Cutting Two Graphs Simultaneously

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Abstract

Consider two graphs, G_1 and G_2, on the same vertex set V, with $|V| = n$ and G_i having m_i edges for $i = 1, 2$. We give a simple algorithm that partitions V into sets A and B such that $e_{G_1}(A, B) \geq m_1/2$ and $e_{G_2}(A, B) \geq m_2/2 - \Delta(G_2)/2$. We also show, using a probabilistic method, that if G_1 and G_2 belong to certain classes of graphs, (for instance, if G_1 and G_2 both have a density of at least $2/3$, or if G_1 and G_2 are both regular of degree at most $(n/16) - 6$ with n sufficiently large) then we can find a partition of V into sets A and B such that $e_{G_i}(A, B) \geq m_i/2$ for $i = 1, 2$.

1 Introduction

Throughout this paper, we shall be concerned with finite simple graphs unless otherwise stated. Given a graph $G = (V, E)$, with A and B disjoint subsets of V, we denote by $E_G(A, B)$ the edges of G that have one end in A and one end in B. Let $e_G(A, B) = |E_G(A, B)|$. For the special case when $B = A^c = V/A$, $E_G(A, A^c)$ is called a *cut* of G. We shall sometimes refer to $E_G(A, A^c)$ as the *cut of G generated by A*. The maximum degree of G will be denoted by $\Delta(G)$.

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It is well known that for any graph G with m edges, there exists a cut of size at least $m/2$. This is achieved by the obvious greedy algorithm. A sharper bound is given by Edwards in [2], [3], where it is shown that every graph G with m edges has a cut of size at least

$$\frac{m}{2} + \sqrt{\frac{m}{8} + \frac{1}{64}} - \frac{1}{8}.$$

Here, the extremal graphs are the complete graphs of odd order.

Henceforth let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be any two graphs on the same vertex set V, with $|V| = n$ and $|E_i| = m_i$ for $i = 1, 2$. In this paper, we shall consider the problem of finding $A \subseteq V$ that generates a large cut both in G_1 and in G_2. This is a problem posed originally by Bollobás and Scott in [1]. More precisely, their problem was the following:

Problem 1 Find the largest integer $f(m)$ such that for every pair of graphs, $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, each with m edges, there exists $A \subseteq V$ with $e_{G_i}(A, A^c) \geq f(m)$ for $i = 1, 2$.

For further details on this and other related problems, see Scott [6].

Bollobás and Scott suggested that perhaps $f(m) = (1 - o(1))m/2$. Kühn and Osthus proved this in [4], where they showed, using probabilistic methods, that if $G_i = (V, E_i)$ with $|E_i| = m_i$ for $i = 1, 2$, then there exists $A \subseteq V$ such that

$$e_{G_i}(A, A^c) \geq \frac{m_i}{2} - \sqrt{m_i} = (1 - o(1))\frac{m_i}{2}$$

for $i = 1, 2$.

In Section 2, we prove the following theorem, which is based on a simple algorithm.

Theorem 1.1 *Let $G_i = (V, E_i)$ with $|E_i| = m_i$ for $i = 1, 2$. Then there exists $A \subseteq V$, with $||A| - |A^c|| \leq 1$, such that*

$$e_{G_1}(A, A^c) \geq \frac{m_1}{2} \quad \text{and} \quad e_{G_2}(A, A^c) \geq \frac{m_2}{2} - \frac{\Delta(G_2)}{2}.$$

A simple modification of the algorithm in Theorem 1.1 yields the following theorem, which again proves that $f(m) = (1 + o(1))m/2$.

Theorem 1.2 *Let G_1 and G_2 be graphs as in Theorem 1.1. Then there exists $A \subseteq V$, with $||A| - |A^c|| \leq 2$, such that*

$$e_{G_1}(A, A^c) \geq \frac{m_1}{2} \quad \text{and} \quad e_{G_2}(A, A^c) \geq \frac{m_2}{2} - \sqrt{m_2}.$$

Both theorems extend easily to graphs with weighted edges. Theorem 1.1 also proves the following conjecture made by Rautenbach and Szigeti in [5].

Conjecture 1 Let $G_i = (V, E_i)$ with $|E_i| = m_i$ for $i = 1, 2$. If both graphs have maximum degree at most Δ then there exists $A \subseteq V$ such that $e_{G_i}(A, A^c) \geq \frac{1}{2}(m_i - \Delta)$ for $i = 1, 2$.

The following conjecture, which is implicit in [4] and [5] but not formally stated, is a natural extension of Conjecture 1.

