The Common Knowledge of Formula Exclusion

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Abstract: A multi-partition with evaluations is defined by two sets S and X, a collection $\mathcal{P}^1, \dots, \mathcal{P}^n$ of partitions of S and a function $\psi : S \to \{0,1\}^X$. To each partition $\mathcal{P}^j$ corresponds a person j who cannot distinguish between any two points belonging to the same member of $\mathcal{P}^j$ but can distinguish between different members of $\mathcal{P}^j$. A cell of a multi-partition is a minimal subset C such that for all j the properties $P \in \mathcal{P}^j$ and $P \cap C \neq \emptyset$ imply that $P \subseteq C$. Construct a sequence $\mathcal{R}_0, \mathcal{R}_1, \dots$ of partitions of S by $\mathcal{R}_0 = \{\psi^{-1}(a) \mid a \in \{0,1\}^X\}$ and x and y belong to the same member of $\mathcal{R}_i$ if and only if x and y belong to the same member of $\mathcal{R}_{i-1}$ and for every person i the members P_x and P_y of $\mathcal{P}^j$ containing x and y respectively intersect the same members of $\mathcal{R}_{i-1}$. Let $\mathcal{R}_\infty$ be the limit of the $\mathcal{R}_i$, namely x and y belong to the same member of $\mathcal{R}_\infty$ if and only if x and y belong to the same member of $\mathcal{R}_i$ for every i. For any set X and number n of persons there is a canonical multi-partition with evaluations defined on a set Ω such that from any multi-partition with evaluations (using the same X and n) there is a canonical map to Ω with the property that x and y are mapped to the same point of Ω if and only if x and y share the same member of $\mathcal{R}_\infty$. We define a cell of Ω to be surjective if every multi-partition with evaluations that maps to it does so surjectively. A cell of a multi-partition with evaluations has finite fanout if every $P \in \mathcal{P}^j$ in the cell has finitely many elements. All cells of Ω with finite fanout are surjective, but the converse does not hold.

Key words: Kripke Structures, Common Knowledge, Baire Category, Cantor Sets

1 Introduction

Common knowledge by persons $1, 2, \ldots, n$ of the event E means that for every string of persons $i_1, i_2, \ldots, i_k$ it holds that i_k knows that i_{k-1} knows that ... i_1 knows that the event E has occurred (Lewis 1969). Formally, what is the definition of knowing an event and what are the collection of events that can be known?

One way to define knowledge and common knowledge is through semantic models, for example through multi-partitions with evaluations, as defined in the abstract. (See also Aumann, 1976.) Let $(S, \mathcal{P}, J, X, \psi)$ stand for a multi-partition with evaluations, where S is the set on which the partitions $\mathcal{P} = (\mathcal{P}^i \mid i \in J)$ are defined, J is the index set of persons, and $\psi : S \rightarrow \{0,1\}^X$ are the evaluations, with ψ^x standing for the function ψ projected to the x coordinate. Applying the above definition of common knowledge to the context of multi-partitions with evaluations, a subset A is known in common by all the persons in J at the point $x \in A$ if the cell containing x is contained in the set A.

Before we can describe our main result, we must define Ω, the canonical multi-partition with evaluations.

Let X be a set of primitive propositions, and let J be a set of agents. Although it is legitimate to consider the case of either X or J infinite, for this paper we will assume throughout that both X and J are finite. Construct the set $\mathcal{L}(X, J)$ of formulas using the sets X and J in the following way:
1) If $x \in X$ then $x \in \mathcal{L}(X, J)$,
2) If $g \in \mathcal{L}(X, J)$ then $(\neg g) \in \mathcal{L}(X, J)$,
3) If $g, h \in \mathcal{L}(X, J)$ then $(g \wedge h) \in \mathcal{L}(X, J)$,
4) If $g \in \mathcal{L}(X, J)$ then $k_j g \in \mathcal{L}(X, J)$ for every $j \in J$,
5) Only formulas constructed through application of the above four rules are members of $\mathcal{L}(X, J)$.
We write simply $\mathcal{L}$ if there is no ambiguity. $\neg f$ stands for the negation of f, $f \wedge g$ stands for both f and g. $f \vee g$ stands for either f or g (inclusive) and $f \rightarrow g$ stands for $\neg f \wedge g$.

