Rendezvous Search with Revealed Information: Applications to the Line

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Abstract

The symmetric rendezvous problem on a network Q asks how two players, forced to use the same mixed strategy, can minimize their expected meeting time, starting from a known initial distribution on the nodes of Q. This minimum is called the (symmetric) 'rendezvous value' of Q.

Traditionally, the players are assumed to receive no information during the play of the game. We consider the effect on rendezvous times of giving the players some information about past actions and chance moves, enabling them to apply Bayesian updates to improve their knowledge of their partner's whereabouts. We consider the case where they are placed a known distance apart on the line graph Q ('symmetric rendezvous on the line'). These techniques can be used to give lower bounds on the rendezvous times of the original game (without any revealed information). Our approach is to concentrate on a general analysis of the effect of revelations, rather than compute the best bounds possible with our techniques.

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1 Introduction

The paper considers how revealing information can affect an important unsolved problem in rendezvous theory: the symmetric rendezvous problem Γ on the line or line graph (whose nodes are the integers Z) posed in [1]. (In fact, with the recent solution by Weber [24] of the symmetric rendezvous problem on the complete graph K_3, Γ is arguably the main unsolved problem.) The problem Γ begins with a chance move, called the *initial configuration* α. This move consists both of the placement of two agents (rendezvousers) I and II at nodes of Z a known distance (taken as 2) apart, and their orientation (facing them left or right, independently). The random orientation models the assumption that the players do not have a common notion of direction along the line. The players then choose paths s_i and s_j with respect to their forward direction, using independent randomization from a *common* mixed strategy μ. This constraint is what makes the game *player-symmetric*, modeling the assumption that the agents cannot meet beforehand to agree a joint strategy. (If they can, this is the *player-asymmetric*, or *distinguishable player*, version studied in [5].) Given these choices, they arrive at some common node at a first (meeting) time $T = T(s_i, s_j, \alpha)$. Their common aim is to minimize (over μ) the expected value of the meeting time T, with the expectation taken with respect to both μ and the given distribution of the initial configurations α. This minimum time is called the (symmetric) *rendezvous value* $v = v(\Gamma)$ of G and the minimizing distribution $\bar{\mu}$ is called the *optimal* mixed strategy. For such symmetric rendezvous problems, one could equally take the viewpoint that μ is chosen by a single optimizer, such as the author of a rendezvous protocol which is routinely given to people such as hikers, or programmed into a control unit of parallel computers to tell them when and where to exchange their information. Such problems were first studied by Anderson and Weber [11] for the complete graphs K_m and a general formulation, including continuous time and more complex notions of orientation, was subsequently given by the author [1].

Symmetric rendezvous problems on graphs [4] are traditionally studied under the assumption that the players receive no information about the other's whereabouts or strategy during the course of the play. In this paper we vary the 'no revealed information' assumption so that, after a known number n of moves, a signal is given to both players which depends on their actions and on chance moves up to that time. Suppose that their possible actions (n-move sequences of forwards or backwards) are $s_1, \ldots, s_m$ and the initial chance moves are $\alpha_1, \ldots, \alpha_K$. The state of the game at time n depends on the triple (s_i, s_j, α_k), where s_i and s_j are the sequences chosen by I and II, and Nature chooses α_k. For some triples, $T(s_i, s_j, \alpha_k) \leq n$, in which case the game is already over at time n. For the rest, forming the *uncertainty set* $\mathcal{U} = \mathcal{U}_n$, the game continues. The signal received by the players consists of being told which element P of a given partition $\mathcal{P}$ of $\mathcal{U}$ contains the state (s_i, s_j, α_k). If the original rendezvous problem (game) is called Γ, we call this revealed information version (where the players know n and $\mathcal{P}$ from the outset) $\Gamma(n, \mathcal{P})$. After learning P, and knowing s_i, Player I (for example) can update his priors on α and II's action s_j, given the knowledge of the common mixed strategy μ that both have used to pick their actions. The players then play optimally in the subgame $\Gamma(n, P)$, which may or may not be a symmetric game, depending on P. The trick is to choose the partition $\mathcal{P}$ so that optimal play in each subgame $\Gamma(n, P)$ can be determined. We can then ensure that the rendezvous value $v(\mathcal{P})$ of the revealed action game $\Gamma(n, \mathcal{P})$ can be calculated (either definitively or in terms of v), yielding the inequality

$$v(\mathcal{P}) \leq v,$$

because the additional information in $\mathcal{P}$ cannot increase the rendezvous time (it can always be ignored). The same observation shows that coarser partitions $\mathcal{P}$ give higher $v(\mathcal{P})$ and hence stronger lower bounds on the original rendezvous value v. Such lower bounds on rendezvous values are usually difficult to come by.

We apply these notions to the symmetric rendezvous problem on the line. The literature on this problem will be discussed in Section 2, and the application of revealed actions to it will be analyzed in Sections 3 to 6, constituting the bulk of the paper. The only versions of revealed information in this context seem to be the papers of Baston and Gal [14] on the line, where the players learn their partner's starting point if they ever reach it (by observing a marker left there) and Anderson and Weber [11] (who mention the possibility of players leaving messages at locations they have visited - but they do not allow this in the versions of the K_m problem that they analyze).

A revealed information version of the Anderson-Weber paper, published about the same time, is the paper of Crawford and Haller [15] on 'repeated coordination games', which considers the complete graph K_m (m locations) with the players' locations (actions) revealed *after each move.* For small m, there are nice simple solutions: If $m = 2$, the best the players can do is randomize forever on the two locations, meeting in the Bernoulli expected time $1/(1/2) = 2$ This is equivalent to Anderson and Weber's result [11] for the complete graph K_2 on two nodes, since in that case the other player's location is 'revealed' (by non-meeting). If $m = 3$ and their first locations are distinct, the players should both go to the unique location unoccupied in the first period where they will meet at time 2 - hence the least expected meeting time is

$$1/3\,(1) + 2/3\,(1+1) = 5/3. \tag{1}$$

If $m = 5$, two strategies are salient: Assuming their initial locations are distinct, they can (i) coordinate on these two locations forever (essentially playing the $m = 2$ game), meeting in expected time $(1/5)(1) + (4/5)(1+2) = \dfrac{13}{5}$; or they can (ii) go randomly among the three unoccupied locations and, if these are distinct, coordinate on the fifth location - with lower expected meeting time

$$\frac{1}{5}(1) + \frac{4}{5}\left(1 + \frac{1}{3}(1) + \frac{2}{3}(2)\right) = \frac{7}{3}.$$

2 The Symmetric Rendezvous Problem Γ on the Line

In this section we formalize the description of Γ, the player-symmetric rendezvous problem on the line, and describe some of the literature giving upper bounds on the rendezvous value v of Γ.