Conjecture 2 Let $G_i = (V, E_i)$ with $|E_i| = m_i$ for $i = 1, 2$. Then there exists $A \subseteq V$ such that $e_{G_i}(A, A^c) \geq \lfloor \frac{1}{2} m_i \rfloor$ for $i = 1, 2$.

Note that Conjecture 2 is false if we replace $\lfloor \frac{1}{2} m_i \rfloor$ with $\frac{1}{2} m_i$. Indeed, let G_1 be a 5-cycle on 5 vertices and let G_2 be its complementary graph (also a 5-cycle). This example is given by Rautenbach and Szigeti in [5], and is the only such example that we know of.

In Section 3, we show, using probabilistic methods similar to those used in [4], that Conjecture 2 holds for certain classes of graphs. More precisely, we prove the following two theorems.

Theorem 1.3 *Let $G_i = (V, E_i)$ with $|V| = n$ and $|E_i| = m_i \geq \frac{1}{3} n^2$ for $i = 1, 2$. Then there exists $A \subseteq V$, with $||A| - |A^c|| \leq 1$, satisfying*

$$e_{G_i}(A, A^c) \geq \frac{m_i}{2}$$

for $i = 1, 2$.

Theorem 1.4 *Let $G_i = (V, E_i)$ with $|V| = n$ and $|E_i| = m_i$ for $i = 1, 2$. If $\Delta(G_i) = r_i < \sqrt{m_i/8} - 2$ for $i = 1, 2$, then there exists $A \subseteq V$, with $||A| - |A^c|| \leq 1$, satisfying*

$$e_{G_i}(A, A^c) \geq \frac{m_i}{2}$$

for $i = 1, 2$.

In particular, the conditions of Theorem 1.4 are met if G_i is r_i-regular, with $r_i \leq \frac{n}{16} - 6$ for $i = 1, 2$, and $n > 128$.

2 A Simple Algorithm

In this section, we present the proof of our main results. The proof of Theorem 1.1 is based on a simple algorithm, which we later adapt to give a slightly sharper result for the case when $\Delta(G_2)$ is large. Before we proceed to the proof of Theorem 1.1, we give one piece of notation. For $G = (V, E)$ a graph, $v \in V$, and $A \subseteq V$, let $d_G(v, A)$ denote the number of neighbours of v in A. Let $d_G(v)$ denote the degree of the vertex v in G.

Proof (of Theorem 1.1) As before, $G_i = (V, E_i)$, with $|V| = n$ and $|E_i| = m_i$ for $i = 1, 2$. We assume that n is even. (If n is odd then we add a vertex to V that is isolated in G_1 and G_2 and apply the theorem for n even.)

For $j = 0, \ldots, n/2$ we inductively construct disjoint subsets A_j and B_j of V as follows. Let $A_0 = B_0 = \phi$ and assume that we have constructed $A_{j-1} = \{a_1, \ldots, a_{j-1}\}$ and $B_{j-1} = \{b_1, \ldots, b_{j-1}\}$.

For each $v \in V$, let

$$d_1^j(v) = d_{G_1}(v, B_{j-1}) - d_{G_1}(v, A_{j-1}).$$

Choose a_j to be any vertex in $V \backslash (A_{j-1} \cup B_{j-1})$ that maximises d_1^j and set $A_j = \{a_1, \ldots, a_j\}$. For each $v \in V$, let

$$d_2^j(v) = d_{G_2}(v, A_j) - d_{G_2}(v, B_{j-1}).$$

Choose b_j to be any vertex in $V \backslash (A_j \cup B_{j-1})$ that maximises d_2^j and set $B_j = \{b_1, \ldots, b_j\}$. Notice, by our choices of a_j and b_j, that for each j, we have

$$d_1^j(a_j) \geq d_1^j(b_j) \quad \text{and} \quad d_2^j(b_j) \geq d_2^j(a_{j+1}) \geq d_2^{j+1}(a_{j+1}).$$

We shall use these inequalities at the end.

After $n/2$ iterations, we obtain $A_{n/2}$ and $B_{n/2}$, sets of equal sizes that partition V. Let $A = A_{n/2}$, so that $A^c = B_{n/2}$. We claim that

$$e_{G_1}(A, A^c) \geq \frac{m_1}{2} \quad \text{and} \quad e_{G_2}(A, A^c) \geq \frac{m_2}{2} - \frac{\Delta(G_2)}{2}.$$

To see this, observe first that

$$m_i = \sum_{j=1}^{n/2} [d_{G_i}(a_j, A_{j-1}) + d_{G_i}(a_j, B_{j-1}) + d_{G_i}(b_j, A_j) + d_{G_i}(b_j, B_{j-1})]$$

and

$$e_{G_i}(A, A^c) = \sum_{i=1}^{n/2} [d_{G_i}(a_j, B_{j-1}) + d_{G_i}(b_j, A_j)].$$