If $\mathcal{K} = (S, \mathcal{P}, J, X, \psi)$ is a multi-partition with evaluations then define a map $\alpha^{\mathcal{K}}$ from $\mathcal{L}(X, J)$ to 2^S, the subsets of S, inductively on the structure of the formulas:

Case 1 $f = x \in X$**:** $\alpha^{\mathcal{K}}(x) := \{s \in S \mid \psi^x(s) = 1\}$.
Case 2 $f = \neg g$**:** $\alpha^{\mathcal{K}}(f) := S \backslash \alpha^{\mathcal{K}}(g)$,
Case 3 $f = g \wedge h$**:** $\alpha^{\mathcal{K}}(f) := \alpha^{\mathcal{K}}(g) \cap \alpha^{\mathcal{K}}(h)$,
Case 4 $f = k_j(g)$**:** $\alpha^{\mathcal{K}}(f) := \{s \mid s \in P \in \mathcal{P}^j \Rightarrow P \subseteq \alpha^{\mathcal{K}}(g)\}$.

There is a very elementary logic defined on the formulas in $\mathcal{L}$ called $S5$. For a longer discussion of the $S5$ logic, see Cresswell and Hughes (1968); and for the multi-person variation, see Halpern and Moses (1992) and also Bacharach, et al, (1997). Briefly, the $S5$ logic is defined by two rules of inference, modus ponens and necessitation, and five types of axioms. Modus ponens means that if f is a theorem and $f \to g$ is a theorem, then g is also a theorem. Necessitation means that if f is a theorem then $k_j f$ is also a theorem for all $j \in J$. The axioms are the following, for every $f, g \in \mathcal{L}(X, J)$ and $j \in J$:
1) all formulas resulting from theorems of the propositional calculus through substitution,
2) $(k_j f \wedge k_j(f \to g)) \to k_j g$,
3) $k_j f \to f$,
4) $k_j f \to k_j(k_j f)$,
5) $\neg k_j f \to k_j(\neg k_j f)$.

A set of formulas $\mathcal{A} \subseteq \mathcal{L}(X, J)$ is called *complete* if for every formula $f \in \mathcal{L}(X, J)$ either $f \in \mathcal{A}$ or $\neg f \in \mathcal{A}$. A set of formulas is called *consistent* if no finite subset of this set leads to a logical contradiction, meaning a deduction of f and $\neg f$ for some formula f. We define

$$\Omega(X, J) := \{S \subseteq \mathcal{L}(X, J) \mid S \text{ is complete and consistent}\}.$$

$\Omega(X, J)$ is itself a multi-partition with evaluation. For every person $j \in J$ we define its corresponding partition $\mathcal{Q}^j(X, J)$ to be that generated by the inverse images of the function $\beta^j : \Omega(X, J) \to 2^{\mathcal{L}(X,J)}$ namely

$$\beta^j(z) := \{f \in \mathcal{L}(X, J) \mid k_j f \in z\},$$

the set of formulas known by person j. Due to the fifth set of axioms $\beta^j(z) \subseteq \beta^j(z')$ implies that $\beta^j(z) = \beta^j(z')$. We will write Ω, $\mathcal{L}$ and $\mathcal{Q}^j$ if there is no ambiguity.

If $\mathcal{K} = (S, \mathcal{P}, J, X, \psi)$ we define a map $\phi^{\mathcal{K}} : S \to \Omega(X, J)$ by

$$\phi^{\mathcal{K}}(s) := \{f \in \mathcal{L}(X, J) \mid \; s \in \alpha^{\mathcal{K}}(f)\}.$$

This is the canonical map referred to in the abstract, also contained in Fagin, Halpern, and Vardi (1991).

As stated in the abstract, a cell C of a multi-partition with evaluations has *finite fanout* if every choice of $i \in J$ and $P \in \mathcal{P}^i$ contained in C the set P has finitely many elements. In Simon (1999) a cell of Ω was defined to be *surjective* if all multi-partitions with evaluations $\mathcal{K}$ that map to it by $\phi^{\mathcal{K}}$ do so surjectively. We construct an example of a countable and surjective cell of Ω that does not have finite fanout. (In Simon (1999) it was proven that any cell of Ω with finite fanout is surjective and any surjective cell of Ω must be countable.)