Γ begins with a chance move in which the two players are placed a distance $D = 2$ apart on the real line (or two nodes away on the line graph Z), and randomly faced in directions that each calls 'forward' forever. From an observer's perspective, Player I is placed on the left of Player 2 and Nature chooses equiprobably among the following four initial configurations α_k, $k = 1, 2, 3, 4$. Each of these gives a ± 1 direction $d_1(k)$ to I and $d_2(k)$ to II. This chance move models the assumption that the players have no common notion of direction along the line, so that their first move must be in a random direction, independent of the direction of their partner's first move.

k	α_k	d_1	d_2
1	$\leftarrow \quad \rightarrow$	-1	$+1$
2	$\rightarrow \quad \rightarrow$	$+1$	$+1$
3	$\leftarrow \quad \leftarrow$	-1	-1
4	$\rightarrow \quad \leftarrow$	$+1$	-1

Figure 1: Initial Configurations

Since we will need to display three dimensional (i, j, k) arrays, we will often put the final coordinate k into an (i, j) entry as a 2×2 matrix in the following form

$\alpha_1 : \ \leftarrow \quad \rightarrow$	$\alpha_2 : \ \rightarrow \quad \rightarrow$
$\alpha_3 : \ \leftarrow \quad \leftarrow$	$\alpha_4 : \ \rightarrow \quad \leftarrow$

(2)

Figure 2: Initial configurations in 2×2 matrix form

In each period $t = 1, 2, \ldots$, a player either moves one unit distance in his forward (F) or backward (B) direction. (A justification of this reduction to discrete moves can be found in [17] and for a related context in [20].) So a pure strategy is an infinite sequence of F's and B's. (By symmetry of the line, we may assume without loss of generality that strategies begin with an F.) For numerical calculations, we take $F = 1$ and $B = -1$. If a player adopts strategy $s = (s(1), s(2), \ldots)$, with $s(t) \in \{F, B\} = \{+1, -1\}$, his net motion at time t from his start, in his forward direction, is given by

$$\hat{s}(t) = \sum_{r=1}^{t} s(r). \tag{3}$$

If I adopts strategy s, II adopts s', and Nature picks the initial configuration α_k, then the directed distance from II to I at time t (II's location minus I's location) is given by

$$D\left(s, s', \alpha_k\right)(t) = \left[d_2(k)\ \ \hat{s}'(t) + 2\right] - \left[d_1(k)\ \ \hat{s}(t)\right] \tag{4}$$

so the players meet at time

$$T\left(s, s', \alpha_k\right) = \min_t \left\{t : D\left(s, s', \alpha_k\right)(t) = 0\right\}. \tag{5}$$

Since α_k is picked equiprobably by Nature (chance), their expected meeting time is given by the averaged function (which we give the same name T),

$$T\left(s, s'\right) = \frac{1}{4} \sum_{k=1}^{4} T\left(s, s', \alpha_k\right). \tag{6}$$

For example, if s begins with FFB and s' begins with FBF, it is easy to see that the meeting times corresponding to the four initial configurations α_k, given in the two dimensional ordering of (2), as

?	2
?	1

where the entry '?' signifies that the players have not met by time 3. That is, $T(s, s', k) > 3$, for $k = 1, 3$.

The problem for the players (or for the author who writes a book which they each read for instructions) is to find the strategy which, when simultaneously adopted by both players, minimizes their expected meeting time. Clearly no pure strategy s is any good for this, as in initial configurations α_2 and α_3 (the ones where they face in the same direction), their initial distance of 2 is preserved, and they never meet. So the problem is to find the mixed strategy μ which minimizes the expected meeting time. (The existence of a minimizing μ is guaranteed by the analysis given in [1].)

If the players *do* have a common sense of direction along the line, we may take this direction to be Player I's forward direction, so we only average over the two cases $k = 2, 4$. The rendezvous value v^c for this problem certainly cannot exceed v, as the players could choose to ignore this information. It has recently been observed by Shmuel Gal, and analyzed in Alpern and Gal [7], that if both players adopt FBB in the first three moves and have not met by then, then they know that $k \in \{2, 3\}$. That is, they they both chose their forward direction the same. Consequently they are now playing the common-direction game Γ^c, with value v^c. It is widely thought that in the symmetric rendezvous problem on the line, having a common sense of direction does not help the players. Weber [24] has shown that on the triangle graph K_3, having a common notion of clockwise does not help the players, which tends to lend credence to the following.

Common Direction Conjecture (CDC): $v^c = v$.

In the author's original paper [1] a very simple strategy was proposed (and naively conjectured to be optimal): In each three-period time interval, choose a random direction to call forward, and move FBB. If k is 2 or 3, the respective meeting times are 1 and 3. In the remaining cases ($k = 2, 4$), the players move in parallel and at the end of the last move they again face the original game Γ. Hence the expected meeting time T for this mixed strategy satisfies

$$T = \frac{1}{4}(1) + \frac{1}{4}(3) + \frac{1}{2}(3 + T), \text{ or } T = 5. \tag{7}$$

Subsequent authors have successively obtained lower expected meeting times by judicious choices of sets of longer move sequences (and appropriate probability distributions): Anderson and Essegaier [9] used $\{FFBBB,\ FBBBFF,\ FBBFBB,\ FBFBBB\}$ and Baston [12] used $\{FBBFFBB,\ FBBBFFF,\ FBFFFBBBF,\ FFBBB\}$. Recent work by Uthaisonbut [23] and further work by Han, Du, Vera and Zuluaga [17] have reduced the best known the upper bound on v to 4.574. The latter have conjectured that $v = v^c = 4.25$.

3 Revealed Information and Partitions

Thus far, the literature of rendezvous search theory has been restricted to the case where the players receive no additional information in the course of play - except of course that they have not yet met their partner. So strategies are simply paths, carried out under the assumption that no meeting has taken place. This is in contrast to most other dynamic games in which information (about other player's actions or chance moves) is revealed at various stages. Cards being turned over constitute an important example of revealed information in parlour games.

Unlike Crawford and Haller, we consider games which give the players information feedback in Γ only once, at a preassigned time n. Take $n = 2$, for example. Recalling our observation that players may begin with an F, there are only two move sequences of length 2, namely $s_1 = FF$ and $s_2 = FB$. So the Action set $\mathcal{A} = \mathcal{A}_2$ which describes all the player and Nature choices up to time 2 is the $2 \times 2 \times 4 = 16$ element set $\{(i, j, k) : i, j \in \{1, 2\}, k \in \{1, 2, 3, 4\}\}$, where (i, j, k) stands for the actions s_i for Player I, s_j for Player II, and initial configuration α_k. If the players have not met by time 2, they know that actual triple (i, j, k) is one for which $T(i, j, k) \equiv T(s_i, s_j, \alpha_k) > 2$. This subset of $\mathcal{A}_2$ is called the *uncertainty set* $\mathcal{U}_2$.