Subtracting 1/2 of the first equation from the second yields

$$e_{G_i}(A, A^c) - \frac{m_i}{2} = \frac{1}{2}\sum_{j=1}^{n/2}\Big([d_{G_i}(a_j, B_{j-1}) - d_{G_i}(a_j, A_{j-1})] + [d_{G_i}(b_j, A_j) - d_{G_i}(b_j, B_{j-1})]\Big).$$

By comparing the terms in square brackets with $d_i^j(a_j)$ and $d_i^j(b_j)$ respectively, and noting for any vertex v that $d_{G_i}(v, A_j) \geq d_{G_i}(v, A_{j-1})$, we obtain that

$$e_{G_i}(A, A^c) - \frac{m_i}{2} \geq \begin{cases} \frac{1}{2}\sum_{j=1}^{n/2}\left(d_1^j(a_j) - d_1^j(b_j)\right) & \text{if } i = 1; \\ \frac{1}{2}\sum_{j=1}^{n/2}\left(d_2^j(b_j) - d_2^j(a_j)\right) & \text{if } i = 2. \end{cases}$$

Using that $d_1^j(a_j) \geq d_1^j(b_j)$ for each j, we see that the first sum is non-negative. Using that $d_2^j(b_j) \geq d_2^{j+1}(a_{j+1})$ for each j , we see that the second sum is at least $-d_2^1(a_1) + d_2^{n/2}(b_{n/2}) \geq -\Delta(G_2)$ as $d_2^1(a_1) = 0$. This completes the proof. $\square$

Examining the proof of Theorem 1.1, we see that it is the last vertex placed that determines the size of $e_{G_2}(A, A^c) - (m_2/2)$. In particular, we can improve on Theorem 1.1 if we can ensure that the degree of $b_{n/2}$ in G_2 is small.

Proof (of Theorem 1.2) Let $v_1, \ldots, v_n$ be an ordering of the vertices of V satisfying $d_{G_2}(v_i) \geq d_{G_2}(v_{i+1})$ for all $i = 1, \ldots, n-1$. Let $V* = \{v_1, \ldots, v_t\}$, where t is an integer to be specified later. For convenience, we ensure that both $|V*|$ and $|V|$ are even by adding isolated vertices to $V*$ and/or $(V*)^c = V \backslash (V*)$ if necessary. After the addition of these isolated vertices, let $t' = |V*|$ and $n' = |V|$. We give a modified version of the algorithm in the proof of Theorem 1.1. The only difference is that initially, we restrict our attention to $V*$, however we describe the algorithm in full for notational convenience.

Let $V_j = V*$ for $j \leq t'/2$ and $V_j = (V*)^c$ for $j > t'/2$. For $j = 0, \ldots, n'/2$, we inductively construct disjoint subsets, A_j and B_j, of V as follows. Let $A_0 = B_0 = \phi$ and assume that we have constructed $A_{j-1} = \{a_1, \ldots, a_{j-1}\}$ and $B_{j-1} = \{b_1, \ldots, b_{j-1}\}$.

For each $v \in V_j$, let

$$d_1^j(v) = d_{G_1}(v, B_{j-1}) - d_{G_1}(v, A_{j-1}).$$

Choose a_j to be any vertex in $V_j \backslash (A_{j-1} \cup B_{j-1})$ that maximises d_1^j and set $A_j = \{a_1, \ldots, a_j\}$.

For each $v \in V_j$, let

$$d_2^j(v) = d_{G_2}(v, A_j) - d_{G_2}(v, B_{j-1}).$$

Choose b_j to be any vertex in $V_j \backslash (A_j \cup B_{j-1})$ that maximises d_2^j and set $B_j = \{b_1, \ldots, b_j\}$.

We iterate $n'/2$ times to obtain sets $A_{n'/2}$ and $B_{n'/2}$. We remove from $A_{n'/2}$ and $B_{n'/2}$ any isolated vertices that we may have added at the beginning to obtain sets A and $B = A^c$ that partition V. Note that $||A| - |A^c|| \leq 2$. This completes the description of the modified algorithm.

Notice, by our choices of a_j and b_j, that for each j we have $d_1^j(a_j) \geq d_1^j(b_j)$, and for each j except $j = t'/2$, we have $d_2^j(b_j) \geq d_2^j(a_{j+1}) \geq d_2^{j+1}(a_{j+1})$.