Central to understanding the relation between surjectivity and finite fanout is point-set topology. For every multi-partition with evaluations $\mathcal{K} = (S, \mathcal{P}, J, X, \psi)$ we define a topology on the set S, the same as in Samet (1990). Let $\{\alpha^{\mathcal{K}}(f) \mid f \in \mathcal{L}\}$ be the base of open sets of S. We call this the topology *induced by the formulas.* The topology of a subset A of S will be the relative topology for which the open sets of A are $\{A \cap O \mid O \text{ is an open set of } S\}$.

Why is our main result surprising? It is closely related to representations of multi-partitions with evaluations through canonical structures indexed by ordinal numbers.

Fagin (1994) defined for any ordinal number γ (and a sets of persons and primitive propositions) a hierarchically constructed canonical multi-partition with evaluations W_γ such that W_ω is Ω, (where ω stands for the first infinite ordinal). This canonical structure represented all the possible truth evaluations with the ordinal numbers representing the levels in the construction of these statements. There are alternative canonical constructions corresponding to the ordinal numbers (Heifetz and Samet 1998, 1999), but with regard to the first infinite ordinal ω they are the same as Fagin's. For every multi-partition with evaluations and ordinal number γ there are canonical maps defined to the canonical structures W_γ.

If there is an ordinal α such that the map to W_α is injective, then the multi-partition with evaluations is called *non-flabby*, and the first such ordinal is called the *distinguishing* ordinal. Otherwise the distinguishing ordinal is the first ordinal α where all pairs of points which get mapped eventually to different places do so to W_α. There is another minimal ordinal β, possibly larger than the distinguishing ordinal, for which the image of the multi-partition with evaluations in W_β can be extended to any W_γ with $\gamma > \beta$ in only one unique way. This ordinal is called the *uniqueness* ordinal. Fagin (1994) proved that the uniqueness ordinal is a limit ordinal and never greater than the next limit ordinal above the distinguishing ordinal. Fagin established that the necessary and sufficient condition for a cell of Ω to have the first infinite ordinal ω as its uniqueness ordinal is that the cell has finite fanout. Without explicitly mentioning topology, Fagin (1994) showed that any member P of some $\mathcal{Q}^j$ is a compact subset of Ω. An extension to $W_{\omega+1}$ of an x in Ω is defined by dense subsets R_j of the various $P_j \in \mathcal{Q}^j$ containing x. Therefore there is a unique extension of a cell of Ω if and only if for every person j every $P_j \in \mathcal{P}^j$ in the cell has only one dense subset, which is equivalent to the cell having finite fanout.

Is the lack of a unique extension from a cell of Ω (equivalently the lack of finite fanout) equivalent to the existence of some multi-partition with evaluations mapping to this cell such that the persons have common knowledge that some set of formulas valid somewhere in the cell are not valid at any point in the original multi-partition with evaluations? The surjective property is exactly the impossibility of such a common knowledge of formula exclusion.

Surprisingly the answer rests upon a property called *centeredness*. The centered property has several equivalent definitions; the most straightforward definition is that a cell of Ω is centered if and only if no other cell of Ω shares the same set of formulas held in common knowledge (Simon 1999). (The set of formulas held in common knowledge is a constant throughout any given cell; see Halpern and Moses 1992). An equivalent formulation of centeredness is that the cell is an open set relative to the closure of itself. The difference between centered and uncentered cells is radical; if a cell is not centered then there are uncountably many other cells sharing the same set of formulas in common knowledge (Simon 1999). Furthermore the converse does hold for centered cells of Ω, namely that a centered cell of Ω is surjective if and only if it has finite fanout (Theorem 3b, Simon 1999).

The lack of finite fanout for a cell C of Ω implies the existence a cluster point y of some $P \in \mathcal{Q}^j$ that is contained in C. Is the point y is a good candidate for the existence of a multi-partition with evaluations that maps to $C \backslash \{y\}$? If C is centered there will be such a multi-partition with evaluations that maps to C but avoids the point y.

In the next section, we provide some more background necessary to understand our solution. In the third and concluding section, we present our example of a cell of Ω that is surjective but without finite fanout.