The information we give the players after move 2 is described by a partition $\mathcal{P}$ of $\mathcal{U}_2$. If $(i, j, k) \in \mathcal{U}_2$ is played, the players are told the element P of $\mathcal{P}$ which contains it. We call such a game $\Gamma(2, \mathcal{P})$ (or more generally $\Gamma(n, \mathcal{P})$, if they are told the element of the partition $\mathcal{P}$ of $\mathcal{U}_n$ after move n). The set $\mathcal{U}_2$ consists of the 10 elements with '?' as the entry in the three dimensional array $T(i, j, k)$ written below in (9), (with the third dimension k written as a 2×2 matrix in the ordering of (2) . That is, $\mathcal{U}_2$ is the set

$$\left\{ \begin{array}{c} (1,1,1), (1,1,2), (1,1,3), (1,2,1), (1,2,3), \\ (2,1,1), (2,1,2), (2,2,1), (2,2,2), (2,2,3) \end{array} \right\} \tag{8}$$

$$\begin{array}{c|c|c|} & s_1 = FF & s_2 = FB \\ \hline s_1 = FF & \begin{array}{|c|c|} \hline ? & ? \\ \hline ? & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline ? & 2 \\ \hline ? & 1 \\ \hline \end{array} \\ \hline s_2 = FB & \begin{array}{|c|c|} \hline ? & ? \\ \hline 2 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline ? & ? \\ \hline ? & 1 \\ \hline \end{array} \\ \hline \end{array} \tag{9}$$

When partitions of a fixed set are considered as a variable, the two to look at first are the partition $\mathcal{P}^0$ into one set and the partition $\mathcal{P}^1$ into singletons. The first gives no new information, so $\Gamma(n, \mathcal{P}^0)$ is the same problem, or game, as Γ. The second, $\mathcal{P}^1$ gives the players complete information, so the resulting subgame at time n is easy to optimize - simply go towards the other player. So if the distance between the players at time n is $d = D(s_i, s_j, \alpha_k)$, they will optimally meet at time $n + d/2$. So if we replace all the '?' entries with $n + d/2$, this represents the best the players can do in each case (i, j, k). In fact, this is what was done by Uthaisonbut [23] to obtain a lower bound for the value v of Γ, without regard to information structures. For $n = 2$, this fills in the table (9) as follows.

$$\begin{array}{c|cc|cc|}
 & s_1 = FF & & s_2 = FB & \\
\hline
\multirow{2}{*}{$s_1 = FF$} & 2+6/2 & 2+2/2 & 2+4/2 & 2 \\
 & 2+2/2 & 1 & 2+4/2 & 1 \\
\hline
\multirow{2}{*}{$s_2 = FB$} & 2+4/2 & 2+4/2 & 2+2/2 & 2+2/2 \\
 & 2 & 1 & 2+2/2 & 1 \\
\hline
\end{array} \tag{10}$$

Averaging the values over $k = 1, 2, 3, 4$ gives a corresponding lower bound matrix for $\Gamma\left(2, \mathcal{P}^1\right)$:

$$B = B\left(2, \mathcal{P}^1\right) = \frac{1}{4}\begin{pmatrix} 12 & 11 \\ 11 & 10 \end{pmatrix}. \tag{11}$$

Now the players know they are playing $\Gamma\left(2, \mathcal{P}^1\right)$ so that they are facing the payoffs B. Hence they choose to play s_i with the probabilities p_i so that the probability vector $p = (p_1, p_2)$ minimizes pBp^T. Thus the value of the game $\Gamma\left(2, \mathcal{P}^1\right)$ is given by

$$V\left(2, \mathcal{P}^1\right) = \min_p pBp^T = (0,1)\,B\,(0,1)^T = 2.5, \tag{12}$$

with the pure strategy $s_2 = FB$ played exclusively. Since the players have (much) more information in the game $\Gamma\left(2, \mathcal{P}^1\right)$ than in the original game Γ, we have the lower bound of [23],

$$V(\Gamma) \geq 2.5. \tag{13}$$

Of course since $\mathcal{P}^1$ is the finest partition, this will be the smallest lower bound obtainable by our methods. By analyzing coarser partitions, we will get higher lower bounds (without increasing n). But to do this we must have methods for analyzing the subgames corresponding to the elements of the partitions that we consider. Some of the subgames that occur will be analyzed in the following section.

A third general partition that we refer to is the one that tells each player the actions of the other. We will call this partition $\mathcal{P}^3$. Its element are the sets $U_{i,j} = \{(i', j', k) \in \mathcal{U} : (i', j') = i, j\}$ which reveal all player actions but no chance moves. To summarize, we have defined three partitions of $\mathcal{U} = \mathcal{U}_n$:

$\mathcal{P}^1$	partition into singletons - all actions by chance and players revealed	(14)
$\mathcal{P}^2$	trivial partition into single element $\mathcal{U}$ - nothing is revealed	
$\mathcal{P}^3$	partition into sets $U_{i,j}$ - elements reveal player actions only	

4 Analysis of Subgames Γ^* , $\bar{\Gamma}(d, e)$and Γ^c

In the matrix (10) of the previous section, we labeled certain entries as $n + d/2$, where d was the distance between the players at time n. The number $d/2$ is trivially the value of the simple rendezvous problem where two players are placed a distance d apart and told the direction to the other player. This game is too simple to be given a name, but for some other subgames that arise we give a name and analyze best play and the corresponding rendezvous value.

The first subgame to be considered here, called Γ^*, is the symmetric game that arises from Γ at time 1 if the players have not met by then. This is in fact a sufficient description of Γ^*, but for completeness we give an explicit definition as well.

Definition 1 *In Γ^* player I is placed at* 0*, facing right. Player II is placed equiprobably in the following three situations: (i) at* 2*, facing right; (ii) at* -2*, facing right; at* -4*, facing left. The initial placement is identical from both players' points of view, so the setup is symmetric, and we require a symmetric solution.*

We next consider a family of asymmetric subgames $\bar{\Gamma}(d,e)$, $d > e \geq 0$, which arise from the partition element $\{(i,j,1),(i,j,3)\}$, with $d = 2 + \triangle_i$ and $e = |\triangle_j|$, taking II's starting position as the origin, and letting

$$\triangle_i = \hat{s}_i(n), \tag{15}$$

be the excess of F's to B's in the $n-$move sequence s_i.