Mimicking the analysis of the algorithm in Theorem 1.1 and noting that $e_{G_i}(A, A^c) = e_{G_i}(A_{n'/2}, B_{n'/2})$, we find that

$$e_{G_1}(A, A^c) - \frac{m_1}{2} \geq \frac{1}{2} \sum_{j=1}^{n'/2} (d_1^j(a_j) - d_1^j(b_j)) \geq 0$$

and

$$\begin{aligned} e_{G_2}(A, A^c) - \frac{m_2}{2} &\geq \frac{1}{2} \sum_{j=1}^{n'/2} (d_2^j(b_j) - d_2^j(a_j)) \\ &\geq \frac{1}{2}(-d_2^1(a_1) + d_2^{t'/2}(b_{t'/2}) - d_2^{(t'/2)+1}(a_{(t'/2)+1}) + d_2^{n'/2}(b_{n'/2})) \\ &\geq \frac{1}{2}\Big(0 - \Big\lfloor \frac{t}{2} \Big\rfloor - \Big\lceil \frac{t}{2} \Big\rceil - d_{G_2}(v_{t+1})\Big) \\ &= -\frac{1}{2}(t + d_{G_2}(v_{t+1})). \end{aligned}$$

Since we are free to choose t as we please, we have that

$$e_{G_2}(A, A^c) - \frac{m_2}{2} \geq -\frac{1}{2} \min_t [t + d_{G_2}(v_{t+1})],$$

where we minimise over $t = 0, \ldots, n-1$. We claim that

$$\min_t [t + d_{G_2}(v_{t+1})] \leq \lfloor 2\sqrt{m_2} \rfloor,$$

which proves the theorem. We prove the claim by contradiction. Suppose that $t + d_{G_2}(v_{t+1}) \geq \lceil 2\sqrt{m_2} \rceil$ for all $t = 0, \ldots, n-1$. Then

$$\begin{aligned}
\sum_{t=0}^{n-1} d_{G_2}(v_{t+1}) &\geq \sum_{t=0}^{n-1} \max[(\lceil 2\sqrt{m_2} \rceil - t), 0] \\
&= \sum_{t=0}^{\lceil 2\sqrt{m_2} \rceil} t \\
&= \frac{1}{2} \lceil 2\sqrt{m_2} \rceil (\lceil 2\sqrt{m_2} \rceil + 1) \\
&> 2m_2,
\end{aligned}$$

which is a contradiction, proving the claim. □

Both Theorem 1.1 and Theorem 1.2 can be extended to graphs with weighted edges. We simply replace each parameter with its weighted counterpart (both in the statements and the proofs of the theorems).

3 Good Simultaneous Cuts For Special Classes of Graphs

In this section, we turn to the problem of finding pairs of graphs, $G_i = (V, E_i)$ with $|E_i| = m_i$ for $i = 1, 2$, for which we can ensure the existence of $A \subseteq V$ such that

$$e_{G_i}(A, A^c) \geq \lfloor m_i/2 \rfloor$$

for $i = 1, 2$. As conjectured earlier, we believe that the above is true for all pairs of graphs. The proofs in this section are of a probabilistic nature.

We first prove that the above is true for graphs of high density, that is, those graphs that give the poorest bounds in Theorem 1.1 and Theorem 1.2. We start with a general lemma.

Lemma 3.1 *Let X be an integer-valued random variable with mean μ and variance σ^2. For $p > 0$, let $r(X, p)$ be maximal such that*

$$\Pr(X \leq r(X, p)) < p.$$

Then

$$r(X, p) + 1 \geq \mu - \sqrt{\frac{1-p}{p}}\sigma.$$

Proof Let Y be the two point random variable taking the value $y_0 = \mathbb{E}(X|X \leq r(X,p)+1)$ with probability $p_0 = \Pr(X \leq r(X,p)+1)$, and taking the value $y_1 = \mathbb{E}(X|X > r(X,p)+1)$ with probability $1-p_0$.

We have that $y_0 \leq r(X,p)+1$, and an easy calculation gives

$$y_0 = \mathbb{E}(Y) - \sqrt{\frac{1-p_0}{p_0} Var(Y)}.$$

Noting that $p_0 \geq p$, $\mathbb{E}(Y) = \mu$, and $Var(Y) \leq \sigma^2$ (the last of these follows from the convexity of x^2), we obtain

$$r(X,p)+1 \geq y_0 = \mathbb{E}(Y) - \sqrt{\frac{1-p_0}{p_0} Var(Y)} \geq \mu - \sqrt{\frac{1-p}{p}}\sigma$$

as required. □

The following corollary is the main probabilistic tool used in the proofs of Theorem 1.3 and Theorem 1.4.

