Two Conjectures on Rendezvous in K_3

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The symmetric rendezvous problem Γ on the triangle K_3 asks how two players, initially randomly placed at distinct vertices, can meet in the minimal expected number of steps v. They must follow a common mixed strategy, with independent randomization. This problem, posed by Alpern and first studied by Anderson and Weber [6] (see also [4],[3]) assumes they have no common notion of a clockwise direction around the triangle - if they do, then the resulting problem Γ_{cc} has a minimum meeting time w which cannot be larger than v.

Despite their apparent simplicity, both problems Γ and Γ_{cc} are open. However there are two widely held conjectures regarding them:

AW (Anderson-Weber) conjecture The rendezvous value of Γ is given by $v = 5/2$. This expected meeting time is obtained by the *stay-search* strategy of choosing, in each two-period interval, to either *stay* where you are for two periods (probability 1/3) or *search* the remaining two vertices in equiprobable order (probability 2/3).

CC (Common-Clockwise) Conjecture Having a common notion of clockwise does not help, that is, $w = v$.

The purpose of this short note is to show that the AW Conjecture implies the CC Conjecture. More generally, we establish an inequality between the two rendezvous values w and v.

Theorem 1 $v \leq \dfrac{2w + 15}{8}$.

Proof. Consider the following mixed strategy for Γ : Use the stay-search strategy for the first two moves. Following 'stay' use any agreed optimal strategy in Γ. Following 'search', it can be seen that your partner also used 'search', and both players moved in the same direction. Call this direction 'clockwise' and now follow an agreed optimal strategy in Γ_{cc}. Clearly v cannot exceed the expected meeting time for this strategy, so we have the equivalent inequaltiy

$$v \leq \frac{4}{9}\left[\frac{1+2}{2}\right] + \frac{1}{9}\left[2+v\right] + \frac{4}{9}\left[\frac{1}{4}(1) + \frac{1}{4}(2) + \frac{1}{2}(2+w)\right],$$

where the bracketed expressions give the expected meeting time conditioned on the players having used stay/search, stay/stay, and search/search in initial two-period randomizations. ■

Corollary 2 *The AW Conjecture implies the CC Conjecture.*

Proof. If CC is false then $w < v$ and hence, by Theorem 1, $v < \dfrac{2v+15}{8}$. Consequently $v < 5/2$, which contradicts AW. ■

A less quantitative proof of this result is simply to observe that if $w < v$, one can improve on the stay/search strategy by playing it until 'search' comes up; then play an agreed optimal strategy in Γ_{cc}.

Similar ideas can be applied to the symmetric rendezvous problem on the line ([1], [5], [7], [2]). In this problem two unit speed agents are placed a distance 2 apart on the line and faced in random directions, so that they have no common notion of direction along the line. If, for example, both players adopt FBB (that is, they move 1 unit in the direction they are facing, and then 2 units the other way) and fail to meet at time 3, they can conclude their 'forward' directions were the same. This observation yields a corresponding estimate to Theorem 1 for the undirected and directed rendezvous values for the line.

References

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