Definition 2 *The game $\bar{\Gamma}(d,e)$, $d > e \geq 0$, is played on a (commonly) labeled line, so the players know where they are. Player I is placed equiprobably at $\pm d$ and Player II is placed equiprobably at $\pm e$.*

In the next section we will need to know the rendezvous values of these two games. They are easy to calculate, as we do in the following.

Lemma 3 *Let v denote the value of the original game Γ, v^* the value of Γ^*, $\bar{v}_{d,e}$ the value of $\bar{\Gamma}(d,e)$, and v^c the value of the common-direction game Γ^c. Then*

1. $v^* = \dfrac{4v-4}{3}$, *and*
2. $\bar{v}_{d,e} = \dfrac{3d-e}{4}$, *for $d > e$.*
3. $v^c \geq 2v - 5$.

Proof.

1. We know by compactness (see [1]) that the value v for the original game exists. We have shown that we can assume that the first move of both players is F. Consider the situation after 1 move. In initial configuration α_4 $(k=4)$ the players are facing each other, and meet in time $T=1$. In the remaining configurations $k = 1,2,3$, $T > 1$. Consider the situation from the point of view of say Player II after 1 move, who now views his current location as 0 and the direction he has just moved as 'right'. If the configuration was $k=1$, then from his point of view the state of I is location -4 and his forward direction must be left; if $k=2$, I must be at -2 with forward direction right; if $k=3$, then I must be at location $+2$ with forward direction right. Since the three possibilities for k are equiprobable, I's relative state to II is as in Γ^*. The same is true for II's state relative to I. So with probability $1/4$ (if $k=4$) we have $T=1$ and with probability $3/4$ $(k=2,3,4)$ the minimum value of T is $1+v^*$ (assuming optimal play in Γ^* starting at real time 1). Hence we have
$$v = \frac{1}{4}(1) + \frac{3}{4}(1+v^*), \quad \text{or } v^* = \frac{4v-4}{3}. \tag{16}$$

2. Since $d > e$, Player I knows (or can infer) the direction of II (namely towards 0), and hence moves in that direction (thin lines in Figure 3). The asymmetric solution is fairly obvious. Assuming (without loss of generality) that II is placed at $+e$ on a vertical line, he goes up to met one path of Player I (starting at $+d$), then follows that path down until it meets the other Player I path (starting at $-d$). This path is indicated by a thick line. In general, against a finite set of known paths (e.g. the two possible paths of Player I), the other player's optimal strategy is of the form where he meets one of the paths as soon as possible, then another, and so on ([5], repeated as Theorem 16.10 of [6]). In the present situation, this means that Player II either meets the path starting at $+d$ and then the one starting at $-d$ (the best strategy, as indicated in the thick line), or the other way around (worse). Consequently with best play the players meet equiprobably at time $(d - e)/2$ and at time d, hence

$$\bar{v}_{d,e} = \frac{1}{2}\left(\frac{d-e}{2}\right) + \frac{1}{2}(d) = \frac{3d-e}{4}. \tag{17}$$

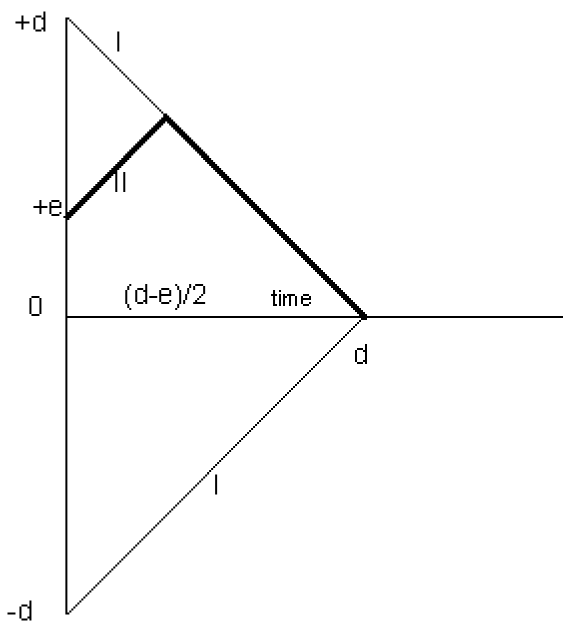


Figure 3: Optimal play in $\Gamma(d, e)$

3. Recall that v^c is the value of the common-direction game Γ^c. Consider the strategy FBB followed by optimal play in the common-direction game Γ^c that results if no meeting takes place by time 3. Clearly v cannot be more than the expected meeting time for this strategy, so

$$v \leq \frac{1}{4}(1) + \frac{1}{4}(3) + \frac{1}{2}(3 + v^c). \tag{18}$$

Solving for v^c gives the required inequality.

$\blacksquare$

5 Analysis of $\Gamma\left(n, \tilde{\mathcal{P}}\right)$

In this section we define an information partition $\tilde{\mathcal{P}} = \tilde{\mathcal{P}}_n$ and find lower bounds for the subgames corresponding to its elements. The partition $\tilde{\mathcal{P}}$ refines the partition $\mathcal{P}^3$ (see (14)) which reveals just player actions. For $n = 2$, $\mathcal{P}^3$ is the partition of the 10 elements of U_2 (the '?'s) into the four boxes in (9). Thus in the partition $\tilde{\mathcal{P}}$, the players are told the moves of the other player and perhaps something about the chance moves. So to define $\tilde{\mathcal{P}}$ we have to define a further partition of each element $U_{i,j}$ of $\mathcal{P}^3$.

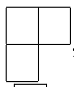
So fix n and for convenience number the 2^{n-1} n-move strategies s_i, $i = 1, \ldots, 2^{n-1}$ (since they all start with F) so that if $i < j$ then a player using s_i, and facing the other player who is facing away from him (initial configuration α_2) will catch him by time n. That is,

$$T(i, j, 2) \le n, \text{or } (i, j, 2) \notin \mathcal{U}, \text{for } i < j. \tag{19}$$

For example, if $n = 3$, we have

$$s_1 = FFF,\ s_2 = FFB,\ s_3 = FBF,\ s_4 = FBB. \tag{20}$$

By symmetry, we need only analyze i, j with $i \le j$. A particular ordering of the s_i which satisfies (19) and in fact gives (20) for $n = 3$ can be defined as follows: Let $\gamma = \gamma(s, s') \equiv \min\{r : s(r) \ne s'(r)\}$ (where $\hat{s}$ is defined as in (3)) and use the order $s \prec s'$ iff $s(\gamma) = F$. We are listing the s_i in dictionary (lexicographic) order (although with the convention that F appears lexicographically before B).