2 More Background

Central to this paper is the first part of **Lemma 5** of Simon (1999), which states that if $\mathcal{K} = (S; J; (\mathcal{P}^j \mid j \in J); X; \psi)$ is a multi-partition with evaluations and P is a member of $\mathcal{P}^j$ for some $j \in J$ then $\phi^{\mathcal{K}}(P)$ is a dense subset of F for some $F \in \mathcal{Q}^j$. This fact was used implicitly by Fagin (1994).

Given a multi-partition with evaluations $\mathcal{K} = (S; J; (\mathcal{P}^j \mid j \in J); X; \psi)$ and a subset $A \subseteq S$, we define the multi-partition with evaluations $\mathcal{V}^{\mathcal{K}}(A) := (A; J; (\mathcal{P}^j|_A \mid j \in J); X; \psi|_A)$ where for all $j \in J$ $\mathcal{P}^j|_A := \{F \cap A \mid F \cap A \neq \emptyset$ and $F \in \mathcal{P}^j\}$. We define a subset $A \subseteq \Omega$ to be *good* if for every $j \in J$ and every $F \in \mathcal{Q}^j$ satisfying $F \cap A \neq \emptyset$ it follows that $F \cap A$ is dense in F. By Lemma 6 of Simon (1999) A is good if and only if for every $z \in A$ $\phi^{\mathcal{V}^{\mathcal{K}}(A)}(z) = z$.

The next lemmatta relate directly the good property to our problem.

Lemma 7 of Simon (1999): If $\mathcal{K} = (S; J; (\mathcal{P}^j | j \in J); X; \psi)$ is a multi-partition with evaluations then $\phi^{\mathcal{K}}(S)$ is a good subset.

Lemma 9 of Simon (1999): If A is a good subset of a cell C and if $A \cap F$ is closed for every $F \in \mathcal{P}^j$ with $A \cap F \neq \emptyset$, then $A = C$.

We need a few more facts about $\Omega(X, J)$ for non-empty X and J. If $|J| \geq 2$ then $\Omega(X, J)$ is topologically equivalent to a Cantor set, (Fagin, Halpern and Vardi 1991). A Cantor set with the usual topology is a metric space. Second we can perceive a Cantor set as $\{0,1\}^{\mathbf{N}}$, where each finite sequence $a = (a^1, a^2, \ldots, a^n)$ defines a cylinder subset $C(a)$ of $\{0,1\}^{\mathbf{N}}$ by $C(a) := \{x \in \{0,1\}^{\mathbf{N}} \mid x^k = a^k \ \forall k \leq n\}$. Furthermore all cylinder subsets are themselves topologically equivalent to Cantor sets, and the same holds for finite unions of cylinder sets. Third, if $|J| \geq 2$ then there exists an uncentered cell of $\Omega(X, J)$ of finite fanout that is dense in $\Omega(X, J)$ (Simon 1999).

Due to topological formulations of the centered property, to demonstrate that there is a surjective cell without finite fanout requires some topological insight. Central to our proof is Theorem 9 of Chapter 12 of E. Moise, (1977):

Let X and Y be two totally disconnected, perfect, compact metric spaces (equivalently Cantor sets) and let X' and Y' be countable and dense subsets of X and Y, respectively. There is a homeomorphism between X and Y that is also a bijection between X' and Y'.

We call a partition $\mathcal{P}$ of a metric space D upper (respectively lower) hemi-continuous if the set valued correspondence that maps every $d \in D$ to the partition member of $\mathcal{P}$ containing d is an upper (respectively lower) hemi-continuous correspondence. (We follow the definitions of Klein and Thompson, 1984.)

Lemma 1: If $\mathcal{K} := (S; J; (\mathcal{P}^j \mid j \in J); X; \psi)$ is a multi-partition with evaluations with a topology (not necessarily that induced by the formulas) such that
1) for every $z \in \{0,1\}^X$ the set $\psi^{-1}(z)$ is clopen (closed and open) and
2) for every $j \in N$ the partition $\mathcal{P}^j$ is lower and upper hemi-continuous,
then the map $\phi^{\mathcal{K}} : S \to \Omega(X, J)$ is continuous.