Corollary 3.2 *Let X_1 and X_2 be integer-valued random variables, and let X_i have mean μ_i and variance σ_i^2 for $i = 1,2$. Then*

$$\Pr(X_1 \geq \mu_1 - \sigma_1,\ X_2 \geq \mu_2 - \sigma_2) > 0.$$

Proof The following easy calculation proves the corollary.

$$\begin{aligned}
&\Pr(X_1 \geq \mu_1 - \sigma_1, X_2 \geq \mu_2 - \sigma_2) \\
&\qquad \geq 1 - \Pr(X_1 \leq \mu_1 - \sigma_1 - 1) - \Pr(X_2 \leq \mu_2 - \sigma_2 - 1) \\
&\qquad \geq 1 - \Pr(X_1 \leq r(X, 1/2)) - \Pr(X_2 \leq r(X, 1/2)) \\
&\qquad > 1 - 1/2 - 1/2 = 0.
\end{aligned}$$

□

The idea of the proof of Theorem 1.3 is an extension of the ideas of Kühn and Osthus in [4].

Proof (of Theorem 1.3) Given graphs G_1 and G_2, pick a subset A of V of size $\lfloor n/2 \rfloor$ uniformly at random and set $X_i = e_{G_i}(A, A^c)$ for $i = 1,2$. Let μ_i and σ_i^2 respectively be the mean and variance of X_i. We show that if G_1 and G_2 are sufficiently dense, then

$$\mu_i - \sigma_i \geq \frac{m_i}{2}.$$

Corollary 3.2 then gives that

$$\Pr(X_1 \geq m_1/2, X_2 \geq m_2/2) > 0,$$

hence there exists some subset of V with the desired property.

It remains only to bound $\mu_i - \sigma_i$. We shall assume that n is even. The case of n odd is proved with a similar calculation to the one below.

We start by computing the expectation and variance of the X_i. Let us focus on X_1. For each $e \in E_1$, define

$$X_e = \begin{cases} 1 \text{ if } e \in E_{G_1}(A, A^c); \\ 0 \text{ otherwise.} \end{cases}$$

Note that $X_1 = \sum_{e \in E_1} X_e$. Using this and the linearity of expectation, we have

$$\mathbb{E}(X_1) = \sum_{e \in E_1} \mathbb{E}(X_e) = \sum_{e \in E_1} \Pr(e \in E_{G_1}(A, A^c)) = \frac{1}{2} m_1 \big(1 + \frac{1}{n-1}\big),$$

since $\Pr(e \in E_{G_1}(A, A^c)) = \frac{1}{2}(1 + \frac{1}{n-1})$.

Next we compute $\mathbb{E}(X_1^2)$. Again, writing X_1 as a sum of indicator functions and expanding, we get

$$\begin{aligned} \mathbb{E}(X_1^2) &= \sum_{e \in E_1} \mathbb{E}(X_e) + \sum_{\substack{e, f \in E_1 \\ e \neq f}} \mathbb{E}(X_e \cdot X_f) \\ &= \mathbb{E}(X) + \sum_{\substack{e, f \in E_1 \\ e \neq f}} \Pr(e, f \in E_{G_1}(A, A^c)). \end{aligned}$$

For two edges, e and f, of a graph, with $e \neq f$, we write e inc f if the edges are incident (meet at exactly one vertex), and we write e ind f if they have no common vertices, that is, they are independent. We split the sum above according to whether or not e and f are incident. Using that

$$\Pr(e, f \in E_{G_1}(A, A^c)) = \begin{cases} \frac{1}{4}\Big(1 + \frac{1}{n-1}\Big)\Big(1 + \frac{1}{n-3}\Big), & \text{if } e \text{ ind } f; \\ \frac{1}{4}\Big(1 + \frac{1}{n-1}\Big), & \text{if } e \text{ inc } f, \end{cases}$$

we obtain

$$\mathbb{E}(X_1^2) = \mathbb{E}(X) + \sum_{\substack{e,f\in E_1\\ e \text{ ind } f}} \Pr(e,f \in E_{G_1}(A,A^c)) + \sum_{\substack{e,f\in E_1\\ e \text{ inc } f}} \Pr(e,f \in E_{G_1}(A,A^c))$$

$$= \frac{1}{2}m_1\big(1+\frac{1}{n-1}\big) + [m_1(m_1-1) - P_2(G_1)]\frac{1}{4}\Big(1+\frac{1}{n-1}\Big)\Big(1+\frac{1}{n-3}\Big)$$

$$+ P_2(G_1)\frac{1}{4}\Big(1+\frac{1}{n-1}\Big)$$

$$= \frac{1}{4}\Big(1+\frac{1}{n-1}\Big)\Big(2m_1 + \Big(1+\frac{1}{n-3}\Big)m_1(m_1-1) - \frac{P_2(G_1)}{n-3}\Big),$$