Note that since $T(i, j, 4) = 1$ for all i, j, the set $U_{i,j}$ is at most the three element set , using the two dimensional ordering on the left of (2). Furthermore, if $i < j$, it is at most  (left column), by (19).

We are now in a position to describe how $\tilde{\mathcal{P}}$ partitions each set $U_{i,j}$. This depends on i and j in four cases described below.

$\mathbf{U_{i,j}} = \{(\mathbf{i}, \mathbf{j}, \mathbf{k})\}$ **,singleton, for some** k In this case both players know the state of the game, namely (i, j, k) , and know the direction and distance

$$\triangle_{i,j,k} = D(s_i, s_j, \alpha_k) > 0 \tag{21}$$

to the other player. Consequently they will go at unit speed towards each other and their meeting time will be $n + \triangle_{i,j,k}/2$. So if I chooses s_i and II chooses s_j, their expected meeting time $b_{i,j}$ in the game $\Gamma\left(n, \tilde{\mathcal{P}}\right)$ is given by

$$4\ b_{i,j} = n + \triangle_{i,j,k}/2 + \sum_{k' \ne k} t_{i,j,k'} \tag{22}$$

In the remaining cases we assume that $U_{i,j}$ has at least two elements, and we describe how they are partitioned in $\tilde{\mathcal{P}}$.

i < j By (19) and the fact that $t_{i,j,4} = 1$ for all i, j, we have $U_{i,j} = \{(i, j, 1), (i, j, 3)\}$ which we have drawn below to illustrate the fact that if we take $U_{i,j}$ as a single element of $\tilde{\mathcal{P}}$, then Player I (left) knows that II is behind him (in his backwards direction).

$$\begin{array}{|cc|}\hline \leftarrow & \rightarrow \\ \hline \leftarrow & \leftarrow \\ \hline \end{array} \qquad (k = 1, 3) \tag{23}$$

$$\tag{24}$$

Both know that I's location at time n is distance $d = 2 + \triangle_i$ from II's start, and that II is distance $e = |\triangle_j|$ from his own start. Hence they know they are playing the game $\bar{\Gamma}(2 + \triangle_i, \triangle_j)$ with value $\bar{v}_{2+\triangle_i,\triangle_j}$. So in this case the meeting times corresponding to the four initial configurations k are

$$\begin{array}{|c|c|}\hline n + \bar{v}_{2+\triangle_i,\triangle_j} & t_{i,j,2} \\ \hline n + \bar{v}_{2+\triangle_i,\triangle_j} & 1 = t_{i,j,4} \\ \hline \end{array}, \tag{25}$$

and the expected meeting time $b_{i,j}$ satisfies

$$4b_{i,j} = 2\left(n + \bar{v}_{2+\triangle_i,\triangle_j}\right) + t_{i,j,2} + 1. \tag{26}$$

i = j, $\mathbf{t_{i,i,1}} \leq$ n Note that if both players use the same move-sequence s_i, they will not meet if they are facing the same direction ($k = 2, 3$), so certainly $U_{i,i} = \{(i, i, 2), (i, i, 3)\}$. In this case we take $U_{i,i}$ as an element of $\tilde{\mathcal{P}}$. If the players are told they are in $U_{i,i}$ they can conclude they are still distance 2 apart and have moved in the same direction. Hence they are playing the common direction game Γ^c and with best play will meet in additional time v^c. Hence the entries for the i, i matrices of this type for $\Gamma\left(n, \tilde{\mathcal{P}}\right)$ are

$$\begin{array}{|c|c|}\hline t_{i,i,1} & n + v^c \\ \hline n + v^c & 1 \\ \hline \end{array} \text{ and } 4b_{ii} = t_{i,i,1} + 2n + 2v^c + 1, \text{ if } t_{i,i,1} \leq n. \tag{27}$$

i = j, $\mathbf{t_{i,i,1}} >$ n, $|\triangle_i| \neq$ 1 Since $t_{i,i,1} > n$, we have $U_{i,i} = \{(i, i, 1), (i, i, 2), (i, i, 3)\}$. Since $t_{i,i,1} > n$, we must also have $\triangle_i \geq 0$. There are two reasonable ways to partition the three element set $U_{ii} = \begin{array}{|cc|cc|}\hline \leftarrow & \rightarrow & \rightarrow & \rightarrow \\ \hline \leftarrow & \leftarrow \\ \cline{1-2} \end{array}$:

$$\{1, 3\}, \{2\} : \begin{array}{|c|c|}\hline \text{a} & \text{b} \\ \hline \text{a} \\ \cline{1-1} \end{array}, \quad \text{or} \quad \{1\}, \{2, 3\} : \begin{array}{|c|c|}\hline \text{c} & \text{d} \\ \hline \text{d} \\ \cline{1-1} \end{array} \tag{28}$$

Consider the first (left) case. If the players are told (a) they are in $\{(i, i, 1), (i, i, 3)\}$, then they can conclude, as above, that they are playing the asymmetric game $\bar{\Gamma}(\triangle_i + 2, \triangle_i)$, with value $\bar{v}_{\triangle_i+2,\triangle_i} = (1 + \triangle_i)/2$. If they are told (b) they are in $\{(i, i, 2)\}$, they are still distance 2 apart and can meet in additional time 1. Thus the sum of the additional meeting times in the three cases ($k = 1, 2, 3$) are

$$2((1 + \triangle_i)/2) + 1 = 2 + \triangle_i. \tag{29}$$

Next consider the second (right) case. If the players are told (c) they are in the set $\{(i,i,1)\}$, they know the other's direction and that their distance at time n is $\triangle_{i,i,1} = 2+2\triangle_i$, so they can meet in additional time $1+\triangle_i$. If they are told they are in $\{(i,i,2),(i,i,3)\}$, they are in the common-direction game Γ^c, so the additional time is v^c. Hence the sum of the three additional times is

$$1+\triangle_i+2v^c. \tag{30}$$

We are trying to maximize the entries (meeting times), so from this point of view we take the first (left) case as our partition of $U_{i,i}$ if and only if $2+\triangle_i \geq 1+\triangle_i+2v^c$, or $1 \geq 2v^c$, which is false. (Note that certainly $v^c \geq 5/3$ (1), the revealed information value, and also $v^c \geq 13/4$, the asymmetric rendezvous value [5].) So in the partition $\tilde{\mathcal{P}}$, we use the second (right) partition in (28) of $U_{i,i}$. Hence we have that $4b_{i.i}$ is the sum of the four entries corresponding to the values of $k=1,2,3,4$, as given in

$$\begin{array}{|c|c|}\hline n+1+\triangle_i & n+v^c \\ \hline n+v^c & 1 \\ \hline\end{array}\text{, so} \tag{31}$$

$$4b_{i,i} \;=\; 3n+2v^c+2+\triangle_i,\ \text{if } t_{i,i,1}>n \text{ and } |\triangle_i| \neq 1. \tag{32}$$