Proof: It suffices to show that $\alpha^{\mathcal{K}}(f)$ is a clopen set for every $f \in \mathcal{L}(X, J)$. We proceed by induction on the structure of formulas. The claim is true for all $x \in X$ by the hypothesis and it is likewise true for $\neg f$ and $f \wedge g$ if it is true for f and g, due to the clopen property being closed under complementation and finite intersection. For some $f \in \mathcal{L}(X, J)$ we assume that $\alpha^{\mathcal{K}}(f)$ is a clopen set. $\alpha^{\mathcal{K}}(k_j f)$ is an open set by the upper semi-continuity of $\mathcal{P}^j$ and the openness of $\alpha^{\mathcal{K}}(f)$. $S \backslash \alpha^{\mathcal{K}}(k_j f) = \alpha^{\mathcal{K}}(\neg k_j f)$ is an open set by the openness of $S \backslash \alpha^{\mathcal{K}}(f)$ and the lower semi-continuity of $\mathcal{P}^j$. □

Lemma 2: Given X and J finite, for every $j \in J$ the partition $\mathcal{Q}^j(X, J)$ of $\Omega(X, J)$ is upper and lower hemi-continuous with respect to the topology induced by the formulas.

Proof: Let $x_1, x_2, \ldots$ be a sequence of points converging to some $x \in P \in \mathcal{Q}^j$ with $x_i \in P_i \in \mathcal{Q}^j$ for every $i = 1, 2, \ldots$.

To prove that $\mathcal{Q}^j$ is upper hemi-continuous it suffices to show that if $y_1, y_2, \ldots$ is a sequence of points converging to y with $y_1 \in P_1, y_2 \in P_2, \ldots$ then y is in P. Let f be any formula such that $k_j f \in y$. Since the y_i converge to y there is an N such that for every $i \geq N$ it must hold that $k_j f$ is in both y_i and

x_i. But this means that $k_j f$ is also in x. The same argument holds for the formula $\neg k_j f$.

To prove that $\mathcal{Q}^j$ is lower hemi-continuous it suffices to show that if $y \in P$ then there is a sequence of $y_1, y_2, \ldots$ in $P_1, P_2, \ldots$ respectively that converges to y. Because there are only countably many formulas and one can create a new sequence from the diagonal of sequences which come closer and closer to y, if the claim were not true then there would be some formula f in y and an N such that f is not in any member of P_i for all $i \geq N$. This would imply also that $k_j(\neg f)$ is in x_i for all $i \geq N$ and likewise that $k_j(\neg f)$ is in x. But this would contradict that the assumption that f is in y and y is in P. $\square$.

3 The Example

Let Ω equal $\Omega(X, \{1,2\})$ with X any finite non-empty set. Let C be an uncentered cell of finite fanout that is dense in Ω. We assume that $\pi : \Omega \to \{0,1\}^{\mathbf{N}}$ is a homeomorphism. For every $n \in \mathbf{N}$ define $\pi_n : \Omega \to \{0,1\}^n$ by $\pi_n(x)$ equaling the $a = (a^1, a^2, \ldots, a^n) \in \{0,1\}^n$ such that $\pi(x) = (a_1, \ldots, a_n, \ldots)$. This means that $\pi_n^{-1} \circ \pi_n(x)$ equals $C(\pi_n(x))$, the corresponding cylinder set. If a is the empty sequence in $\{0,1\}^0$ then define $\pi_0(x) := a$ and $\pi_0^{-1} \circ \pi_0(x) = \Omega$ for all $x \in \Omega$.

