc} & \begin{array}{cc} H & U \end{array} \\ \begin{array}{c} H \\ Y \end{array} & \left[\begin{array}{cc} A & B \\ C & D \end{array}\right] \end{array},$$

one has $S^{-1} = \mathbb{T} \cap \rho(A)$ and

$$g(z) = D + Cz(I - zA)^{-1}B, \quad z \in \mathbb{D}. \tag{4.1}$$

Proof. Let Ω and $R(\theta)$ be constructed as in the proof of Theorem 1.2, and let $H = L^2(\mathbb{T}; Y)$. Define $B \in \mathcal{L}(U, H)$ by

$$(Bu)(e^{i\theta}) = \frac{g(R(\theta)e^{i\theta})(R(\theta) - iR'(\theta))}{(R(\theta))^2 e^{i\theta}} u, \quad \theta \in [0, 2\pi), \quad u \in U,$$

Let $A \in \mathcal{L}(H)$ be the multiplication operator by $1/\varphi(e^{i\theta})$:

$$(Af)(e^{i\theta}) = f(e^{i\theta})/(R(\theta)e^{i\theta}), \quad \theta \in [0, 2\pi), \quad f \in L^2(\mathbb{T}; Y).$$

Define $C \in \mathcal{L}(H;Y)$ by

$$Cf = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta, \quad f \in L^2(\mathbb{T};Y).$$

Finally, let $D := g(0) \in \mathcal{L}(U,Y)$. Let $\Lambda \in Y^*$ and $u \in U$. By applying Theorem 1.2 to $\Lambda g(\cdot)u \in H^\infty(\mathbb{D})$, we see that

$$\Lambda g(z)u = \Lambda Du + \Lambda Cz(I - zA)^{-1}Bu = \Lambda(D + Cz(I - zA)^{-1}B)u, \quad z \in \mathbb{D},$$

and that $S^{-1} = \mathbb{T} \cap \rho(A)$. Since the choice of $\Lambda \in Y^*$ and $u \in U$ was arbitrary, it follows that (4.1) holds. □

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