$i=j, t_{i,i,1}>n$, $\mathbf{i=j, t_{i,i,1}> n, |\triangle_i|=1}$ Note that $|\triangle_i|=1$ implies that n is odd, so this case can only occur for odd n. As in the previous case, we have $U_{i,i}=\{(i,i,1),(i,i,2),(i,i,3)\}$. Take $U_{i,i}$ as an element of $\tilde{\mathcal{P}}$. If the players are told that they are in this element of $\tilde{\mathcal{P}}$, they can conclude that they are in the symmetric game Γ^* defined in the previous section, and hence that the minimal remaining expected time to meet is v^*. Hence we have

$$4b_{i,i}=3(n+v^*)+1, \text{if } t_{i,i,1}>n \text{ and } |\triangle_i|=1. \tag{33}$$

Observe that this is a larger value than in the previous case (31), as

$$\begin{aligned} & [3(n+v^*)+1]-[3n+2v+2+\triangle_i] && (34)\\ \geq\;& \left(3\left(n+\frac{4v-4}{3}\right)+1\right)-(3n+2v+2) && (35)\\ =\;& 2v-5>2(13/4)-5>0, && (36)\end{aligned}$$

since the value v of the symmetric game Γ cannot be smaller than the value 13/4 of the corresponding asymmetric game (see [5] or Theorem 9 of [6]).

For the ordering (20) when $n=3$, the partition and the numbers making up $b_{i,j}$ can be seen from the following table. The set $\mathcal{U}_3$ consists of all the entries which are not written as a single number (including entries like $3+8/2$). The partition $\tilde{\mathcal{P}}$ of the non-number entries in each 2×2 is based on identical form entries. By symmetry of the players (and the matrix B), we can confine our analysis to strategy pairs $i \leq j$.

$$
\begin{array}{lcccc}
i\backslash j & 1{:}FFF & 2{:}\ FFB & 3{:}\ FBF & 4{:}\ FBB \\
1{:}\ FFF & \begin{array}{|c|c|}\hline 3+8/2 & 3+v^c \\ \hline 3+v^c & 1 \\ \hline\end{array} & \begin{array}{|c|r|}\hline 3+\bar{v}_{5,1} & 3 \\ \hline 3+\bar{v}_{5,1} & 1 \\ \hline\end{array} & \begin{array}{|c|r|}\hline 3+\bar{v}_{5,1} & 2 \\ \hline 3+\bar{v}_{5,1} & 1 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 3+\bar{v}_{5,1} & 2 \\ \hline 3+\bar{v}_{5,1} & 1 \\ \hline\end{array} \\
2{:}\ FFB & & \begin{array}{|c|c|}\hline 3+v^* & 3+v^* \\ \hline 3+v^* & 1 \\ \hline\end{array} & \begin{array}{|c|r|}\hline 3+\bar{v}_{3,1} & 2 \\ \hline 3+\bar{v}_{3,1} & 1 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 3+\bar{v}_{3,1} & 2 \\ \hline 3+\bar{v}_{3,1} & 1 \\ \hline\end{array} \\
3{:}\ FBF & & & \begin{array}{|c|c|}\hline 3+v^* & 3+v^* \\ \hline 3+v^* & 1 \\ \hline\end{array} & \begin{array}{|c|r|}\hline 3+\bar{v}_{3,1} & 3 \\ \hline 3+\bar{v}_{3,1} & 1 \\ \hline\end{array} \\
4{:}\ FBB & & & & \begin{array}{|c|c|}\hline 3 & 3+v^c \\ \hline 3+v^c & 1 \\ \hline\end{array}
\end{array}
\tag{37}
$$

If we combine all the cases, use (16) and (17) to evaluate v^* and $\bar{v}$ in terms of v, assume the CDC ($v^c = c$) we obtain the following.

Theorem 4 *Let $\tilde{\mathcal{P}}$ be the partition of $\mathcal{U}_n$ described above, and let $\Gamma\left(n, \tilde{\mathcal{P}}\right)$ be the game in which the players play symmetrically as in Γ for n moves and if they haven't met by then are told the element of the $\tilde{\mathcal{P}}$ to which the action triple (i, j, k) belongs. If the players choose $n-$move strategies s_i and s_j and then play optimally if they haven't met by time n, then assuming CDC, the expected time $\tilde{b}_{i,j}$ to meet $(i \le j)$ is given by*

$$
\tilde{b}_{i,j} = \frac{1}{4}\begin{cases}
2n + (3\triangle_i - |\triangle_j|)/2 + t_{i,j,2} + 4, & \text{if } i < j\ ,\ t_{i,j,1} > n,\ t_{i,j,3} > n \\
[n + (\triangle_i + \triangle_j + 2)/2] + t_{i,j,2} + t_{i,j,3} + 1, & \text{if } i < j\ ,\ t_{i,j,1} > n,\ t_{i,j,3} \le n \\
n + (\triangle_i - \triangle_j)/2 + t_{i,j,1} + t_{i,j,2} + 2, & \text{if } i < j\ ,\ t_{i,j,1} \le n,\ t_{i,j,3} > n \\
t_{i,j,1} + t_{i,j,2} + t_{i,j,3} + 1, & \text{if } i < j\ ,\ t_{i,j,1} \le n,\ t_{i,j,3} \le n \\
t_{i,i,1} + 2(n + v) + 1, & \text{if } i = j,\ t_{i,i,1} \le n \\
3n + 2v + 2 + \triangle_i, & \text{if } i = j, t_{i,i,1} > n, |\triangle_i| \ne 1 \\
3n + 4v - 3, & \text{if } i = j, t_{i,i,1} > n, |\triangle_i| = 1
\end{cases}
\tag{38}
$$

The matrix $\tilde{B} = \tilde{B}_n = \left\{\tilde{b}_{i,j}\right\}$ is symmetric. It has entries which are constants or linear functions of v, with positive coefficients of v. The(rendezvous) value $\tilde{v} = \tilde{v}_n$ of the symmetric rendezvous problem $\Gamma\left(n, \tilde{\mathcal{P}}\right)$ is given by

$$
\tilde{v}_n = \min_{p: p_i \ge 0, \sum p_i = 1} p\tilde{B}_n p^T \le v. \tag{39}
$$

If we don't assume CDC the result remains the same, with v replaced by v^c in the next to last row (case) in the definition of $\tilde{b}_{i,j}$.