Let z be any member of C and for every $i = 1, 2, \ldots$ let z_i be a member of C such that $\pi_{2i-2}(z_i) = \pi_{2i-2}(z)$ but $\pi_{2i}(z_i) \neq \pi_{2i}(z)$. For every i define non-empty and mutually disjoint sets $A_{i,1}, A_{i,2}, \ldots A_{i,i}$ in the following way. Let $A_{1,1}$ equal $\Omega \backslash\ (\pi_2^{-1} \circ \pi_2(z_1) \cup \pi_2^{-1} \circ \pi_2(z))$. For $1 \leq k < i$ let $A_{i,k} := \pi_{2i-2}^{-1} \circ \pi_{2i-2}(z_k) \backslash\ \pi_{2i}^{-1} \circ \pi_{2i}(z_k)$ and let $A_{i,i} := \pi_{2i-2}^{-1} \circ \pi_{2i-2}(z) \backslash\ (\pi_{2i}^{-1} \circ \pi_{2i}(z_i) \cup \pi_{2i}^{-1} \circ \pi_{2i}(z))$. Because for every $a \in \{0,1\}^{2i}$ there are four members b of $\{0,1\}^{2i+2}$ such that $a = \pi_{2i} \circ \pi_{2i+2}^{-1}(b)$, all the sets $A_{i,j}$ are non-empty and homeomorphic to Cantor sets. By Proposition 1, for every $i \geq 1$ and $1 \leq k \leq i$ there is a homeomorphism $f_k : A_{i,1} \to A_{i,k}$ such that f_k maps $C \cap A_{i,1}$ bijectively to $C \cap A_{i,k}$. This implies for every $i \geq 1$ that there exists an upper and lower semi-continuous partition $\mathcal{P}^i$ of $C \cap (\cup_{k=1}^i A_{i,k})$ such that every partition member of $\mathcal{P}^i$ has i members, one member in $A_{i,k}$ for every $1 \leq k \leq i$. Notice that all the $A_{i,k}$ are mutually disjoint, meaning that $A_{i,k} = A_{i^*,j^*}$ if and only if $i = i^*$ and $k = k^*$. Furthermore the disjoint union $\cup_{i \geq 1} \cup_{1 \leq k \leq i} A_{i,k}$ is equal to $\Omega \backslash\ \{z, z_1, z_2, \ldots\}$. Let $\mathcal{P}$ be

$(\cup_{i=1}^{\infty} \mathcal{P}^i) \cup \{z, z_1, z_2, \ldots\}$, a partition of C. It is straightforward to check that $\mathcal{P}$ is upper and lower semi-continuous. We define $\mathcal{A}$ be the multi-partition with evaluations $(C; \{1,2,3\}; \mathcal{Q}^1|_C, \mathcal{Q}^2|_C, \mathcal{P}; X, \psi|_C)$, with the partition $\mathcal{P}$ corresponding to the third person.

Theorem: $\phi^{\mathcal{A}}$ maps C bijectively to a cell of $\Omega(\{1,2,3\})$ that is surjective but without finite fanout.

Proof: We have by Lemma 1 that $\phi^{\mathcal{A}} : C \rightarrow \Omega(X, \{1,2,3\})$ is continuous. Since every member of $\mathcal{Q}^1|_C$, $\mathcal{Q}^2|_C$, or $\mathcal{P}$ is compact, their images in $\Omega(X, \{1,2,3\})$ are also compact. By Lemma 9 of Simon (1999) $\phi^{\mathcal{A}}$ maps C surjectively to a cell $\phi^{\mathcal{A}}(C)$ of $\Omega(X, \{1,2,3\})$. Between any two points of $\phi^{\mathcal{A}}(C)$ there is an adjacency path using images of members of $\mathcal{Q}^1|_C$ and $\mathcal{Q}^2|_C$, all finite possibility sets of $\Omega(X, \{1,2,3\})$ – therefore there can be no proper good subset of $\phi^{\mathcal{A}}(C)$. By Lemma 7 of Simon (1999) this implies that $\phi^{\mathcal{A}}(C)$ is a surjective cell. Since for every $f \in \mathcal{L}(X, \{1,2\})$ $\alpha^{\Omega(X,\{1,2\})}(f)$ gets mapped to $\alpha^{\Omega(X,\{1,2,3\})}(f)$, $\phi^{\mathcal{A}}$ is an injective and an open map (meaning that open sets are mapped to open sets), and therefore the map $\phi^{\mathcal{A}}$ is also a homeomorphism of C to $\phi^{\mathcal{A}}(C)$. Therefore the image of the one infinite set in $\mathcal{P}$ is also an infinite set in the cell $\phi^{\mathcal{A}}(C)$, which implies that this cell of $\Omega(X, \{1,2,3\})$ does not have finite fanout. q.e.d.

4 Acknowledgements

Many helped in finding the best citation for Proposition 1; the lemma has many interesting variations, including the same conclusion for open intervals proven by G. Cantor (1895).

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