where $P_2(G_1)$ denotes the number of (ordered) pairs of incident edges in G_1. Alternatively, $P_2(G_1)$ is twice the number of paths of length 2 in G_1, and we can bound it as follows. Let $v_1,\dots,v_n$ be the vertices in V and let d_i be the degree of v_i in G_1. Then

$$P_2(G_1) = \sum_{i=1}^{n} d_i(d_i-1)$$

$$= \sum_{i=1}^{n} d_i^2 - 2m_1$$

$$\geq n\Big(\frac{1}{n}\sum_{i=1}^{n} d_i\Big)^2 - 2m_1 \text{ (Cauchy-Schwarz inequality)}$$

$$= \frac{4m_1^2}{n} - 2m_1.$$

Using this bound, together with the expressions for $\mathbb{E}(X_1^2)$, we find that

$$\mathbb{E}(X_1^2) \leq \frac{1}{4}\Big(1+\frac{1}{n-1}\Big)\Big(m_1 + \Big(1+\frac{1}{n-3}\Big)m_1^2 - \frac{4m_1^2}{n(n-3)} + \frac{m_1}{n-3}\Big).$$

Using our expression for $\mathbb{E}(X_1)$, we obtain

$$Var(X_1) \leq \frac{1}{4}\Big(1+\frac{1}{n-1}\Big)\Big(m_1 + \frac{2m_1^2}{(n-1)(n-3)} - \frac{4m_1^2}{n(n-3)} + \frac{m_1}{n-3}\Big),$$

and similarly for $Var(X_2)$. It is sufficient to show that

$$Var(X_i) \leq \Big(\mu_i - \frac{m_i}{2}\Big)^2.$$

Substituting the expression for μ_i and the bound for σ_i^2, we find it is sufficient to show that

$$\frac{1}{4}\Big(1+\frac{1}{n-1}\Big)\Big(m_i + \frac{2m_i^2}{(n-1)(n-3)} - \frac{4m_i^2}{n(n-3)} + \frac{m_i}{n-3}\Big) \leq \frac{m_i^2}{4(n-1)^2}.$$

For $n \geq 3$ and $m_i > 0$, the above inequality holds if and only if

$$m_i \geq \frac{n(n-1)(n-2)}{3n-7},$$

which holds if $m_i \geq \frac{1}{3}n^2$ for $i = 1, 2$. □

Next we prove a theorem showing that pairs of graphs with small maximum degree (relative to the number of edges in the graphs) also satisfy Conjecture 1. The proof of the theorem broadly follows that of the previous theorem, the only difference being the way in which the random cut is constructed.

Going into more detail, the random cut is constructed as follows. We first deterministically pair up the vertices of our vertex set V so that a large proportion of the pairs form edges of our graphs. We then partition V randomly, ensuring that vertices of each pair are in different parts.

This motivates the following lemma and its corollary.

Lemma 3.3 *For graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, let $A_i \subseteq E_i$ be sets of independent edges for $i = 1, 2$. Then there exists a set, $A \subseteq A_1 \cup A_2$, of independent edges such that*

$$|A \cap A_i| \geq \left\lfloor \frac{1}{2}|A_i| \right\rfloor - 1.$$

Proof Observe that each edge of $A_1 \cap A_2$ is independent of all other edges in $A_1 \cup A_2$. Let $B_i = A_i \backslash (A_1 \cap A_2)$. Then it is sufficient to find a set $B \subseteq B_1 \cup B_2$ of independent edges such that $|B \cap B_i| \geq \lfloor |B_i|/2 \rfloor - 1$ (then set $A = B \cup (A_1 \cap A_2)$).

We construct B as follows. Assume, without loss of generality, that $|B_2| = |B_1| + b$ where b is a non-negative integer. Note that $B_1 \cup B_2$ is a disjoint union of paths and cycles where edges alternate between being in B_1 and being in B_2. Let S be the set of these paths and cycles.

A path in S whose first and last edges are both in B_1 (resp. B_2) will be referred to as a 1-path (resp. 2-path). Let P^1 (resp. P^2) be the set of 1-paths (resp. 2-paths). Any other path in S is necessarily a path with an even number of edges, so we call it an even path. Let P^e be the set of even paths in S. Let C be the set of cycles in S (each of which necessarily has an even number of edges).