We illustrate the construction of $\tilde{B}_n$ and the calculation of $\tilde{v}_n$ by considering the case $n = 3$, with the strategy ordering $s_1 = FFF$, $s_2 = FFB$, $s_3 = FBF$, $s_4 = FBB$. Using the formula for $\tilde{b}_{i,j}$ given above, the form (37), we obtain

$$\tilde{B} = \tilde{B}(v) = \frac{1}{4}\begin{pmatrix} 2v^c+14 & 17 & 16 & 16 \\ 17 & 4v+6 & 13 & 13 \\ 16 & 13 & 4v+6 & 14 \\ 16 & 13 & 14 & 2v^c+10 \end{pmatrix} \tag{40}$$

$$= \frac{1}{4}\begin{pmatrix} 2v+14 & 17 & 16 & 16 \\ 17 & 4v+6 & 13 & 13 \\ 16 & 13 & 4v+6 & 14 \\ 16 & 13 & 14 & 2v+10 \end{pmatrix}, \text{ under CDC.} \tag{41}$$

We want to find the value $\tilde{v} = \tilde{v}_3$ in terms of v, that is

$$\tilde{v}_n(v) \equiv g(v) = \min_p p\tilde{B}p^T, \tag{42}$$

over probability 4-vectors p. In particular, under the CDC ($v^c = c$) we find the unique minimizing q to be

$$p_1 = 0,\ p_2 = \frac{4v-5}{16v-26},\ p_3 = \frac{4v-7}{16v-26},\ p_4 = \frac{8v-14}{16v-26}, \tag{43}$$

$$g(v) = \frac{16v^2+160v-303}{4(16v-26)}. \tag{44}$$

Since the game $\Gamma\left(3, \tilde{\mathcal{P}}\right)$ gives the players more information than in Γ, we have $v \geq g(v)$, which holds for

$$v \geq \frac{11+2\sqrt{5}}{4} \approx 3.868. \tag{45}$$

So, assuming the CDC, we have $v \geq 3.868$. The corresponding estimate if we reveal everything (use partition $\mathcal{P}^1$) at time $n = 3$ is $v \geq 3$, as obtained by Uthaisombut [23]. A related bound of $v^c > 3.5869$ for $n = 3$ has been obtained by Han, Du, Vera and Zuluaga [17] (under the assumption that the players have a common notion of direction along the line) by revealing everything if different move sequences are used and telling the agents that they have used the same move sequence, if that has occurred. Substituting this lower bound of [17] for v^c for $n = 3$ in (40) gives the bound

$$\tilde{v}_3 \geq 3.829, \tag{46}$$

which is obtained by considering only move sequences of length 3 (for both common direction and no common direction). By formulating a relaxed version of the problem as one of semi-definite programming, Han, Du, Vera and Zuluaga [17] are able to handle the large matrices that occur for n up to 7, obtaining for $n = 7$ the bound $v^c \geq 4.1520$. Bootstrapping their estimates on v^c obtained for these n would give higher lower bounds for $\tilde{v}_n$ in each case, but apparently the improvement lessens for higher n. In a literal sense, we could use these bounds to obtain better estimates for $\tilde{v}_3$, but this would be a rather unfair estimate for comparison, as those bounds require the analysis of long (n) move sequences, while (46), as mentioned above, requires only the analysis of 3-move sequences.

6 Partitions with Incomplete Information Subgames

We now consider revealed information variants $\Gamma(n,\mathcal{P})$ of the symmetric rendezvous problem Γ on the line, without the assumption that $\mathcal{P}$ refines $\mathcal{P}^3$. That is, we do not assume that the players are given information which includes the past actions of their partner. If the game is stopped at time n, then without any revealed information the private knowledge of Player I is simply the $n-$move strategy sequence he has used, namely i (for s_i). He does not know the other player's strategy sequence j, nor Nature's (chance) move k. If the partition $\mathcal{P}$ does refine $\mathcal{P}^3$, then all the subgames $\Gamma(n,P)$, $P\in\mathcal{P}$, have complete information.

Suppose however, that $\mathcal{P}$ has an element (set) P which contains elements of the information space $\mathcal{U}_n$ with distinct Player I strategies, say i and i'. In the subgame $\Gamma(n,P)$ corresponding to P, Player I knows his *type* (i or i', he knows his previous moves), but II does not know this. So the subgame has incomplete information. However, given the common distribution p over n-move strategies that both players are employing, the players can update the distribution of the other's type. So the game $\Gamma(n,P)$ is a Bayesian game.

For example when $n=3$ the elements of $\mathcal{U}_3$ for $i=1$ (FFF) and 2 (FFB) and $j=4$ (FBB) are the four elements $(1,4,1),(1,4,3),(2,4,1),(2,4,3)$. In the partition $\tilde{\mathcal{P}}$ of the previous section, these four elements were partitioned into two sets $\{(1,4,1),(1,4,3)\},\{(2,4,1),(2,4,3)\}$. That is, i and j were revealed, but not k. Recall that in the first instance this led to the subgame $\Gamma(5,1)$, with value $\bar{v}_{5,1}=7/2$; and in the second to $\Gamma(3,1)$, with value $\bar{v}_{3,1}=2$. In both games Player I knows the direction to II, so to him the distance is strategically irrelevant - he simply moves in that direction. How does this uncertainty affect II? Let p_1 and p_2 denote the probability with which I plays s_1 and s_2, respectively. So II knows that with probability p_1 Player I is starting the subgame distance five (either direction) from II's starting point in the original game. And II is distance 1 from that point (and he knows the direction).