We have that S is the disjoint union of C, P^e, P^1 and P^2. For $s \in S$, $|s|$ refers to the number of edges in s. Let

$$\begin{aligned} C &= \{c_1, c_2, \ldots, c_j\} \text{ with } |c_1| \geq |c_2| \geq \ldots \geq |c_j|, \\ P^e &= \{p_1^e, p_2^e, \ldots, p_k^e\} \text{ with } |p_1^e| \geq |p_2^e| \geq \ldots \geq |p_k^e|, \\ P^1 &= \{p_1^1, p_2^1, \ldots, p_l^1\} \text{ with } |p_1^1| \geq |p_2^1| \geq \ldots \geq |p_l^1|, \\ P^2 &= \{p_1^2, p_2^2, \ldots, p_m^2\} \text{ with } |p_1^2| \geq |p_2^2| \geq \ldots \geq |p_m^2|, \end{aligned}$$

and note that the number of 2-paths exceeds the number of 1-paths by $b = |B_2| - |B_1|$, hence $l + b = m$. We order the elements of S as follows,

$$c_1, c_2, \ldots, c_j, p_1^e, p_2^e, \ldots, p_k^e, p_1^2, p_1^1, p_2^2, p_2^1, \ldots, p_l^2, p_l^1, p_{l+1}^2, p_{l+2}^2, \ldots, p_m^2$$

and call this ordering O_S. For each $s \in S$, fix an ordering, $f_1, \ldots, f_q$, of the edges of s such that f_i and f_{i+1} are incident for $i = 1, \ldots, q-1$, and if s is an even path or cycle, then $f_1 \in B_2$. Concatenate these orderings of elements of S according to O_s to give an ordering, $e_1, \ldots, e_t$, of the edges of $B_1 \cup B_2$. Note that the edges in our ordering $e_1, \ldots, e_t$ alternate between B_1 and B_2 except at a transition between P_{l+z}^2 and P_{l+z+1}^2 ($z = 1, \ldots, b-1$), where we have two consecutive edges in B_2. We call such a transition, a P^2-transition.

Choose x minimal such that $|\{e_1, \ldots, e_x\} \cap B_1| = \lfloor |B_1|/2 \rfloor - 1$ and let $B_1' = \{e_1, \ldots, e_x\} \cap B_1$. Let $B_2' = \{e_{x+2}, \ldots, e_t\} \cap B_2$ and let $B = B_1' \cup B_2'$. It is not too difficult to see that B is a set of independent edges.

It remains only to show that $|B \cap B_2| = |B_2'| \geq \lfloor |B_2|/2 \rfloor - 1$. Let y be the number of P^2-transitions in $e_1, \ldots, e_x$. Since $p_{l+1}^2, \ldots, p_m^2$ are ordered according to size in O_s, we find that $y \leq b/2 - 1$, otherwise $|B_1'| \geq |B_1|/2$. Using this, we get

$$\begin{aligned} |B_2'| &= |B_2| - |\{e_1, \ldots, e_x\} \cap B_2\}| - 1 \\ &= |B_2| - (|B_1'| + y) - 1 \\ &\geq |B_2| - \frac{1}{2}(|B_1| + b) - 1 \\ &\geq \left\lfloor \frac{1}{2}|B_2| \right\rfloor - 1 \end{aligned}$$

as required. □

Corollary 3.4 *Let* $G_1 = (V, E_1)$ *and* $G_2 = (V, E_2)$ *be graphs with* $|V| = n$ *and* $|E_i| = m_i$. *If* $\Delta(G_i) = r_i$ *for* $i = 1, 2$, *then there exists a pairing,*

$P = \{(v_1, w_1), \ldots, (v_{\lfloor n/2 \rfloor}, w_{\lfloor n/2 \rfloor})\}$, *of the vertices in* V *such that for* $i = 1, 2$, *we have*

$$|P \cap E_i| \geq \left\lfloor \frac{m_i}{2(r_i + 1)} \right\rfloor - 1.$$

Proof By Vizing's Theorem, we can find an $(r_i + 1)$-colouring of the edges of G_i and so, in each of the graphs G_i, we can find an independent set of edges of size at least $m_i/(r_i + 1)$. Let A_i be such a set of independent edges for graph G_i. By Lemma 3.3, we know there exists a set $A \subseteq A_1 \cup A_2$ of independent edges such that $|A \cap A_i| \geq \lfloor |A_i|/2 \rfloor - 1$. This proves the corollary since the edges in A induce a partial pairing of V and we extend this (in any way) to a total pairing, P, with the desired property. □

We are now ready to prove Theorem 1.4.