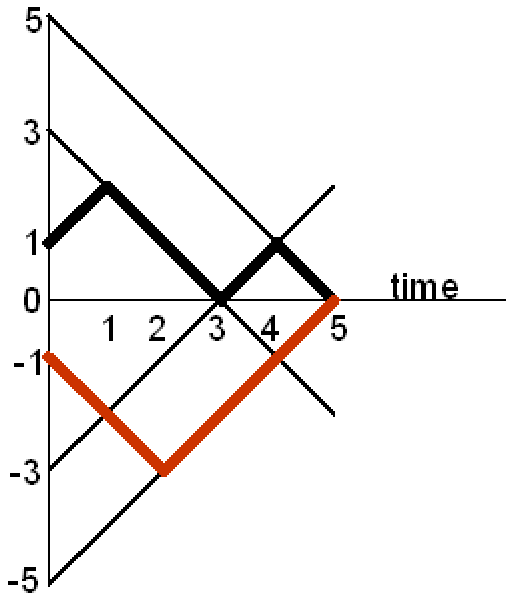


Figure 4: II knows I starts at ±3 or ±5

So I's four possible paths start at ±5 or ±3 as indicated by the four thin lines. If p_1 is sufficiently large then II clearly should take the line (which we have drawn starting at -1) which minimize the expected time to reach the equiprobable paths of I starting from ±5. Similarly if p_2 is large he should take the thick line which we have drawn starting at $+1$. (We could have drawn both II's paths starting from the same point, but it would have been less clear.) The expected meeting times for the two paths are given by

$$\begin{aligned} \textit{top} &= p_1\left(\frac{1}{2}\ \ 4+\frac{1}{2}\ \ 5\right)+p_2\left(\frac{1}{2}\ \ 1+\frac{1}{2}\ \ 3\right)=\frac{9p_1+4p_2}{2}, && (47)\\ \textit{bottom} &= p_1\left(\frac{1}{2}\ \ 2+\frac{1}{2}\ \ 5\right)+p_2\left(\frac{1}{2}\ \ 1+\frac{1}{2}\ \ 4\right)=\frac{7p_1+5p_2}{2} && (48)\end{aligned}$$

Consequently II should play the top strategy if $p_2 \geq 2p_1$ and the bottom strategy otherwise.

Now consider the partition $\tilde{\mathcal{P}}$ indicated by identical forms in (37), and change it so that all the twelve cells with entry $3+\bar{v}_{5,1}$ or $3+\bar{v}_{3,1}$ form a single element of a new partition. Player II can calculate the probability q_1 that $d=5$ or the probability q_2 that $d=3$ (in the subgame $\bar{\Gamma}(d,e)$) based on the strategy s_j that he has played up to time 3. For instance, if he has played strategy $j=4$, then $q_1 = p_1/(p_1+p_2+p_3)$. This partition (and generalizations to higher n) gives Player II less information than he had in $\Gamma\left(n,\mathcal{P}^3\right)$, and cannot result in smaller rendezvous values. For $n=3$ it does not help any, because $p_1=0$ at the optimum, but in general this idea will give higher lower bounds than can be obtained using the $b_{i,j}$ from Theorem 4.

7 Conclusions

Giving the rendezvousers some (revealed) information during the course of their search can be used both to describe real problems, where such information is available or can be sent, and also to obtain lower bounds on rendezvous values of existing problems which have no such information. In the latter case the information partition $\mathcal{P}$, describing the information to be given at time n, must be chosen carefully so that it is as coarse as possible while still having all its subgames $\Gamma(n, P)$, $P \in \mathcal{P}$, amenable to analysis. This will probably be a useful technique for certain rendezvous problems and not for others.

As observed by an anonymous referee, the information revelation described in this paper is not the only sort that might be considered. We could also consider that the revealed information depends on the mixed strategy chosen by the rendezvous team, although such a model would take us out of the usual game tree description.

References

[1] Alpern, S. (1995). The rendezvous search problem. SIAM J. Control & Optimization 33, 673-683.

[2] Alpern, S. (2002a). Rendezvous search: A personal perspective. Operations Research 50, No. 5, 772-795.

[3] Alpern, S. (2002b). Rendezvous search on labelled graphs. Naval Research Logistics 49, 256-274.

[4] Alpern, S., Baston, V., and Essegaier, S. (1999). Rendezvous search on a graph. J. Appl. Prob. 36, no. 1, 223-231.

[5] Alpern, S., and Gal, S. (1995). Rendezvous search on the line with distinguishable players. SIAM J. Control & Optimization 33, 1270-1276.

[6] Alpern, S., and Gal, S (2003). The theory of search games and rendezvous. International Series in Operations Research and Management Science, Volume 55, Kluwer Academic Publishers, Boston, 336 pp.

[7] Alpern, S., and Gal, S (2006). Two conjectures on rendezvous in K_3. Department of Mathematics, London School of Economics, LSE-CDAM-2006-21 http://www.cdam.lse.ac.uk/Reports/reports2006.html.

[8] Alpern, S., and Pikounis, M. (2000). The telephone coordination game. Game Theory Appl. 5, 1-10.

[9] Anderson, E. J., and Essegaier, S. (1995). Rendezvous search on the line with indistinguishable players, SIAM Journal of Control and Optimization 33, 1637-1642.

[10] Anderson, E. J., and Fekete, S. (2001). Two dimensional rendezvous search. Operations Research 49, 107-118.

[11] Anderson, E. J., and R. R. Weber. (1990). The rendezvous problem on discrete locations. J. Appl. Prob. 28, 839-851.

[12] Baston, V. J. (1999). Two rendezvous search problems on the line, Naval Research Logistics 46, 335-340.

[13] Baston, V. J. and Gal, S. (1998). Rendezvous on the line when the players' initial distance is given by an unknown probability distribution. SIAM Journal of Control and Optimization 36, no. 6, 1880-1889.

[14] V. Baston and S. Gal (2001). Rendezvous search when marks are left at the starting points, Naval Res. Log. 48 722-731.

[15] V. P. Crawford and H. Haller (1990). Learning how to cooperate: optimal play in repeated coordination games. Econometrica 58, No. 3, 571-596.

[16] Gal, S., (1999). Rendezvous search on the line. Operations Research 47, 974-976.

[17] Han, Q.;Du, Donglei; Vera, J. and Zuluaga, Luis F. (2006). Improved bounds for the symmetric rendezvous value on the line. Preprint, October 2006..

[18] Howard, J. V. (1999). Rendezvous search on the interval and circle. Operations Research 47, No. 4, 550-558.

[19] Lim, W. S. (1997). A rendezvous-evasion game on discrete locations with joint randomization. Adv. Appl. Probab. 29, 1004-1017.

[20] Lim, W. S., Alpern, S. and Beck, A. (1997). Rendezvous search on the line with more than two players. Operations Research 45, 357-364.

[21] Ruckle, W. (1983). Pursuit on a cyclic graph. Int. J. Game Theory 10, 91-99.

[22] Thomas, L. C. and Hulme, P. B. (1997). Searching for targets who want to be found. Journal of the O. R. Society 48 , Issue 1, Pages 44 - 50.

[23] Uthaisombut, P. (2006). Symmetric rendezvous search on the line using moving patterns with different lengths. Preprint, Department of Computer Science, University of Pittsburgh. www.cs.pitt.edu/~utp/papers/rendezvous.pdf

[24] Weber, R. (2006). The optimal strategy for symmetric rendezvous search on K_3. Preprint, Cambridge University. http://www.statslab.cam.ac.uk/~rrw1/research/weber-k3.pdf