Proof (of Theorem 1.4) Assume n is even (if n is odd, add a vertex to V (isolated in G_1 and G_2), and apply the theorem for the case when n is even). By Corollary 3.4, there exists a pairing $P = \{(v_1, w_1), \ldots, (v_{n/2}, w_{n/2})\}$ of the vertices of V such that $k_i = |P \cap E_i| \geq \lfloor \frac{m_i}{2(r_i+1)} \rfloor - 1$ for $i = 1, 2$. Let A be a random subset of V constructed as follows. For each pair (v_i, w_i) of P, we either choose $v_i \in A$, $w_i \notin A$ or $v_i \notin A$, $w_i \in A$, each with probability 1/2. The choices for each $i = 1, \ldots, \lfloor n/2 \rfloor$ are made independently of one another. Let $X_i = e_{G_i}(A, A^c)$ and let X_i have mean μ_i and variance σ_i^2. By Corollary 3.2, it is sufficient to prove that

$$\mu_i - \sigma_i \geq \frac{1}{2} m_i,$$

for $i = 1, 2$. As before we compute μ_i and σ_i^2.

Let $G'_i = (V, E'_i) = (V, E_i \backslash P)$ with $m'_i = |E'_i| = m_i - k_i$, and let $X'_i = e_{G'_i}(A, A^c)$. Let X'_i have mean $\mu'_i = \mu_i - k_i$ and variance $\sigma_i'^2 = \sigma_i^2$.

For $e \in E'_i$, we have that

$$\Pr(e \in E_{G_i}(A, A^c)) = 1/2,$$

so as in Theorem 1.3, we have

$$\mathbb{E}(X'_i) = \sum_{e \in E'_i} \Pr(e \in E_{G'_i}(A, A^c)) = \frac{m'_i}{2}.$$

Two edges e, f ($e \neq f$) in E'_i are said to be *linked* if there exists $p_1, p_2 \in P$ such that $e \cup f \subseteq p_1 \cup p_2$. For $e, f \in E'_i$, we have

$$\Pr(e, f \in E_{G_i}(A, A^c)) = \begin{cases} \frac{1}{2} & \text{if } e, f \text{ are linked and not incident;} \\ 0 & \text{if } e, f \text{ are linked and incident;} \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$

For any edge $e \in E_i'$, there is at most one edge $f \in E_i'$ that is linked and not incident to e. Hence there are at most m_i' (ordered) pairs of edges of E_i' that are linked and not incident. As in the proof of Theorem 1.3, we have

$$\begin{aligned}\mathbb{E}(X_i'^2) &= \mathbb{E}(X_i') + \sum_{\substack{e,f\in E_i'\\ e\neq f}} \Pr(e,f \in E_{G_i'}(A,A^c)) \\ &\leq \frac{1}{2}m_i' + \frac{1}{4}[m_i'(m_i'-1) - m_i'] + \frac{1}{2}m_i' \\ &= \frac{1}{4}m_i'^2 + \frac{1}{2}m_i',\end{aligned}$$

and

$$\begin{aligned}\sigma_i'^2 &= \mathbb{E}(X_i'^2) - \mathbb{E}(X_i')^2 \\ &\leq \frac{1}{2}m_i'.\end{aligned}$$

Therefore $\mu_i = \frac{1}{2}(m_i + k_i)$ and $\sigma_i^2 \leq \frac{1}{2}(m_i - k_i)$. We find that $\mu_i - \sigma_i \geq m_i/2$ if $\sigma_i^2 \leq \frac{1}{4}k_i^2$, i.e. if

$$m_i \leq \frac{1}{2}k_i^2 + k_i.$$

Given that $k_i \geq \lfloor \frac{m_i}{2(r_i+1)} \rfloor - 1$, it is easy to check that the above holds if $r_i \leq \sqrt{m_i/8} - 2$. □

Note that the condition $\Delta(G_i) \leq \sqrt{m_i/8} - 2$ is only used at the end of the proof in order to bound k_i. More generally, any pair of graphs G_i, $i = 1,2$, satisfying the condition that $m_i \leq \frac{1}{2}k_i^2 + k_i$ will satisfy Conjecture 2.

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References

[1] B. Bollobás and A. D. Scott. Judicious partitions of bounded-degree graphs. *J. Graph Theory*, 46(2):131–143, 2004.

[2] C. S. Edwards. Some extremal properties of bipartite subgraphs. *Canad. J. Math.*, 25:475–485, 1973.

[3] C. S. Edwards. An improved lower bound for the number of edges in a largest bipartite subgraph. In *Recent advances in graph theory (Proc. Second Czechoslovak Sympos., Prague, 1974)*, pages 167–181. Academia, Prague, 1975.

[4] Daniela Kühn and Deryk Osthus. Maximising several cuts simultaneously. *Combinatorics, Probability and Computing, to appear*, 2006.

[5] D. Rautenbach and Z. Szigeti. Simultaneous large cuts. *manuscript*, 2004.

[6] Alex Scott. Judicious partitions and related problems. In *Surveys in combinatorics 2005*, volume 327 of *London Math. Soc. Lecture Note Ser.*, pages 95–117. Cambridge Univ. Press, Cambridge, 2005.